Generalization in Large scale MDPs

Wen Sun

CS 6789: Foundations of Reinforcement Learning

Recap on Bellman Error and Bellman Operator

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If $BE(s,a) \neq 0$, then $f \neq Q^\star$

Notations

Probability of π visiting (s, a) at time step h: $d_h^\pi(s, a)$

Question for Today

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In other words, what structural conditions permit RL generalization, provably?

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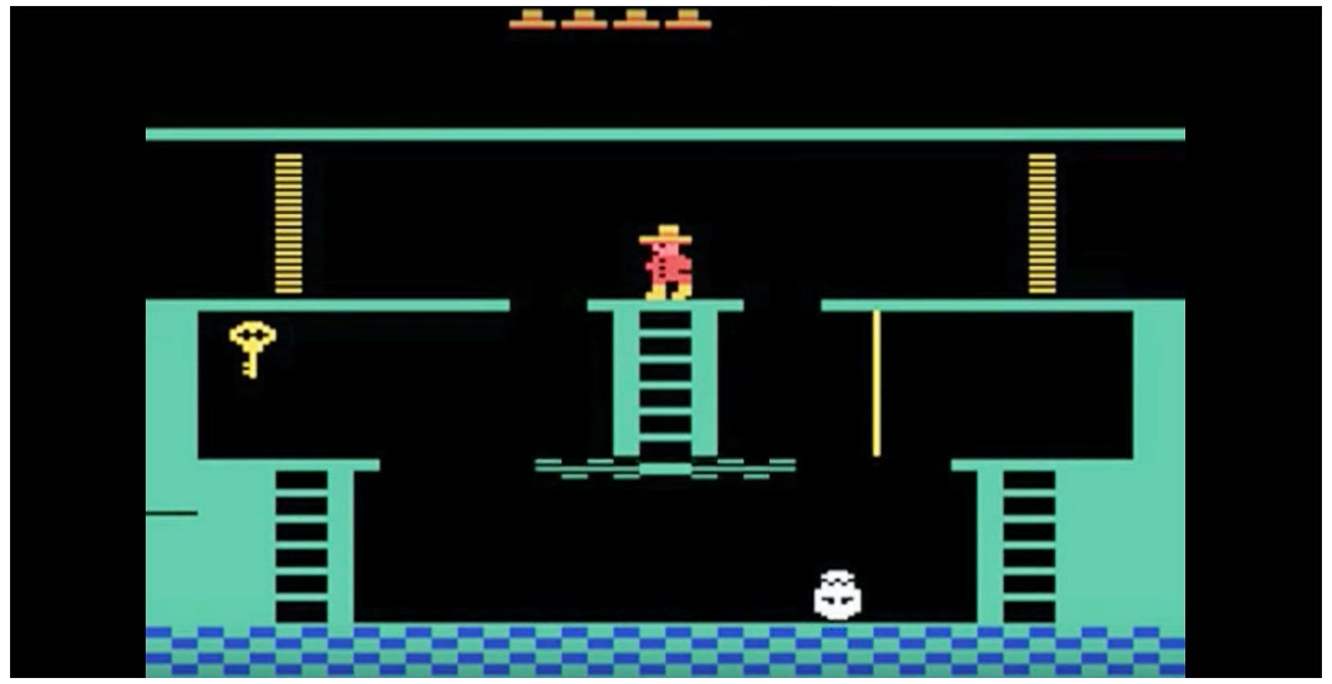
1. Bellman rank Definitions

2. Examples that are captured by the Bellman rank framework

Setting

Finite horizon episodic MDP $\left\{\{S_h\}_{h=0}^{H}, \{A_h\}_{h=0}^{H-1}, H, s_0, r, P\right\}$

State space S_h is extremely large:

$s =$ 

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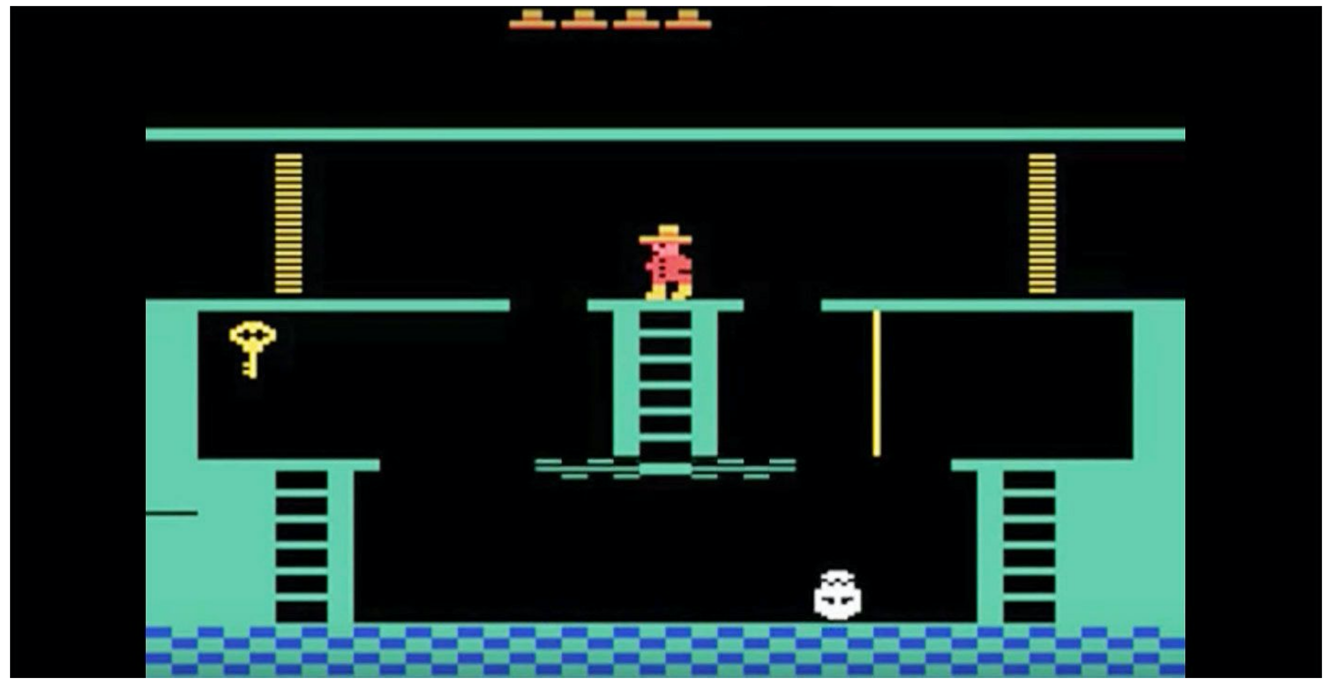
$s =$ 

Not acceptable: poly $\left(|S|\right)$

Need to generalize via (nonlinear) function approximation

Let’s set up function class in RL setting

We will consider **Q function class**

$$\mathcal{F} \subset S \times A \mapsto [0,H]$$

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Define **value function class**: $\mathcal{V} = \{V_f : V_f(s) = \max_{a} f(s,a) \,|\, f \in \mathcal{F}\}$

Learning Goal:

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Given approximation error ϵ and failure prob δ,
can we learn ϵ *near optimal policy* (i.e., $V^{\hat\pi} \geq V^\star - \epsilon$) in # of samples scaling
poly* with all relevant parameters *(here, we need poly in $\ln(|\mathscr{F}|)$)

How to check if a Q-approximator is good?

We define **average** Bellman error of a Q-estimate g below:

$$\mathcal{E}(g; f, h) = \mathbb{E}_{s_h, a_h \sim d_h^{\pi_f}} \left[g(s_h, a_h) - r(s_h, a_h) - \mathbb{E}_{s_{h+1} \sim P(\cdot | s_h, a_h)} \left[\max_{a \in \mathcal{A}} g(s_{h+1}, a) \right] \right]$$

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The Q / V-Bellman rank

$$\forall h : \mathscr{E}_h \in \mathbb{R}^{|\mathscr{F}|\times|\mathscr{F}|}$$

$$\begin{array}{c|c|c|c|c|c|c|}
 & & g & f & & & \\
\hline
 & & & & & & \\
\hline
 & & & & & & \\
\hline
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Rank of this Matrix is defined as Bellman Rank

The Q / V-Bellman rank

In other words, there are two mappings $W_h : \mathcal{F} \mapsto \mathbb{R}^d, \quad X_h : \mathcal{F} \mapsto \mathbb{R}^d$ (d = Bellman-rank)

$$\forall f, g \in \mathcal{F} : \mathcal{E}(g; f, h) = \left\langle W_h(g), X_h(f) \right\rangle$$

Note, we just assume the existence of W, X, but they are unknown

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$$\forall h, \exists w \in \mathbb{R}^d, s.t., w^\top \phi(s, a) = r(s, a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} \theta^\top \phi(s', a'), \forall s, a$$

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Note linear Bell-completion captures tabular / linear mdp already

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As we will see, linear Q*&V* is learnable, and recall linear Q* is not…

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Claim: this model has Q-Bellman rank $|Z||A| + |Z|$

We can show that this model is captured by linear $Q^\star$ & $V^\star$

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$P_h(s' \mid s, a) = \mu_h^\star(s')^\top \phi_h^\star(s, a)$ (neither $\mu^\star$ nor $\phi^\star$ is known)

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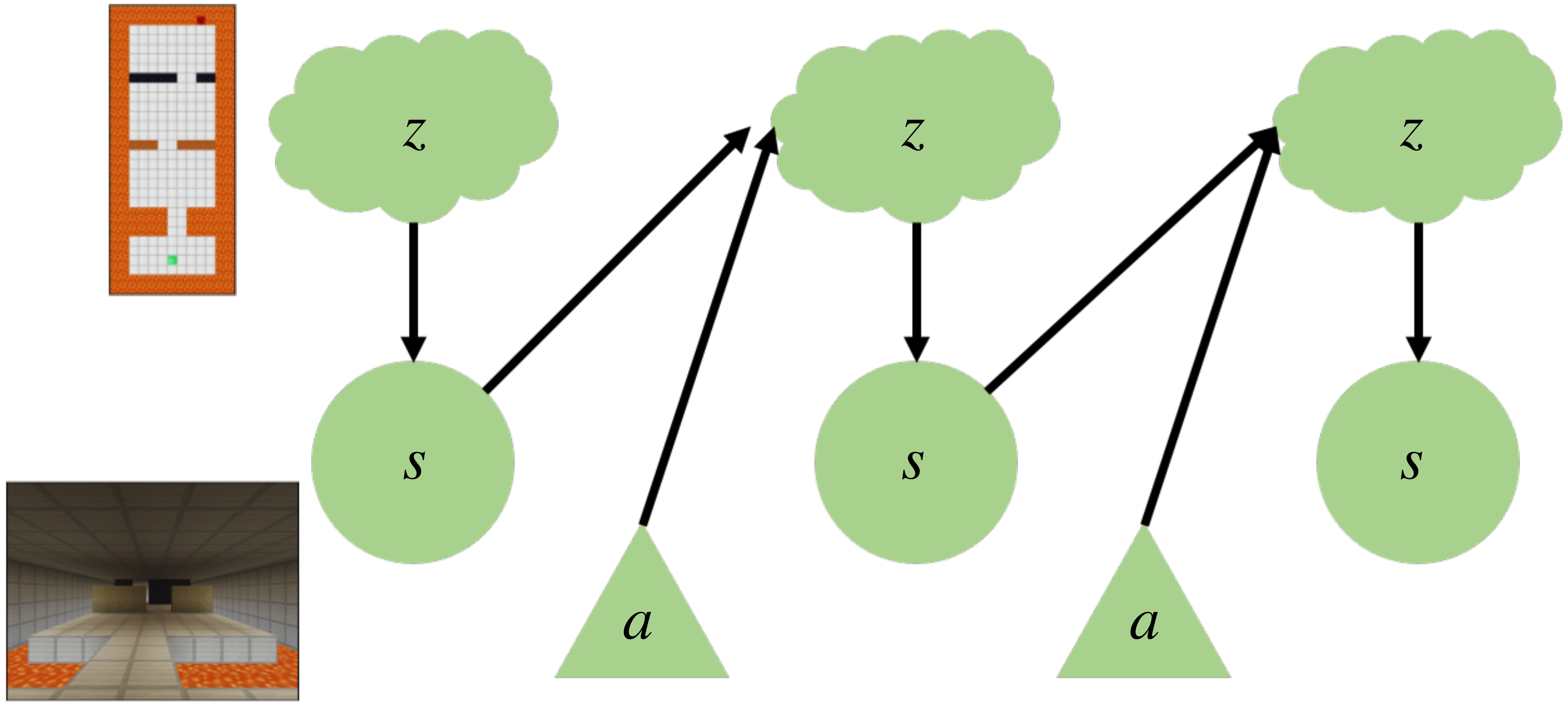
Latent variable MDP

Latent variable MDP is captured by low-rank MDP, so it has small V-Bellman rank…



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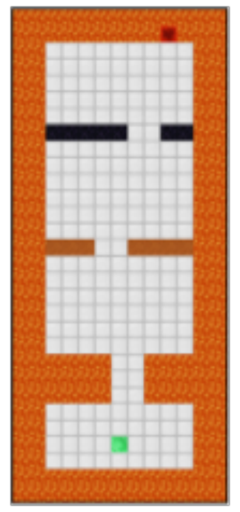

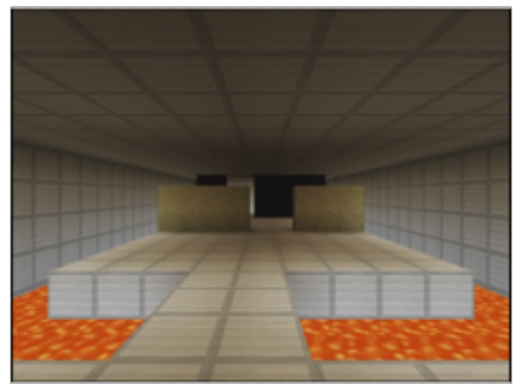

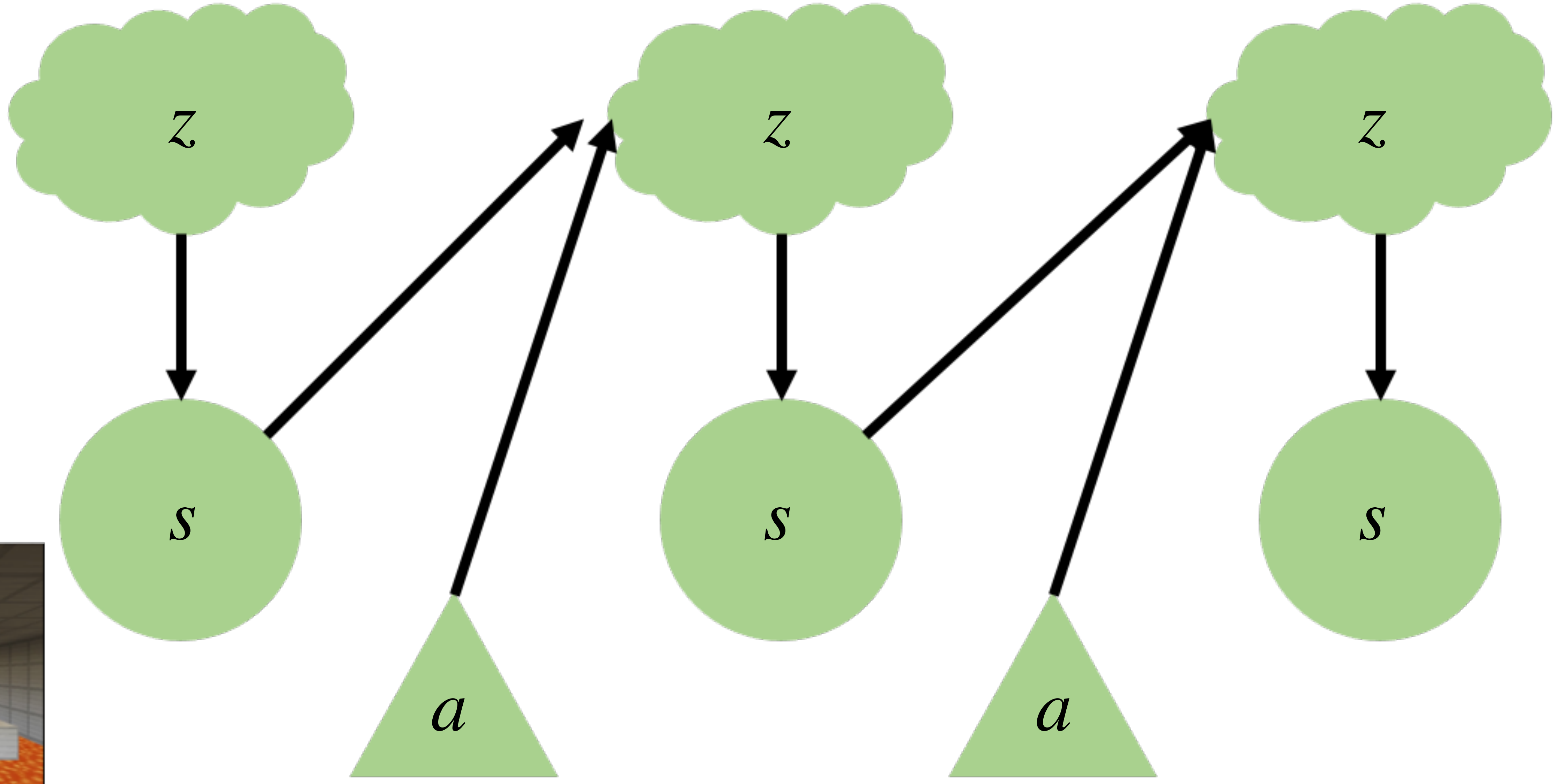

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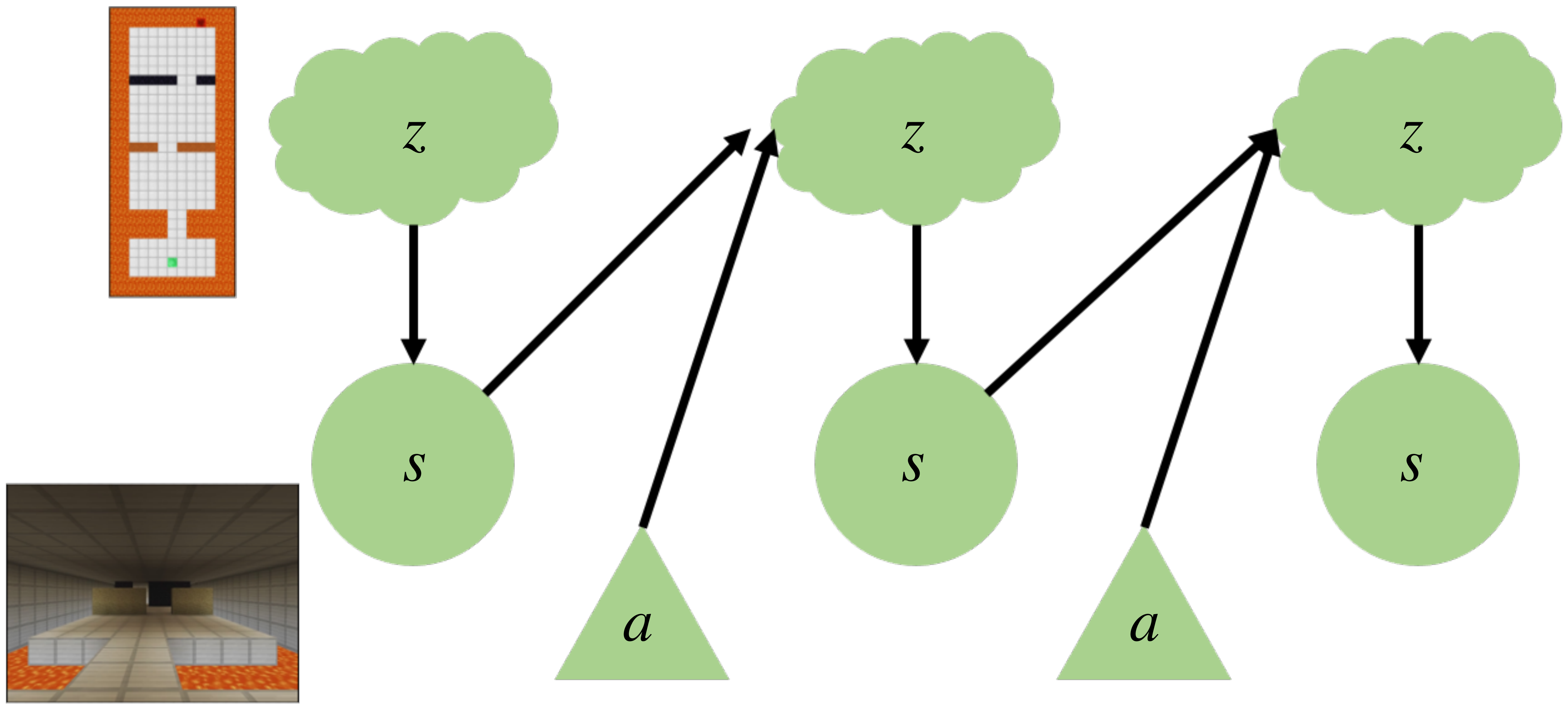

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V-Bellman rank = Number of latent states

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$$\mathscr{E}(g; f, h) = \mathbb{E}_{s_h, a_h \sim d_h^{\pi_f}} \left[g(s_h, a_h) - r(s_h, a_h) - \mathbb{E}_{s_{h+1} \sim P(\cdot | s_h, a_h)} \left[\max_{a \in \mathscr{A}} g(s_{h+1}, a) \right] \right]$$

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where $X_h(f), W_h(f)$ are low-dim vectors

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4. Many models (more in the book chapter) indeed have low-Q or V Bellman rank

Next week:

A general algorithm that can learn an ϵ near optimal policy w/ # of samples

$$\text{poly}(H, 1/\epsilon, \ln(|\mathscr{H}|), \text{b-rank})$$