Linear Bandits

Wen Sun

CS 6789: Foundations of Reinforcement Learning

Recap on MAB

Setting:

We have K many arms: $a_1, \ldots, a_K$

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We have K many arms: $a_1, \ldots, a_K$

Each arm has a unknown reward distribution, i.e., $\nu_i \in \Delta([0,1])$,
w/ mean $\mu_i = \mathbb{E}_{r \sim \nu_i}[r]$

Regret

More formally, we have the following learning objective:

$$\mu^\star = \max_{i \in [K]} \mu_i$$

$$\text{Regret}_T = T\mu^\star - \sum_{t=0}^{T-1} \mu_{I_t}$$

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Total expected reward if we pulled best arm over T rounds

Total expected reward of the arms we pulled over T rounds

Goal: no-regret, i.e., $\text{Regret}_T / T \to 0$, as $T \to \infty$

UCB: Optimism in the face of Uncertainty

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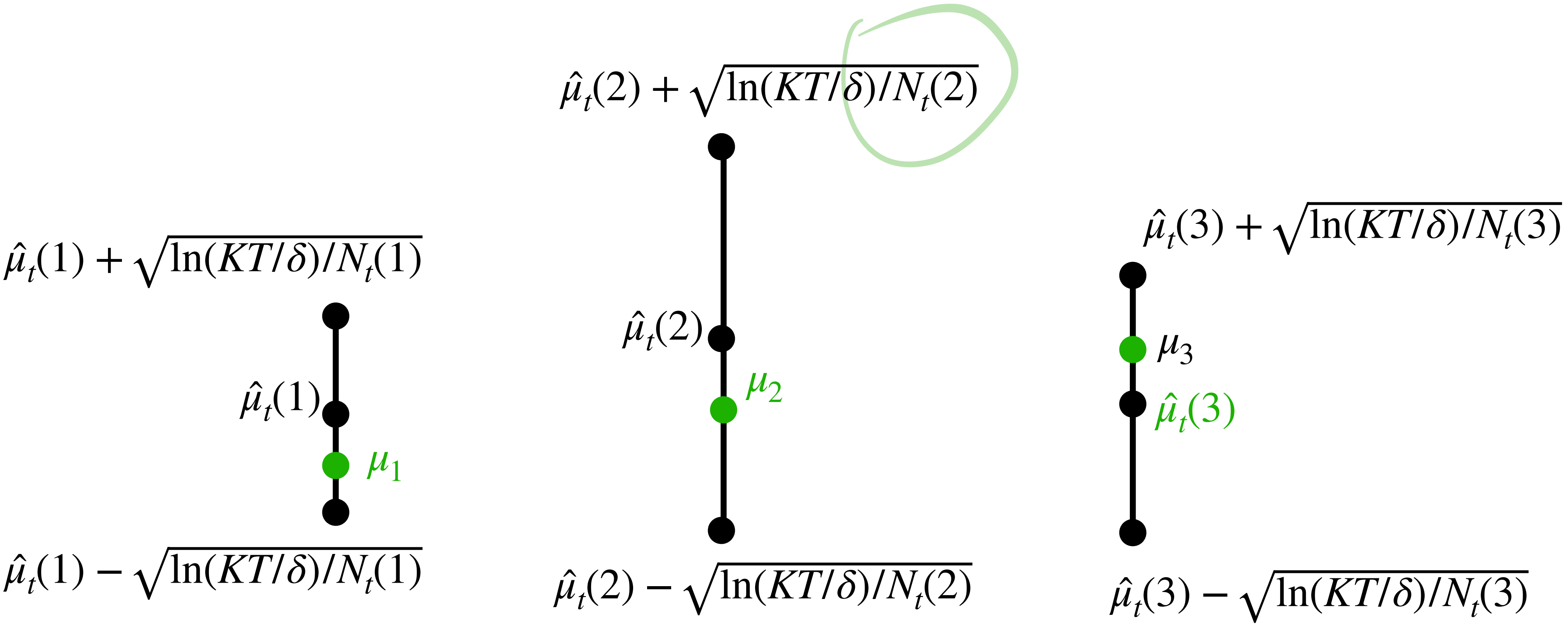

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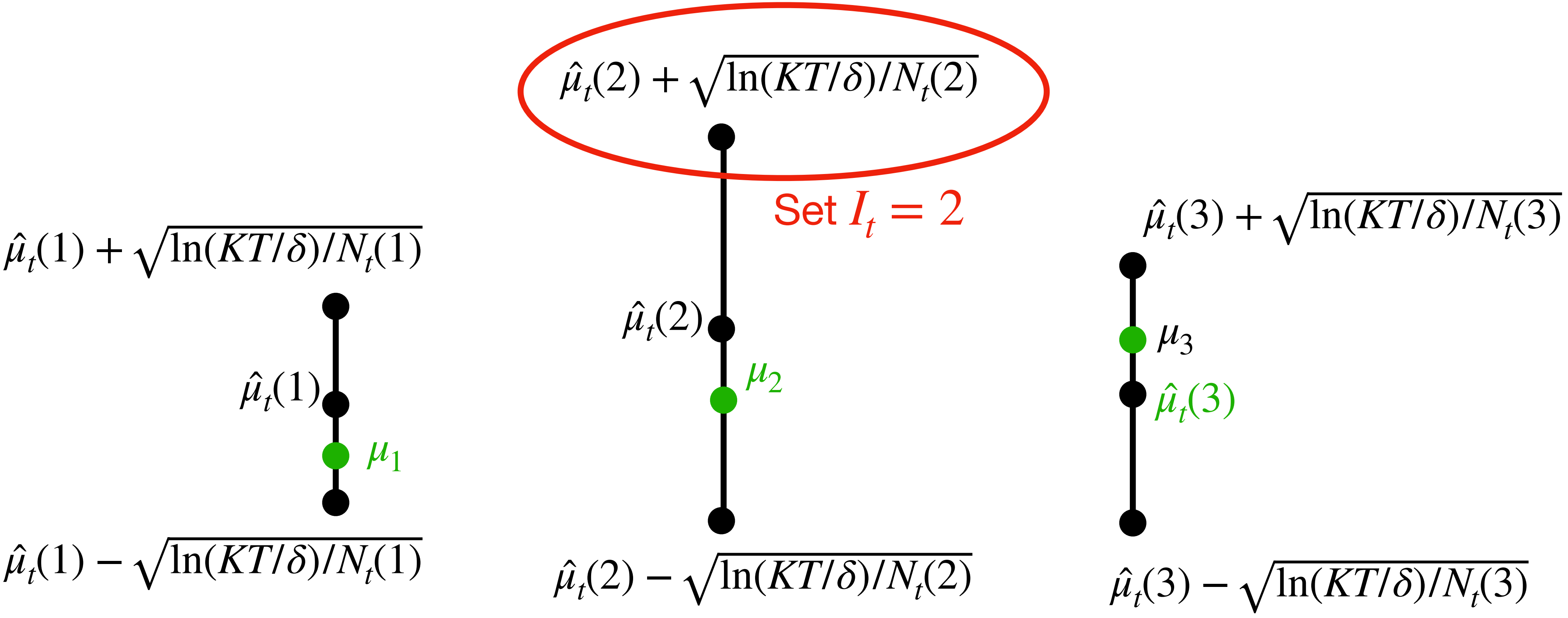

Optimism in the face of uncertainty

Denote the optimal arm $I^\star = \arg\max_{i \in [K]} \mu_i$; recall $I_t = \arg\max_{i \in [K]} \hat{\mu}_t(i) + \sqrt{\frac{\ln(KT/\delta)}{N_t(i)}}$

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$$\text{Regret-at-t} = \mu^\star - \mu_{I_t}$$

$$\leq \hat{\mu}_t(I_t) + \sqrt{\frac{\ln(TK/\delta)}{N_t(I_t)}} - \mu_{I_t}$$

$$\leq 2 \cdot \sqrt{\frac{\ln(\cdots)}{N_t(I_t)}}$$

Optimism in the face of uncertainty

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$$\le \hat{\mu}_t(I_t) + \sqrt{\frac{\ln(TK/\delta)}{N_t(I_t)}} - \mu_{I_t} \le 2\sqrt{\frac{\ln(TK/\delta)}{N_t(I_t)}}$$

optimism

$$\sum_{t=0}^{T} \text{Regret-at-}t \le \sum_{t=0}^{T} \sqrt{\frac{\ln(\ldots)}{N_t(I_t)}}$$

$$\le \sqrt{KT}$$

Today:

MAB w/ K arms has regret $O(\sqrt{KT})$

What if there are infinitely many actions?

Introducing structures in the reward function

Outline for Today:

1. Linear Bandit Setting

2. Algorithm: LinUCB

3. Regret analysis of LinUCB

Linear Bandit Setting

We have an action set $D \subset \mathbb{R}^d$

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Every time we pick an action $x \in D$, we observe a noisy reward

$$r = \mu^\star \cdot x + \eta$$

$a \cdot b = a^\top b$

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Zero mean i.i.d noise

Learning protocol and goal:

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Goal: minimize regret

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Expected Reward of x_t

$$x^\star = \arg\max_{x \in D} \mu^\star \cdot x$$

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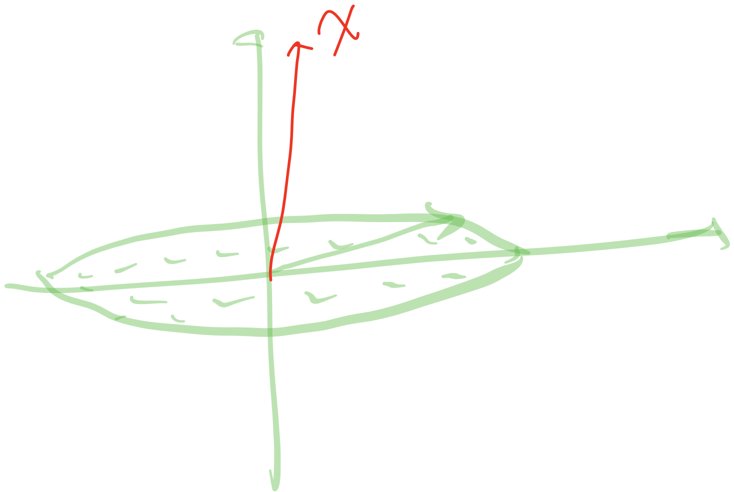
LinUCB algorithm

Overall idea:

Ridge linear regression for learning $\mu^\star$ + design exploration bonus



MAB:

$\hat{\mu} + \sqrt{\frac{1}{N_t(\cdot)}}$

LinUCB algorithm

In iteration t:

1. Perform Ridge LR on data $\{x_i, r_i\}_{i=0}^{t-1}$:

   Set $\hat{\mu}_t := \arg\min_{\mu} \sum_{i=0}^{t-1} (\mu^\top x_i - r_i)^2 + \lambda \|\mu\|_2^2$

LinUCB algorithm

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Set $\hat{\mu}_t := \arg\min_{\mu} \sum_{i=0}^{t-1} (\mu^\top x_i - r_i)^2 + \lambda \|\mu\|_2^2$

$$\Sigma_t = \sum_{i=0}^{t-1} x_i x_i^\top + \lambda I$$

2: Set exploration bonus: $b_t(x) = \beta \sqrt{x^\top \Sigma_t^{-1} x}$

LinUCB algorithm

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2: Set exploration bonus: $b_t(x) = \beta \sqrt{x^\top \Sigma_t^{-1} x}$

3: Play optimistically, i.e., $x_t = \arg\max_{x \in D} \hat{\mu}_t^\top x_t + b_t(x)$

Outline for Today:

1. Linear Bandit Setting

2. Algorithm: LinUCB

$\tilde{O}\sqrt{2^d \cdot T}$

↑ Discretization

3. Regret analysis of LinUCB

$\tilde{O}\sqrt{d^2 T}$

Analysis of Ridge Linear Regression

Recall $\hat{\mu}_t := \arg\min_{\mu} \sum_{i=0}^{t-1} (\mu^\top x_i - r_i)^2 + \lambda \|\mu\|_2^2$

Analysis of Ridge Linear Regression

$\hat{\mu}_t - \mu^*$

Recall $\hat{\mu}_t := \arg\min_{\mu} \sum_{i=0}^{t-1} (\mu^\top x_i - r_i)^2 + \lambda \|\mu\|_2^2$

$$\hat{\mu}_t = \Sigma_t^{-1} \sum_{i=0}^{t-1} x_i r_i$$

$$\Sigma_t = \sum_{i=0}^{t-1} x_i x_i^\top + \lambda I$$

Regularization

$r_i = \mu^{*\top} x_i + \eta_i$

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$$\hat{\mu}_t = \Sigma_t^{-1} \sum_{i=0}^{t-1} x_i r_i$$

$$= \Sigma_t^{-1} \sum_{i=0}^{t-1} x_i (x_i^\top \mu^\star + \eta_i)$$

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$$= \Sigma_t^{-1} \sum_{i=0}^{t-1} x_i (x_i^\top \mu^\star + \eta_i) \;= \Sigma_t^{-1} (\Sigma_t - \lambda I) \mu^\star + \Sigma_t^{-1} \sum_{i=0}^{t-1} x_i \eta_i$$

$\sum_{i=0}^{t-1} x_t x_t^\top$

Analysis of Ridge Linear Regression

Recall $\hat{\mu}_t := \arg\min_{\mu} \sum_{i=0}^{t-1} (\mu^\top x_i - r_i)^2 + \lambda \|\mu\|_2^2$

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$$= \mu^\star - \lambda \Sigma_t^{-1} \mu^\star + \Sigma_t^{-1} \sum_{i=0}^{t-1} x_i \eta_i$$

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$$= \mu^\star - \lambda \Sigma_t^{-1} \mu^\star + \Sigma_t^{-1} \sum_{i=0}^{t-1} x_i \eta_i$$

$$\hat{\mu}_t - \mu^\star = - \lambda \Sigma_t^{-1} \mu^\star + \Sigma_t^{-1} \sum_{i=0}^{t-1} x_i \eta_i$$

Analysis of Ridge Linear Regression

$$\Sigma_t = \sum_{i=0}^{t-1} x_t x_t^\top + \lambda I$$

$$\hat{\mu}_t - \mu^\star = -\lambda \Sigma_t^{-1} \mu^\star + \Sigma_t^{-1} \sum_{i=0}^{t-1} x_i \eta_i$$

Let us look at the training error:

$$\sqrt{(\hat{\mu}_t - \mu^\star)^\top \Sigma_t (\hat{\mu}_t - \mu^\star)}$$

$$\Rightarrow \sum_{i=1}^{t-1} \left((\hat{\mu}_t - \mu^\star)^\top x_i \right)^2 + \| (\hat{\mu}_t - \mu^\star) \|^2 \cdot \lambda$$

Analysis of Ridge Linear Regression

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Let us look at the training error:

$$\sqrt{(\hat{\mu}_t - \mu^\star)^\top \Sigma_t (\hat{\mu}_t - \mu^\star)} \leq \left\| \lambda \Sigma_t^{-1/2} \mu^\star \right\| + \left\| \Sigma_t^{-1/2} \sum_{i=0}^{t-1} \eta_i x_i \right\|$$

Triangle Ineq.

$$\left\| \Sigma_t^{\frac{1}{2}} (\hat{\mu}_t - \mu^\star) \right\| = \left\| -\lambda \Sigma_t^{-\frac{1}{2}} \mu^\star + \Sigma_t^{-\frac{1}{2}} \sum_{i=0}^{t-1} x_i \eta_i \right\|$$

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$$\leq \sqrt{\lambda} \|\mu^\star\| + ???$$

$$\sigma_{max}\left(\Sigma_t^{-\frac{1}{2}}\right) \leq \frac{1}{\sqrt{\lambda}}$$

Analysis of Ridge Linear Regression

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Self-normalized Martingale bound

$$\left\| \Sigma_t^{-\frac{1}{2}} \sum_{i=0}^{t-1} \eta_i x_i \right\|^2 \approx d \log(t)$$

Self-normalized Bound for Vector-valued Martingales

Suppose $\{\eta_i\}_{i=0}^{\infty}$ are mean zero random variables, and $|\eta_i| \leq \sigma$;

Let $\{x_i\}_{i=0}^{\infty}$ be any sequence of random vectors with $\|x_i\| \leq 1$, then w/ prob $1-\delta$, for all $t \geq 1$,

$$\left\| \Sigma_t^{-1/2} \sum_{i=0}^{t-1} x_i \eta_i \right\|^2 \leq \sigma^2 d \cdot \left(\ln\left(\frac{t}{\lambda} + 1\right) + \ln(1/\delta) \right)$$

$$\left(\sum_{i=0}^{t-1} x_i \eta_i\right)^\top \Sigma_t \left(\sum_{i=0}^{t-1} x_i \eta_i\right) \quad \Sigma_t = \sum_{i=0}^{t-1} x_i x_i^\top + \lambda I$$

Analysis of Ridge Linear Regression (Continue)

$$\hat{\mu}_t - \mu^\star = -\lambda \Sigma_t^{-1} \mu^\star + \Sigma_t^{-1} \sum_{i=0}^{t-1} x_i \eta_i$$

Let us look at the training error:

$$\sqrt{(\hat{\mu}_t - \mu^\star)^\top \Sigma_t (\hat{\mu}_t - \mu^\star)} \leq \left\| \lambda \Sigma_t^{-1/2} \mu^\star \right\| + \left\| \Sigma_t^{-1/2} \sum_{i=0}^{t-1} \eta_i x_i \right\| \approx \sqrt{d \ln(T)}$$

$$\lesssim \sqrt{\lambda} + \sigma \sqrt{d \cdot \ln(T/(\lambda\delta))}$$

Summary for Ridge Linear Regression

$$\hat{\mu}_t - \mu^\star = -\lambda \Sigma_t^{-1} \mu^\star + \Sigma_t^{-1} \sum_{i=0}^{t-1} x_i \eta_i$$

$$\sqrt{(\hat{\mu}_t - \mu^\star)^\top \Sigma_t (\hat{\mu}_t - \mu^\star)} \lesssim \sqrt{\lambda} + \sigma \sqrt{d \ln(T/(\lambda\delta))}$$

Optimism

$$\sqrt{(\hat{\mu}_t - \mu^\star)^\top \Sigma_t (\hat{\mu}_t - \mu^\star)} \lesssim \sqrt{\lambda} + \sigma^2 d \ln(T/(\lambda\delta))$$

Let's construct uncertainty quantification for each action $x \in D$

$$|\hat{\mu}_t \cdot x - \mu^\star \cdot x|$$

Optimism

$$\sqrt{(\hat{\mu}_t - \mu^\star)^\top \Sigma_t (\hat{\mu}_t - \mu^\star)} \lesssim \sqrt{\lambda} + \sqrt{\sigma^2 d \ln(T/(\lambda\delta))}$$

Let's construct uncertainty quantification for each action $x \in D$

CS

$$|\hat{\mu}_t \cdot x - \mu^\star \cdot x| \leq \|\hat{\mu}_t - \mu^\star\|_{\Sigma_t} \cdot \|x\|_{\Sigma_t^{-1}}$$

$\leq \sqrt{\lambda} + \sqrt{\sigma^2 d \ln(T)}$

Optimism

$$\sqrt{(\hat{\mu}_t - \mu^\star)^\top \Sigma_t (\hat{\mu}_t - \mu^\star)} \lesssim \sqrt{\lambda} + \sigma^2 d \ln(T/(\lambda\delta))$$

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$$b_t(x) := \beta \cdot \|x\|_{\Sigma_t^{-1}}$$

Optimism

$$\sqrt{(\hat{\mu}_t - \mu^\star)^\top \Sigma_t (\hat{\mu}_t - \mu^\star)} \lesssim \sqrt{\lambda} + \sigma^2 d \ln(T/(\lambda\delta))$$

$\forall x \in D$

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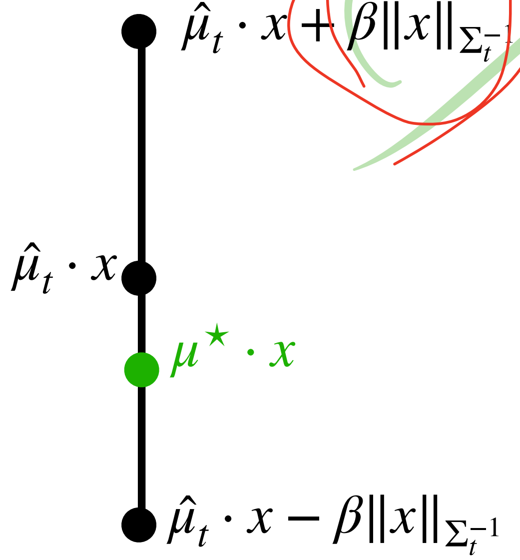
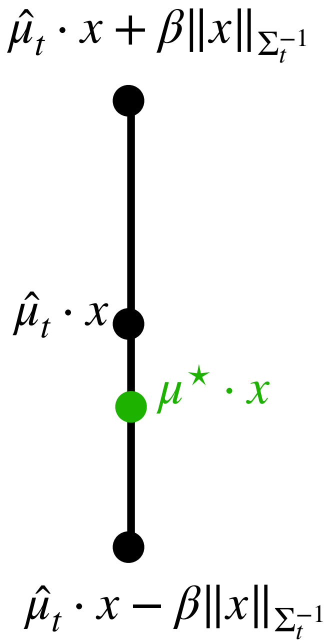


Optimism

Optimism: $\mu^\star \cdot x^\star \leq \hat{\mu}_t \cdot x_t + \beta \|x_t\|_{\Sigma_t^{-1}}$

$\forall x \in D$



Proof:

$x_t = \arg\max \hat{\mu}_t \cdot x + \beta \|x\|_{\Sigma_t^{-1}}$

Regret

Regret-at-t $= \mu^\star \cdot x^\star - \mu^\star \cdot x_t$

Regret

$$\text{Regret-at-t} = \mu^\star \cdot x^\star - \mu^\star \cdot x_t$$

$$\leq \hat{\mu}_t^\top x_t + \beta \|x_t\|_{\Sigma_t^{-1}} - \mu^\star \cdot x_t$$

$$\left| \hat{\mu}_t^\top x_t - \mu^{\star\top} \cdot x_t \right| \leq \beta \|x_t\|_{\Sigma_t^{-1}}$$

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Intuitively this should be convincing already:

$$\mu^\star \cdot x^\star - \mu^\star \cdot x_t \geq \delta$$

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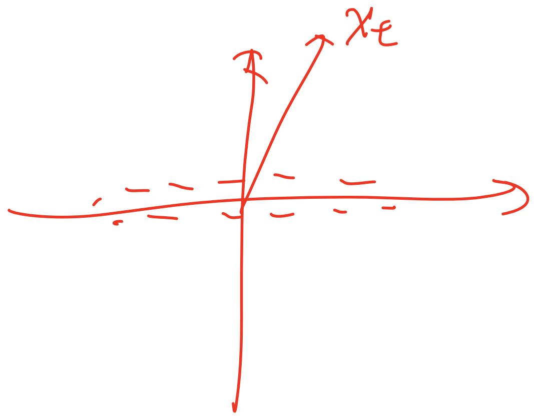
Case 1: x_t is a bad arm, i.e., $2\beta \|x_t\|_{\Sigma_t^{-1}} \geq \mu^\star \cdot (x^\star - x_t) \geq \delta$



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x_t falls in the subspace where "data is sparse", i.e., we explored!

Regret

Regret-at-t $= \mu^\star \cdot x^\star - \mu^\star \cdot x_t$

$$\leq \hat{\mu}_t^\top x_t + \beta \|x_t\|_{\Sigma_t^{-1}} - \mu^\star \cdot x_t \quad \leq 2\beta \|x_t\|_{\Sigma_t^{-1}}$$

$\leq$ Small

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Case 2: confidence interval $\|x_t\|_{\Sigma_t^{-1}}$ is small

Regret

Regret-at-t $= \mu^\star \cdot x^\star - \mu^\star \cdot x_t$

$$\leq \hat{\mu}_t^\top x_t + \beta \|x_t\|_{\Sigma_t^{-1}} - \mu^\star \cdot x_t \ \leq 2\beta \|x_t\|_{\Sigma_t^{-1}}$$

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Case 1: x_t is a bad arm, i.e., $2\beta \|x_t\|_{\Sigma_t^{-1}} \geq \mu^\star \cdot (x^\star - x_t) \geq \delta$

x_t falls in the subspace where “data is sparse”, i.e., we explored!

Case 2: confidence interval $\|x_t\|_{\Sigma_t^{-1}}$ is small

Then regret at this round is small too, i.e., we exploited!

Regret

$$\text{Regret-at-t} = \mu^\star \cdot x^\star - \mu^\star \cdot x_t$$

$$\leq \hat{\mu}_t^\top x_t + \beta \|x_t\|_{\Sigma_t^{-1}} - \mu^\star \cdot x_t \leq 2\beta \|x_t\|_{\Sigma_t^{-1}}$$

More formally, we can show:

$$\text{Regret} \leq \beta \sum_{t=0}^{T-1} \|x_t\|_{\Sigma_t^{-1}}$$

Regret

$$\text{Regret-at-t} = \mu^\star \cdot x^\star - \mu^\star \cdot x_t$$

$$\leq \hat{\mu}_t^\top x_t + \beta \|x_t\|_{\Sigma_t^{-1}} - \mu^\star \cdot x_t \quad \leq 2\beta \|x_t\|_{\Sigma_t^{-1}}$$

More formally, we can show:

$$\text{Regret} \leq \beta \sum_{t=0}^{T-1} \|x_t\|_{\Sigma_t^{-1}} \leq \beta \sqrt{T} \cdot \sqrt{\sum_{t=0}^{T-1} \|x_t\|^2_{\Sigma_t^{-1}}}$$

$$\sum_{t=0}^{T-1} \|x_t\|^2_{\Sigma_t^{-1}} \leq d \ln(T)$$

$$\beta = \sqrt{\lambda} + \sigma \sqrt{d \ln(T)}$$

$$= d\sqrt{T}$$

Regret

Regret-at-t $= \mu^\star \cdot x^\star - \mu^\star \cdot x_t$

$$\le \hat{\mu}_t^\top x_t + \beta \|x_t\|_{\Sigma_t^{-1}} - \mu^\star \cdot x_t \quad \le 2\beta \|x_t\|_{\Sigma_t^{-1}}$$

More formally, we can show:

$$\text{Regret} \le \beta \sum_{t=0}^{T-1} \|x_t\|_{\Sigma_t^{-1}} \le \beta\sqrt{T} \cdot \sqrt{\sum_{t=0}^{T-1} \|x_t\|^2_{\Sigma_t^{-1}}}$$

$$\lesssim \beta\sqrt{T} \cdot \sqrt{d \ln(T/\lambda + 1)} \qquad \forall \lambda \ge 1$$

$$\sum_{t=0}^{T-1} \|x_t\|^2_{\Sigma_t^{-1}} \le O(d \ln T)$$

Summary

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x

$\phi(x)$

$K(x, x')$

$= \langle \phi(x), \phi(x') \rangle$

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4. Regret is upper bounded by $\beta \sum_t \|x_t\|_{\Sigma_t} \leq \beta \sqrt{T} \sqrt{\sum_t \|x_t\|^2_{\Sigma_t^{-1}}}$