Multi-armed Bandits

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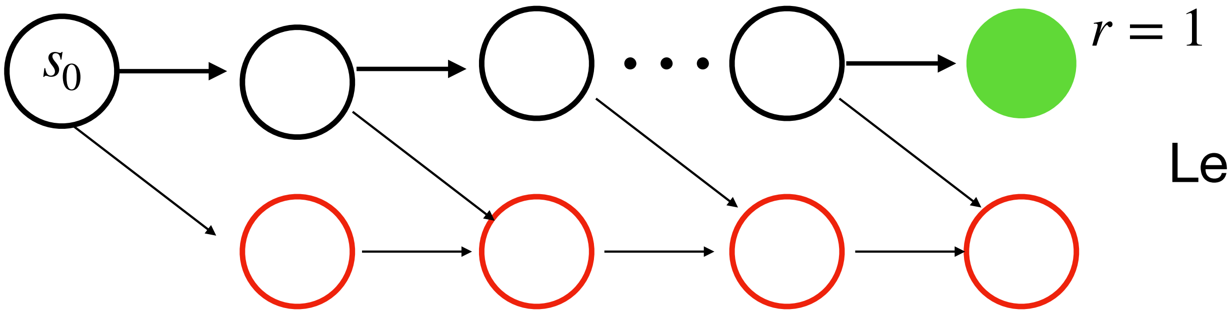
CS 6789: Foundations of Reinforcement Learning

The need for Exploration in RL:

The Combination Lock Example (i.e., the sparse reward problem)

(1) We have reward zero everywhere except at the goal (the right end);
(2) Every black node, one of the two actions will lead the agent to the dead state (red)



Length: H

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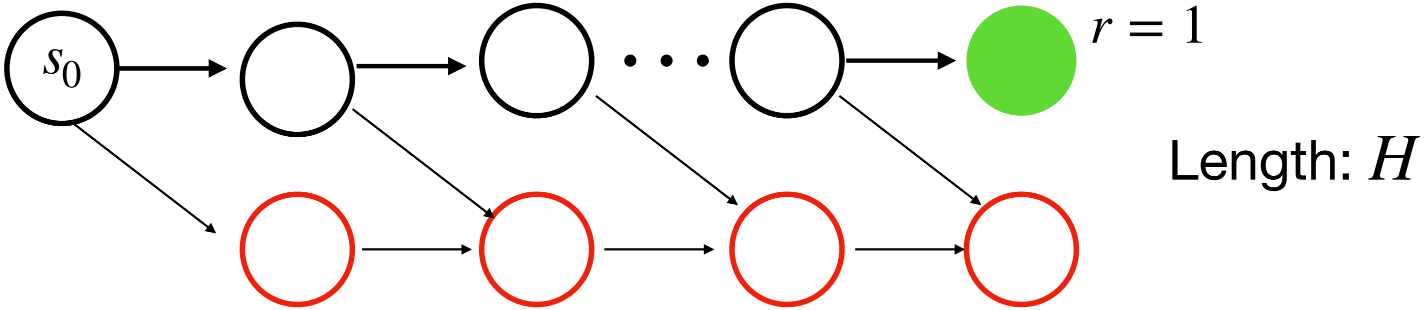


What is the probability of a random policy generating a trajectory that hits the goal?

Exploration!

We need to perform systematic exploration,
i.e., remember where we visited, and purposely try to visit unexplored regions..

What we will do today:

Study Exploration in a very simple MDP:

$$\mathscr{M} = \{s_0, \{a_1, \ldots, a_K\}, H = 1, R\}$$

i.e., MDP with one state, one-step transition, and K actions

This is also called Multi-armed Bandits

Plan for today:

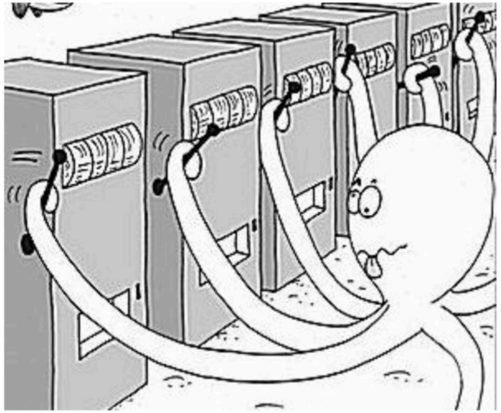
1. Introduction of MAB

2. Attempt 1: Greedy Algorithm (a bad algorithm)

3. Attempt 2: Explore and Commit

4. Attempt 3: Upper Confidence Bound (UCB) Algorithm

Intro to MAB



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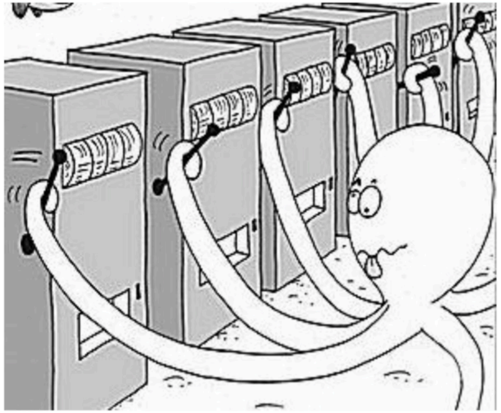

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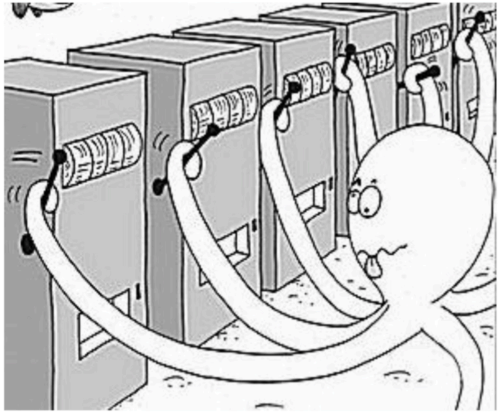

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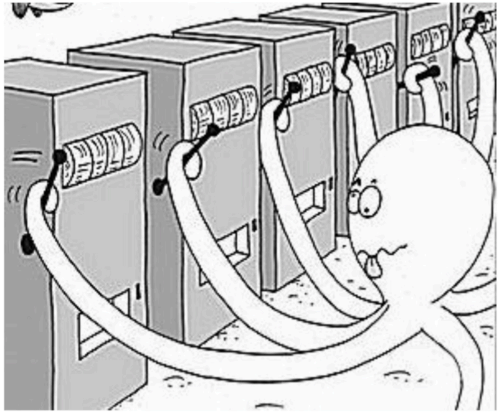

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Every time we pull arm a_i, we observe an i.i.d reward $r = \begin{cases} 1 & \text{w/ prob } p \\ 0 & \text{w/ prob } 1 - p \end{cases}$

Intro to MAB

Applications on online advertisement:

Arms correspond to Ads

Each arm has **click-through-rate** (CTR): probability of getting clicked (unknown)

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3. **Update**: Decide what ad to recommend for next round

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For $t = 0 \rightarrow T - 1$

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Note: each iteration, we do not observe rewards of arms that we did not try

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$$\text{Regret}_T = T\mu^\star - \sum_{t=0}^{T-1} \mu_{I_t}$$

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← arms we tried At Round t

Total expected reward if we pulled best arm over T rounds

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Goal: no-regret, i.e., $\text{Regret}_T / T \to 0$, as $T \to \infty$

Intro to MAB

Why the problem is hard?

Exploration and Exploitation Tradeoff:

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Exploration and Exploitation Tradeoff:

Every round, we need to ask ourselves:

Should we pull arms that are less frequently tried in the past (i.e., **explore**),
Or should we commit to the current best arm (i.e., **exploit**)?

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Q: what could be wrong?

A bad arm (i.e., low μ_i) may generate a high reward by chance!
(recall we have $r \sim \nu$, i.i.d)

Attempt 1: Greedy Algorithm

More concretely, let's say we have two arms a_1, a_2:

Reward dist for a_1: w/ prob 60%, $r = 1$; else $r = 0$

Reward dist for a_2: w/ prob 40%, $r = 1$; else $r = 0$

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The greedy alg will pick a_2—**loosing expected reward 0.2 every time in the future**

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(a bad algorithm: constant regret)

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What lessons we learned from the Greedy Alg:

Due to randomness in the reward distribution, trying each arm once is not enough, i.e., observed single reward may be far away from the mean

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Q: what's the fix here?

Yes, let's (1) try each arm multiple times, (2) compute the empirical mean of each arm, (3) commit to the one that has the highest empirical mean

Alg: Explore and Commit:

Algorithm hyper parameter $N < T/K$ (we assume $T >> K$)

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Calculate arm k's empirical mean: $\hat{\mu}_k = \sum_{i=1}^{N} r_i / N$

$N \to \infty$

$\hat{\mu}_k \to \mu_k$

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Q: how to set N?

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Empirical Avg

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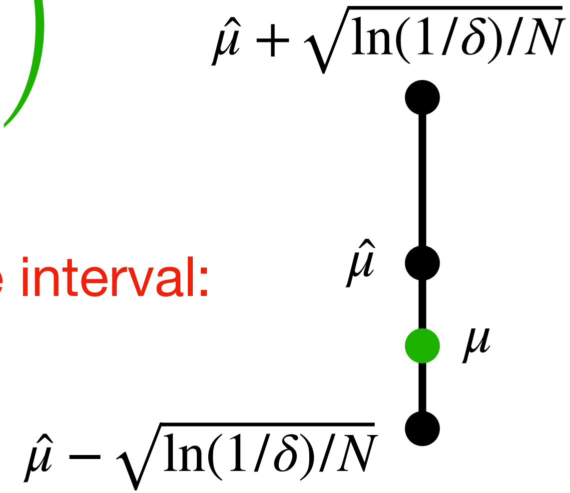

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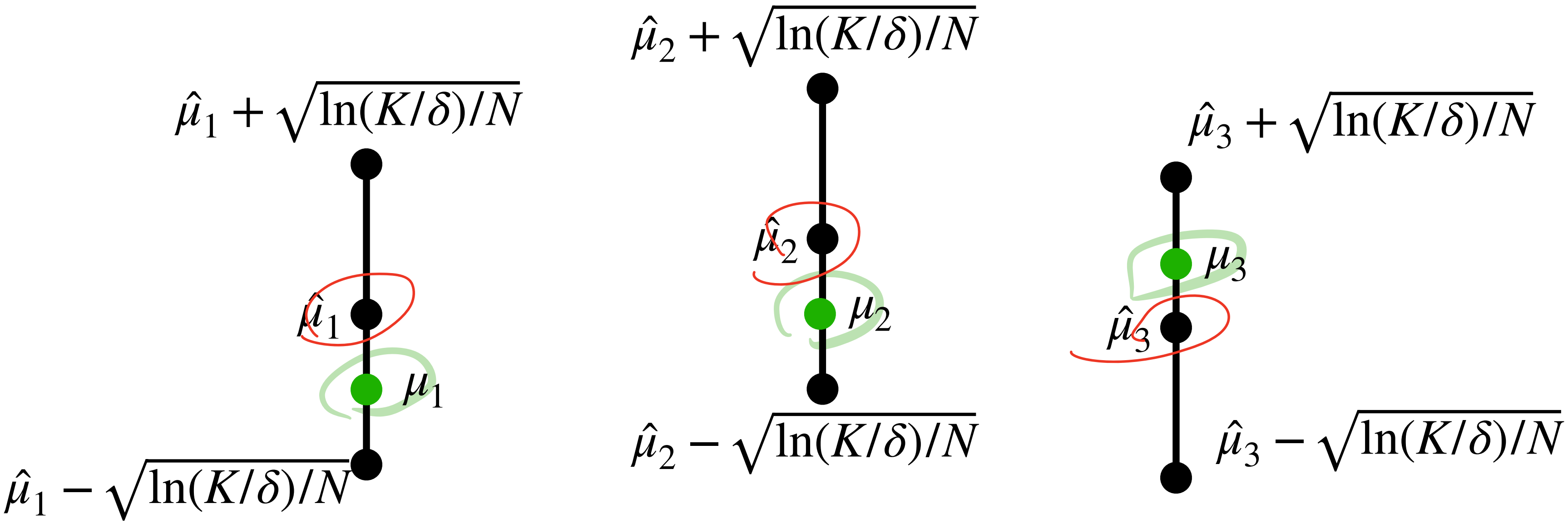

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rounds for Exploit

Gap

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Let's now bound $\text{Regret}_{exploit}$

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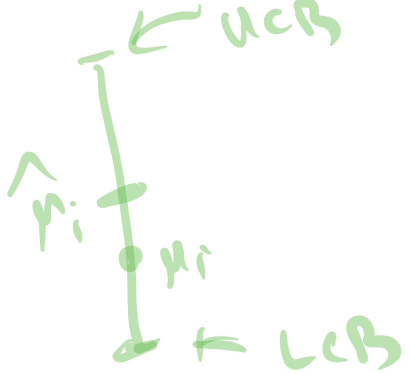

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≤ 0

Q: why?

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$$\leq 2\sqrt{\ln(K/\delta)/N}$$

$$\text{Regret}_{exploit} \leq (T - NK)\left(\mu_{I^\star} - \mu_{\hat{I}}\right) \leq 2T\sqrt{\frac{\ln(K/\delta)}{N}}$$

$\leq T$

Finally, combine two regret together:

$$\text{Regret}_{explore} \leq N(K-1) \leq NK$$

$$\text{Regret}_{exploit} \leq (T-NK)\left(\mu_{I^\star} - \mu_{\hat{I}}\right) \leq T\sqrt{\frac{\ln(K/\delta)}{N}}$$

$$\text{Regret}_T = \text{Regret}_{explore} + \text{Regret}_{exploit} \leq NK + 2T\sqrt{\frac{\ln(K/\delta)}{N}}$$

Explore exploit

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Minimize the upper bound via optimizing N:

Set $N = \left(\frac{T\sqrt{\ln(K/\delta)}}{2K}\right)^{2/3}$, we have:

$$\text{Regret}_T \leq O\left(T^{2/3}K^{1/3} \cdot \ln^{1/3}(K/\delta)\right)$$

$O(T^{2/3} \cdot K^{1/3})$

Avg-Reg $= T^{-1/3} K^{1/3}$

To conclude on Explore then Commit:

[Theorem] Fix $\delta \in (0,1)$, set $N = \left(\frac{T\sqrt{\ln(K/\delta)}}{2K}\right)^{2/3}$, with probability at least $1-\delta$, **Explore and Commit** has the following regret:

$$\text{Regret}_T \leq O\left(T^{2/3}K^{1/3} \cdot \ln^{1/3}(K/\delta)\right)$$

Q: can we do better, particularly, can we get $\sqrt{T}$ regret bound?

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# of Times we tried i

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and its empirical mean $\hat{\mu}_t(i)$ so far;

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At the beginning of iteration t, for all $i \in [K]$, # of times we have tried arm i,

i.e., $N_t(i) = \sum_{\tau=0}^{t-1} \mathbf{1}\{I_\tau = i\}$

and its empirical mean $\hat{\mu}_t(i)$ so far;

i.e., $\hat{\mu}_t(i) = \sum_{\tau=0}^{t-1} \mathbf{1}\{I_\tau = i\} r_\tau / N_t(i)$

Recall the Tool for Building Confidence Interval:

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Thus, we can show that for all iteration t, we have the for all $k \in [K]$, w/ prob $1 - \delta$,

$$|\hat{\mu}_k(i) - \mu_k| \leq \sqrt{\frac{\ln(KT/\delta)}{N_t(k)}}$$

Hoeffding

+ Union Bound

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Proving this result actually requires reasoning **Martinalges**, as samples are not i.i.d, i.e., whether or not you pull arm k in this round depends on previous random outcomes (See Ch 6 for more details)

UCB: Optimism in the face of Uncertainty

Given the confidence interval, we pick arm that has the **highest Upper-Conf-Bound:**

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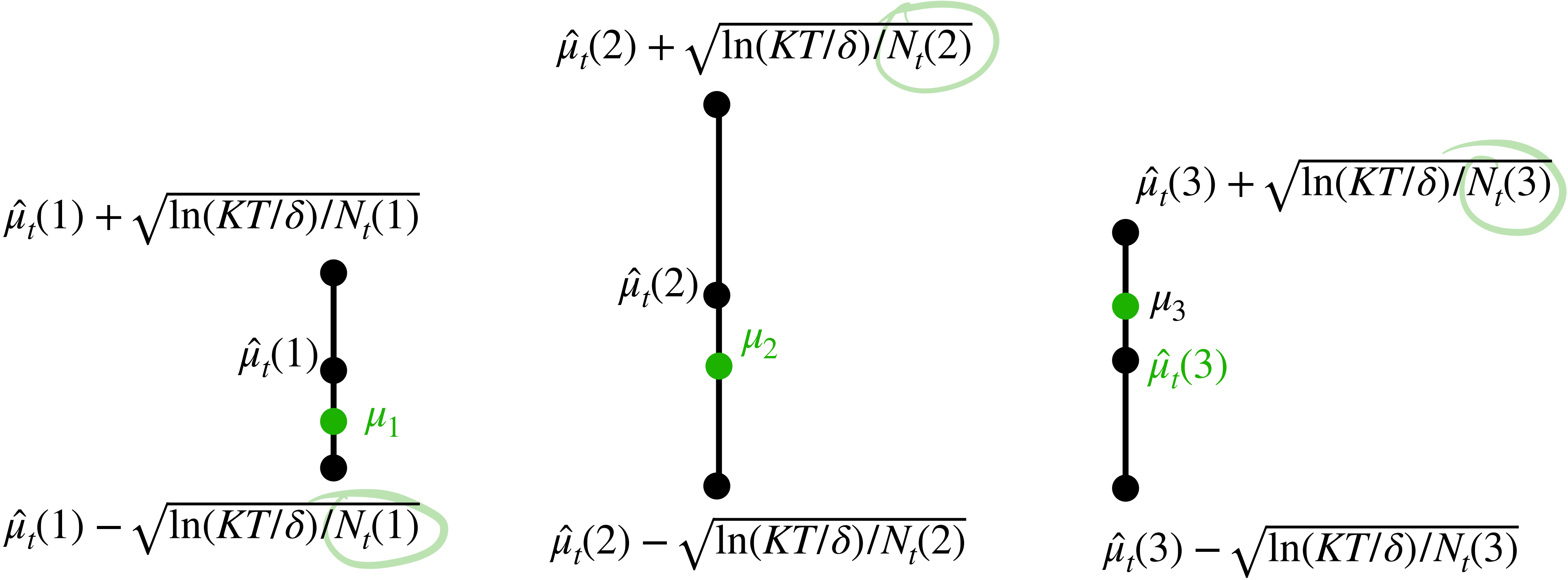

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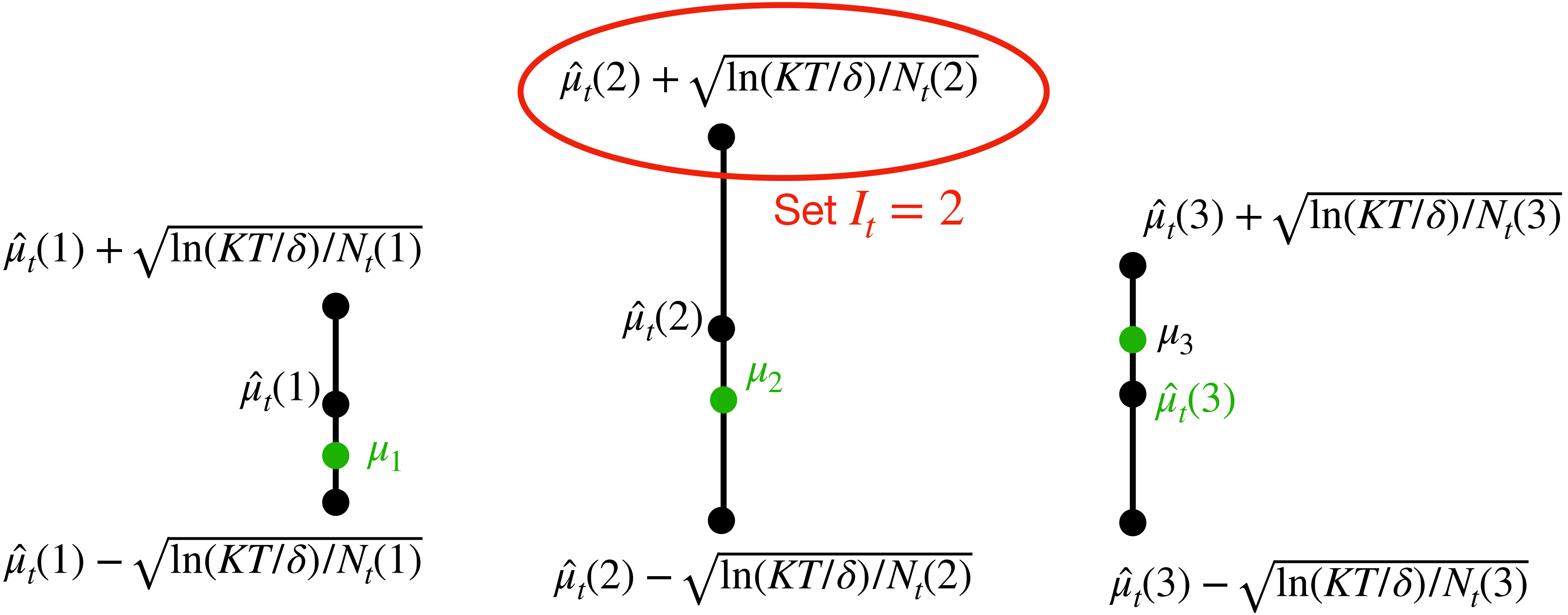


Put things together: UCB Algorithm:

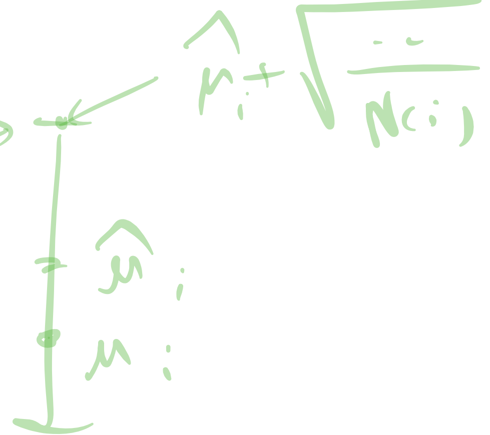
For $t = 0 \rightarrow T - 1$:

$$I_t = \arg\max_{i \in [K]} \left(\hat{\mu}_t(i) + \sqrt{\frac{\ln(KT/\delta)}{N_t(i)}} \right)$$

UCB

Receive $r \sim \nu_{I_t}$

$N_{t+1}(I_t) = N_t(I_t) + 1$



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(# Upper-conf-bound of arm i)

"Reward Bonus": $\sqrt{\frac{\ln(KT/\delta)}{N_t(i)}}$

UCB Regret:

[Theorem (informal)] With high probability, UCB has the following regret:

$$\text{Regret}_T = \widetilde{O}\left(\sqrt{KT}\right)$$

Intuitive Explanation of UCB

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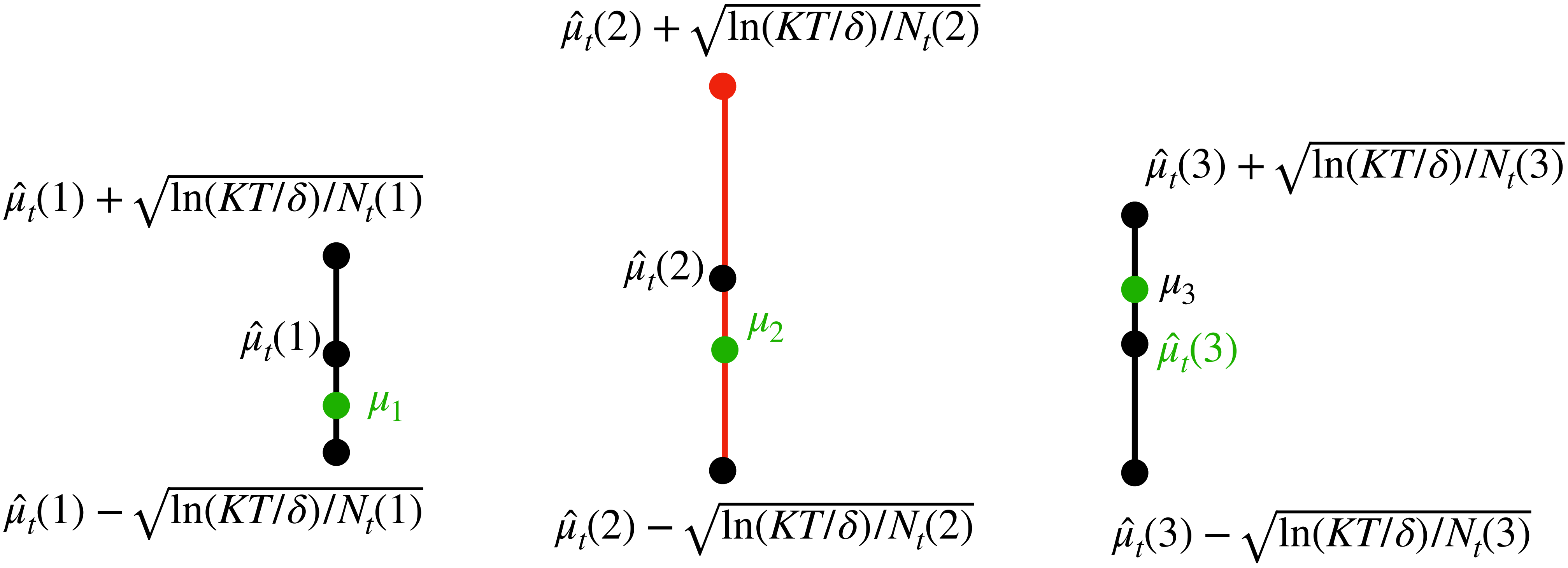
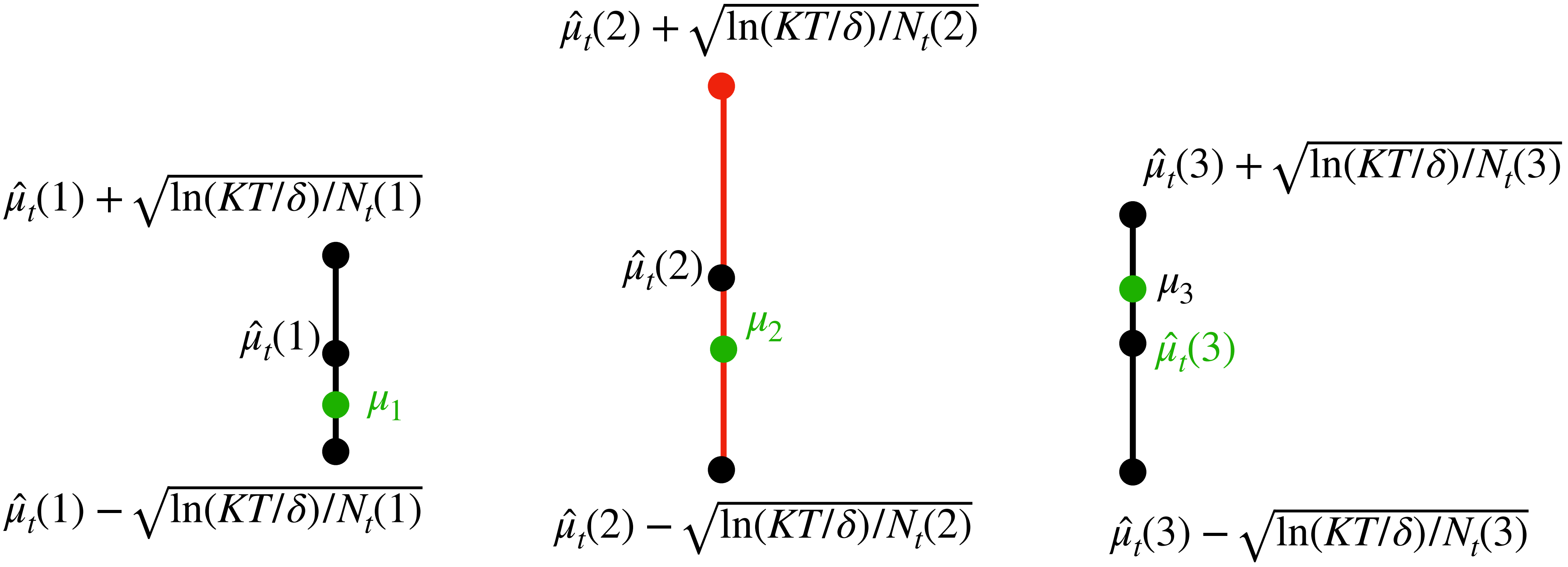


Intuitive Explanation of UCB

Case 1: it has large conf-interval, which means that it has not been tried many times yet (high uncertainty)



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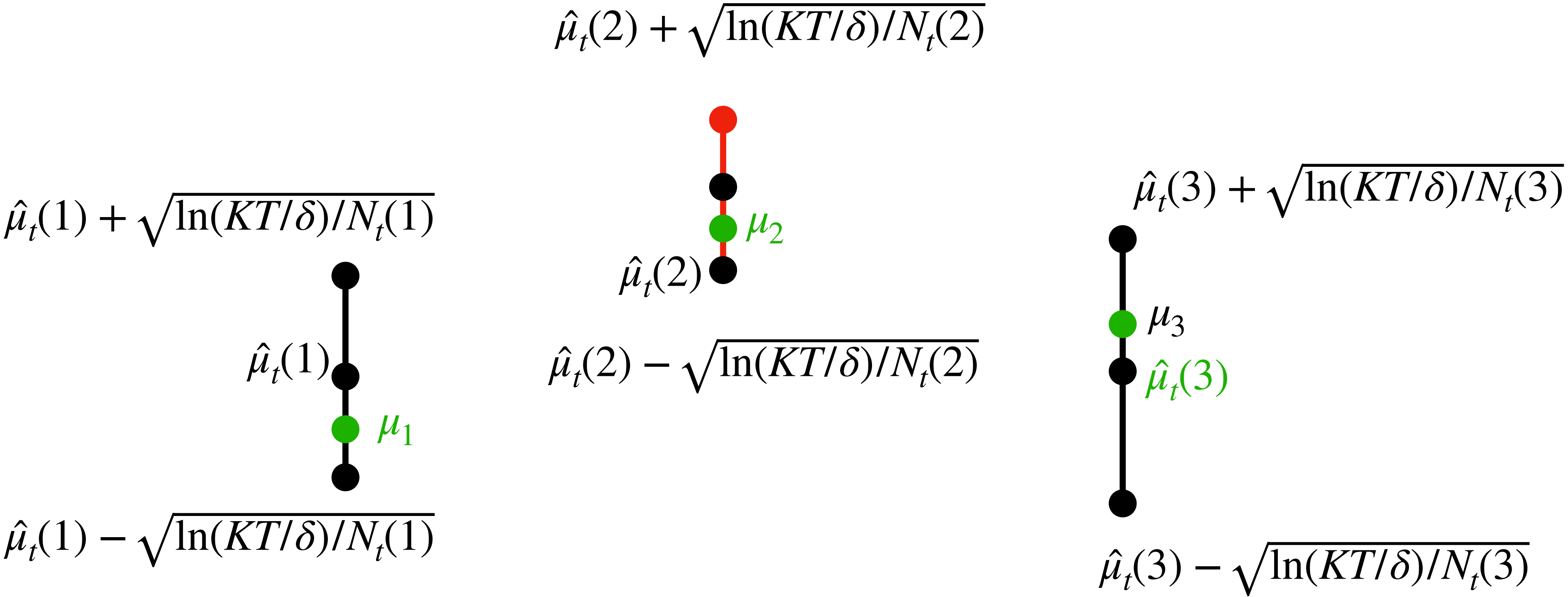
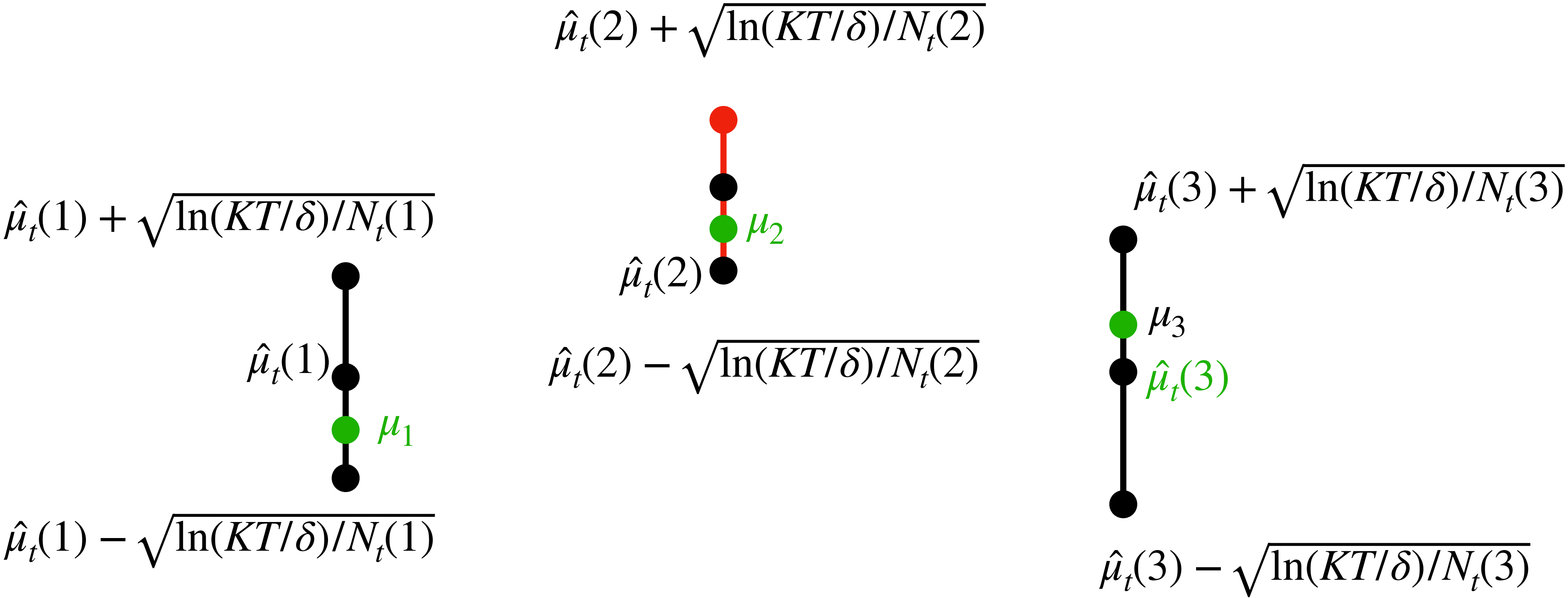


Intuitive Explanation of UCB

Case 2: it has low uncertainty, then it is simply a good arm, i.e., it's true mean is high!



Explore and Exploration Tradeoff

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Case 2: I_t has small conf-interval, then it is simply a good arm, i.e., it's true mean is pretty high!

Thus, we do exploitation in this case!

Let’s formalize the intuition

Denote the optimal arm $I^\star = \arg\max_{i \in [K]} \mu_i$; recall $I_t = \arg\max_{i \in [K]} \hat{\mu}_t(i) + \sqrt{\frac{\ln(KT/\delta)}{N_t(i)}}$

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$$\text{Regret-at-t} = \mu^\star - \mu_{I_t}$$

$$\leq \hat{\mu}_t(I_t) + \sqrt{\frac{\ln(TK/\delta)}{N_t(I_t)}} - \mu_{I_t}$$

$UCB(I_t) \geq UCB(I^\star)$

$\geq \mu(I^\star)$

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$$\leq 2\sqrt{\frac{\ln(TK/\delta)}{N_t(I_t)}}$$

$$\left| \hat{\mu}_t(I_t) - \mu_{I_t} \right| \lesssim \sqrt{\frac{\quad}{N_t(I_t)}}$$

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Case 1: $N_t(I_t)$ is small

(i.e., uncertainty about I_t is large);

We pay regret, BUT we **explore** here,

as we just tried I_t at iter t!

Let's formalize the intuition

Denote the optimal arm $I^\star = \arg\max_{i \in [K]} \mu_i$; recall $I_t = \arg\max_{i \in [K]} \hat{\mu}_t(i) + \sqrt{\frac{\ln(KT/\delta)}{N_t(i)}}$

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Case 2: $N_t(I_t)$ is large, i.e., conf-interval of I_t is small,

Then we **exploit** here, as I_t is pretty good (the gap between $\mu^\star$ & μ_{I_t} is small)!

Let's formalize the intuition

Finally, let's add all per-iter regret together:

$$\text{Regret}_T = \sum_{t=0}^{T-1} \left(\mu^\star - \mu_{I_t} \right)$$

$$\leq \sum_{t=0}^{T-1} 2\sqrt{\frac{\ln(TK/\delta)}{N_t(I_t)}}$$

$$\leq 2\sqrt{\ln(TK/\delta)} \cdot \sum_{t=0}^{T-1} \sqrt{\frac{1}{N_t(I_t)}}$$

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Lemma: $\sum_{t=0}^{T-1} \sqrt{\frac{1}{N_t(I_t)}} \leq O\left(\sqrt{KT}\right)$

$$\sum_{i=1}^{N} \sqrt{\frac{1}{i}} \le O\left(\sqrt{N}\right)$$

$$i: \sqrt{\tfrac{1}{1}},\ \sqrt{\tfrac{1}{2}},\ \sqrt{\tfrac{1}{3}},\ \sqrt{\tfrac{1}{4}},\ \dots,\ \sqrt{\tfrac{1}{N_T(i)}}$$

$$\Rightarrow \sum_{i=1}^{N_T(i)} \sqrt{\frac{1}{i}} \le \sqrt{N_T(i)}$$

$$= \sum_{k=1}^{K} \sum_{t=0}^{T-1} \mathbb{1}(I_t = k) \sqrt{\frac{1}{N_t(k)}}$$

$$\le \sum_{k=1}^{K} \sqrt{N_T(k)} \le \sqrt{K} \sqrt{\sum_{k=1}^{K} N_T(k)}, \quad \sum_{k=1}^{K} N_T(k) = T$$

Summary

1. Setting of Multi-armed Bandit: MDP with one state, and K actions, $H = 1$

2. Need to carefully balance exploration and exploitation

3. The Principle of Optimism in the face of Uncertainty