Exploration in Linear MDPs

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CS 6789: Foundations of Reinforcement Learning

Recap: linear MDP definition

Finite horizon time-dependent episodic MDP $\mathscr{M} = \{S, A, H, \{r\}_h, \{P\}_h, s_0\}$

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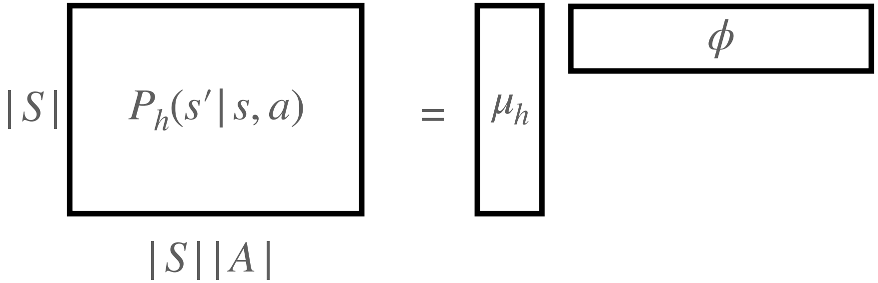
Low-Rank Decomposition:



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Finite horizon time-dependent episodic MDP $\mathscr{M} = \{S, A, H, \{r\}_h, \{P\}_h, s_0\}$

$$P_h(s' \mid s, a) = \mu_h^\star(s') \cdot \phi(s, a), \quad \mu_h^\star \in S \mapsto \mathbb{R}^d, \phi \in S \times A \mapsto \mathbb{R}^d$$

$$r(s, a) = \theta_h^\star \cdot \phi(s, a), \quad \theta_h^\star \in \mathbb{R}^d$$

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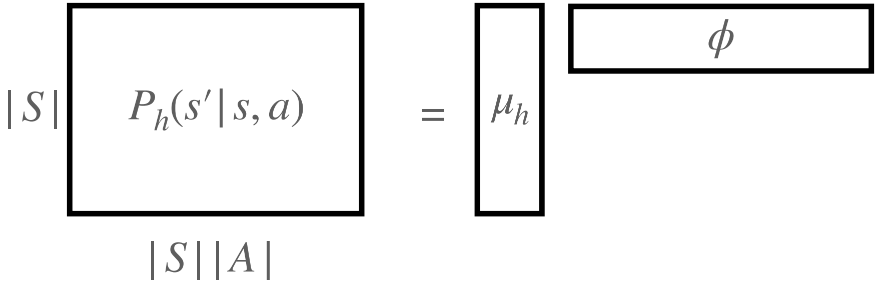

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Since given $s, a, s' \sim P_h^\star(\cdot \mid s, a)$, we know $\mathbb{E}_{s' \sim P_h^\star(\cdot \mid s, a)} \delta(s') = P_h^\star(\cdot \mid s, a)$,

$\delta \in \mathbb{R}^{|S|}$

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Given dataset $\{s_h^i, a_h^i, s_{h+1}^i\}_{i=1}^{n-1}$

Ridge Linear Regression:

$$\min_{\mu} \sum_{i=1}^{n-1} \|\mu \phi(s_h^i, a_h^i) - \delta(s_{h+1}^i)\|_2^2 + \lambda \|\mu\|_F^2$$

$$\mathbb{E}_{s a}[\delta] = P^\star(\cdot \mid s a)$$

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$$\widehat{\mu}_h^n = \sum_{i=1}^{n-1} \delta(s_{h+1}^i) \phi(s_h^i, a_h^i)^\top (\Lambda_h^n)^{-1}; \quad \widehat{P}_h^n(\cdot \mid s, a) = \widehat{\mu}_h^n \phi(s, a)$$

Today:

Regret bound for the UCBVI algorithm for Linear MDP and its proof sketch

Outline:

1. Model fitting and its guarantee $(\hat{P}(\cdot|s,a) - P(\cdot|s,a))^\top V$, for some fixed V

2. Covering argument to bound $(\hat{P}(\cdot|s,a) - P(\cdot|s,a))^\top V$, for ALL $V \in \mathcal{F}$

3. UCBVI revisit and its guarantee

1. Model Learning in Linear MDPs

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Cannot directly bound $\|\widehat{P}^n(\cdot \mid s, a) - P(\cdot \mid s, a)\|_1$ since $P(\,.\mid s, a)$ is in $\mathbb{R}^{|S|}$

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As in tabular-UCBVI and Generative Model, we care **average model error:**

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As in tabular-UCBVI and Generative Model, we care **average model error:**

Consider a fixed function $V : S \mapsto [0,H]$, we can bound:

$$\left| \left(\widehat{P}_h^n(\cdot \mid s,a) - P_h(\cdot \mid s,a) \right) \cdot V \right|$$

1. Model Learning in Linear MDPs

Ridge Linear Regression: $$\min_{\mu} \sum_{i=1}^{n-1} \|\mu\phi(s_h^i, a_h^i) - \delta(s_{h+1}^i)\|_2^2 + \|\mu\|_F^2$$

$$\widehat{\mu}_h^n = \sum_{i=1}^{n-1} \delta(s_{h+1}^i)\phi(s_h^i, a_h^i)^\top(\Lambda_h^n)^{-1}, \quad \Lambda_h^n = \sum_{i=1}^{n-1} \phi(s_h^i, a_h^i)\phi(s_h^i, a_h^i)^\top + \lambda I$$

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Lemma [Model Average Error under a fixed V]:

Consider a fixed $V : S \to [0,H]$. With probability at least $1 - \delta$, for any s, a, h, n, we have:

$$\left| \left(\widehat{P}_h^n(\cdot \mid s, a) - P_h(\cdot \mid s, a) \right) \cdot V \right| \leq \|\phi(s,a)\|_{(\Lambda_h^n)^{-1}} \times \left(2H \sqrt{\ln \frac{H \det(\Lambda_h^n)^{1/2} \det(\lambda I)^{-1/2}}{\delta}} + H\sqrt{\lambda d} \right)$$

1. Model Learning in Linear MDPs: proof

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

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1. Model Learning in Linear MDPs: proof

$$\mathbb{E}_{sa}\left[\delta(s')\right] = P(\cdot \mid sa)$$

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$$\widehat{\mu}_h^n = \sum_{i=1}^{n-1} \left(P_h(\cdot \mid s_h^i, a_h^i) + \epsilon_h^i\right) \phi(s_h^i, a_h^i)^\top (\Lambda_h^n)^{-1}$$

$$\epsilon = \delta(s') - P(\cdot \mid sa)$$

$$\mathbb{E}[\epsilon] = 0$$

1. Model Learning in Linear MDPs: proof

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$= P(\cdot \mid s_h^i, a_h^i)$

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$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda\mu_h^\star\left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1}\epsilon_h^i\phi(s_h^i, a_h^i)^\top\left(\Lambda_h^n\right)^{-1}$$

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$$= \mu_h^\star\left(\sum_{i=1}^{n-1}\phi(s_h^i, a_h^i)\phi(s_h^i, a_h^i)^\top\right)(\Lambda_h^n)^{-1} + \sum_{i=1}^{n-1}\epsilon_h^i\phi(s_h^i, a_h^i)^\top(\Lambda_h^n)^{-1}$$

$+\lambda I - \lambda I$

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$$= \mu_h^\star - \lambda \mu_h^\star (\Lambda_h^n)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top (\Lambda_h^n)^{-1}$$

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$$a^\top b \leq \|a\| \, \|b\|$$

$$\left| \lambda \phi(s,a)^\top \left(\Lambda_h^n\right)^{-1} \left(\mu_h^\star\right)^\top V \right| \leq \lambda \|\phi(s,a)^\top (\Lambda_h^n)^{-1/2}\|_2 \|(\Lambda_h^n)^{-1/2} (\mu_h^\star)^\top V\|_2$$

$$\lambda \left(\phi(s,a)^\top (\Lambda_h^n)^{-\frac{1}{2}}\right) \left((\Lambda_h^n)^{-\frac{1}{2}} (\mu^\star)^\top V \right)$$

1. Model Learning in Linear MDPs

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1. Model Learning in Linear MDPs

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$\sigma_{min}(\Lambda_h^n) \geq \lambda$

$$\left| \lambda \phi(s,a)^\top \left(\Lambda_h^n\right)^{-1} \left(\mu_h^\star\right)^\top V \right| \leq \lambda \|\phi(s,a)^\top (\Lambda_h^n)^{-1/2}\|_2 \|(\Lambda_h^n)^{-1/2}(\mu_h^\star)^\top V\|_2$$

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Normalization assumption on $\mu_h^\star$

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Normalization assumption on $\mu_h^\star$

1. Model Learning in Linear MDPs

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$$\left| \left((\widehat{\mu}_h^n - \mu_h^\star) \phi(s,a) \right)^\top V \right| \leq H\sqrt{\lambda d} \|\phi(s,a)\|_{(\Lambda_h^n)^{-1}} + \left| \sum_{i=1}^{n-1} \phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s_h^i, a_h^i) (\epsilon_h^i)^\top V \right|$$

1. Model Learning in Linear MDPs

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$$(\Lambda^n)^{-\frac{1}{2}} (\Lambda^n)^{-\frac{1}{2}}$$

$$\|\phi(s,a)\|_{\Lambda_h^{n^{-1}}}$$

1. Model Learning in Linear MDPs

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$$\left| \sum_{i=1}^{n-1} \phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right| \leq \|\phi(s,a)^\top (\Lambda_h^n)^{-1/2}\|_2 \left\| \sum_{i=1}^{n-1} (\Lambda_h^n)^{-1/2} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right\|_2$$

$$= \|\phi(s,a)\|_{(\Lambda_h^n)^{-1}} \left\| \sum_{i=1}^{n-1} \phi(s_h^i, a_h^i)\left((\epsilon_h^i)^\top V\right) \right\|_{(\Lambda_h^n)^{-1}}$$

$E(\varepsilon) = 0$

$E[\varepsilon^\top V] = 0$

1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

$$\left| \left((\widehat{\mu}_h^n - \mu_h^\star)\phi(s,a) \right)^\top V \right| \leq H\sqrt{\lambda d} \|\phi(s,a)\|_{(\Lambda_h^n)^{-1}} + \left| \sum_{i=1}^{n-1} \phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right|$$

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$$= \|\phi(s,a)\|_{(\Lambda_h^n)^{-1}} \left\| \sum_{i=1}^{n-1} \phi(s_h^i, a_h^i) \left((\epsilon_h^i)^\top V \right) \right\|_{(\Lambda_h^n)^{-1}}$$

$$\mathbb{E}\left[(\epsilon_h^i)^\top V \mid \mathcal{H}_{i,h} \right] = 0, \quad |(\epsilon_h^i)^\top V| \leq \|\epsilon_h^i\|_1 \|V\|_\infty \leq 2H$$

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With prob $1 - \delta, \ \forall n$

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1. Model Learning in Linear MDPs

Lemma [Model Average Error under a fixed V]:

Consider a fixed $V : S \to [0,H]$. With probability at least $1-\delta$, for all s, a, n, h, we have:

$$\left| \left(\widehat{P}_h^n(\cdot \mid s,a) - P_h(\cdot \mid s,a) \right) \cdot V \right| \le \|\phi(s,a)\|_{(\Lambda_h^n)^{-1}} \times \left(2H\sqrt{\ln \frac{H \det(\Lambda_h^n)^{1/2} \det(\lambda I)^{-1/2}}{\delta}} + H\sqrt{\lambda d} \right)$$

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$$\det(\Lambda_h^n)^{1/2} \det(\lambda I)^{-1/2} \le (n+\lambda)^{d/2} \lambda^{-d/2} = (N/\lambda + 1)^{d/2}$$

$\sigma_{max}(\Lambda_h^n)$

$\Lambda_h^n = \sum_{i=1}^{N} \phi_i \phi_i^T + \lambda I$

$\mathrm{Det}(\Lambda_h^n) \le (N+\lambda)^d$

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$\forall s a$

$$\left| \left(\widehat{P}_h^n(\cdot \mid s,a) - P_h(\cdot \mid s,a) \right) \cdot V \right| \le \| \phi(s,a) \|_{(\Lambda_h^n)^{-1}} \left(2H \sqrt{d \ln\left(\frac{NH}{\lambda} + 1\right) + \ln\left(\frac{1}{\delta}\right)} + H\sqrt{\lambda d} \right)$$

$$= \widetilde{O}\left(H\sqrt{d} + H\sqrt{\ln(1/\delta)} \right) \| \phi(s,a) \|_{(\Lambda_h^n)^{-1}}$$

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Q: Can we get a uniform convergence argument for a function class $\mathcal{F}$?

$V \in \mathcal{F}$

Outline:

✓ 1. Model fitting and its guarantee $(\hat{P}(.\,|\,s,a) - P(.\,|\,s,a))^\top V$, for some fixed V

2. Covering argument to bound $(\hat{P}(.\,|\,s,a) - P(.\,|\,s,a))^\top V$, for ALL $V \in \mathcal{F}$

3. UCBVI revisit and its guarantee

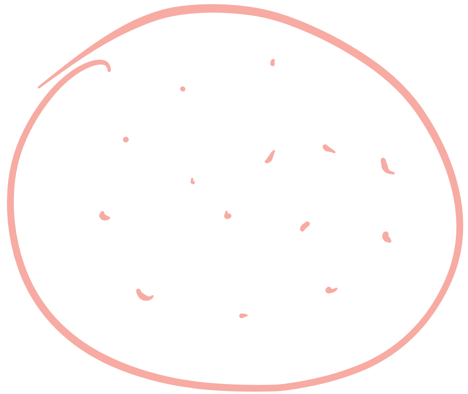
Detour: Covering Number

Consider the ball $\Theta = \{\theta : \theta \in \mathbb{R}^d, \|\theta\|_2 \leq R\}$.

Denote ϵ**-Net** as a subset $\mathcal{N}_\epsilon \subseteq \Theta$, such that $\forall \theta \in \Theta$:

$\exists \theta' \in \mathcal{N}_\epsilon$, s.t. $\|\theta' - \theta\|_2 \leq \epsilon$.

Denote ϵ**-cover** as the smallest $\mathcal{N}_\epsilon$



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Then $(\epsilon/L)-$Net on Θ gives us an $\epsilon-$Net on $\mathcal{F}$ with $d(f_{\theta_1}, f_{\theta_2}) := \|f_{\theta_1} - f_{\theta_2}\|_\infty$

Detour: Covering Number and An Example

Consider a specific parameterization $\theta = (w, \beta, \Lambda)$,

$$\Theta = \{(w, \beta, \Lambda) : \|w\|_2 \leq L, \beta \in [0,B], \sigma_{\min}(\Lambda) \geq \lambda\}$$

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$$f_{w,\beta,\Lambda}(s) := \min \left\{ \max_a \left(w^\top \phi(s,a) + \beta \sqrt{\phi(s,a)^\top \Lambda^{-1} \phi(s,a)} \right), H \right\}$$

Q^*

bonus

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Denote $\mathcal{F} = \{f_{w,\beta,\Lambda} : \|w\|_2 \leq L, \beta \in [0,B], \sigma_{\min}(\Lambda) \geq \lambda\}$, **what's the covering number of $\mathcal{F}$ under ℓ_∞**

$|f_\theta - f_{\theta'}|_\infty \leq L \cdot \|\theta - \theta'\|$

Detour: Covering Number

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Lemma: Denote $\mathcal{F} = \{f_{w,\beta,\Lambda} : \|w\|_2 \leq L, \beta \in [0,B], \sigma_{\min}(\Lambda) \geq \lambda\}$,

under ℓ_∞ we have: $\ln|\mathcal{N}_\epsilon| \leq d\ln(1 + 6L/\epsilon) + 2d^2 \ln(1 + 18B^2\sqrt{d}/(\lambda\epsilon^2)) = \widetilde{O}\left(d^2\right)$

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Key step in the proof:

$$\left|f_\theta(s) - f_{\hat{\theta}}(s)\right| \leq \|w - \hat{w}\|_2 + |\beta - \hat{\beta}|/\sqrt{\lambda} + B\sqrt{\|\Lambda^{-1} - \hat{\Lambda}^{-1}\|_F}$$

ε-Net

ε·√λ -Net

√ε/B -Net

Detour: Uniform Convergence

$$f_{w,\beta,\Lambda} :, f_{w,\beta,\Lambda}(s) := \min \left\{ \max_a \left(w^\top \phi(s,a) + \beta \sqrt{\phi(s,a)^\top \Lambda^{-1} \phi(s,a)} \right), H \right\}$$

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Lemma [uniform convergence]: With probability at least $1-\delta$, for all s, a, h, n, **and ALL** $f \in \mathcal{F}$:

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(fixed V: $H\sqrt{d} \, \|\phi\|_{\Lambda^{-1}}$)

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Proof sketch: the model error we had for a fixed V + ϵ-Net argument
(Same high level steps as the ones we used for HW1's last question)

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This will be our bonus term

Proof sketch: the model error we had for a fixed V + ϵ-Net argument
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Summary of Covering Argument

Covering allows us to build a uniform convergence result (i.e., $\forall f \in \mathcal{F}$)
over a infinite hypothesis class
(Intuitively, log of covering number scales w.r.t to the # of parameters)

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Algorithm: UCBVI in Linear MDPs

At the beginning of iteration n:

1. Learn transition model $\{\widehat{P}_h^n\}_{h=0}^{H-1}$ from all previous data via Ridge linear regression

comment: $\min_{\mu} \sum_{i=1}^{n-1} \|\mu\phi(s_h^i, a_h^i) - \delta(s_{h+1}^i)\|_2^2 + \lambda\|\mu\|_F^2$

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4. Execute π^{n+1} for H steps

Regret bound for UCBVI in linear MDP

$$\mathbb{E}\left[\sum_{n=1}^{N}\left(V^{\star} - V^{\pi^n}\right)\right] \leq \widetilde{O}(H^2 d^{1.5}\sqrt{N})$$

No S or A polynomial dependence!

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$$= \theta_h^\star \cdot \phi(s,a) + \beta \sqrt{\phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s,a)} + \left(\widehat{\mu}_h^n \phi(s,a) \right)^\top \widehat{V}^{\,n}_{h+1}$$

2. Value Iteration in the Learned Model w/ Reward Bonus

$$\pi^{n+1} = \text{Value-Iter}\left(\{ \widehat{P}^n \}_h, \{ r_h + b_h^n \} \right)$$

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linear in ϕ

$$= \beta\sqrt{\phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s,a)} + \left(\theta_h^\star + (\widehat{\mu}_h^n)^\top \widehat{V}_{h+1}^n(s') \right)^\top \phi(s,a)$$

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$$\widehat{V}^{\,n}_h(s) = \min \left\{ \max_a \left(\phi(s,a)^\top \widehat{w}^{\,n}_h + \beta \sqrt{\phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s,a)} \right), H \right\}, \quad \pi_h^n(s) = \arg\max_a \widehat{Q}^{\,n}_h(s,a)$$

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Lemma [Optimism]: with high probability, for all n, h, s:

$$\widehat{V}_h^n(s) \geq V_h^\star(s)$$

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Valid Transition

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NOTE this is different from what we did in tabular MDP!!!

$\widehat{P}$ is not a valid Transition

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$$\geq b_h^n(s,a) - \left| \left(\widehat{P}_h^n(\cdot \mid s,a) - P_h(\cdot \mid s,a) \right) \cdot \widehat{V}_{h+1}^n \right|$$

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$$\geq 0$$

NOTE this is different from what we did in tabular MDP!!!

$\leq Hd \|\phi\|_{\Lambda^{-1}}$

4. Regret Decomposition

Conditioned on history up to the end of episode n-1:

$$V_0^{\star}(s_0) - V_0^{\pi^n}(s_0) \le \widehat{V}_0^{n}(s_0) - V_0^{\pi^n}(s_0)$$

optimism

value of π^n in (P, r)

$\pi_n = VI(\widehat{P}, r+b)$

value of π_n in $(\widehat{P}, r+b)$

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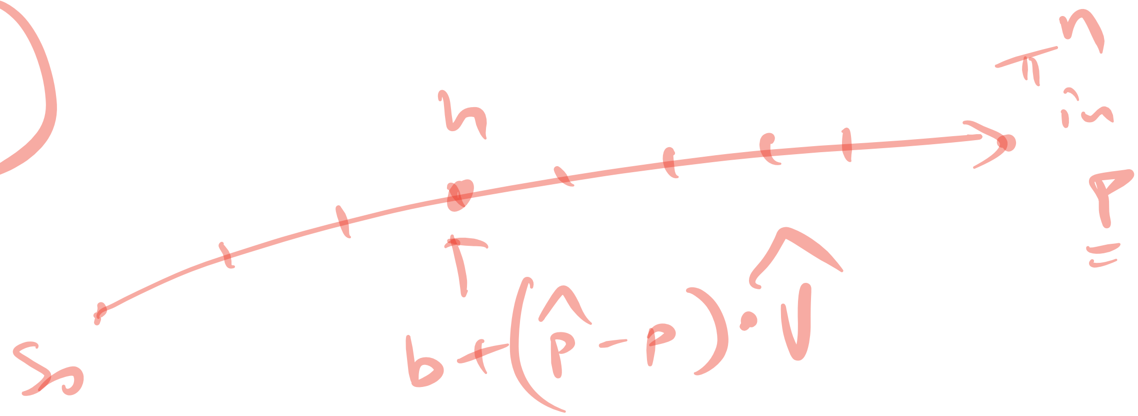

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$\leq b_h^n(s_h, a_h)$

$$\lesssim 2 \sum_{h=0}^{H-1} \mathbb{E}_{s_h,a_h \sim d_h^{\pi^n}} \left[b_h^n(s_h,a_h) \right]$$

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Conditioned on history up to the end of episode n-1:

$\delta \le$ Reg-n

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4. Concluding the Regret Computation

$$\mathbb{E}\left[\sum_{n=1}^{N}\left(V_0^{\star}(s_0) - V_0^{\pi^n}(s_0)\right)\right] = \mathbb{E}\left[\mathbf{1}[\text{good event holds}]\sum_{n=1}^{N}\left(V_0^{\star}(s_0) - V_0^{\pi^n}(s_0)\right)\right] + \mathbb{E}\left[\mathbf{1}[\text{good event doesn't hold}]\sum_{n=1}^{N}\left(V_0^{\star}(s_0) - V_0^{\pi^n}(s_0)\right)\right]$$

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$$\lesssim \widetilde{O}(H^2 d^{1.5}\sqrt{N})$$