

Learning with Linear Bellman Completion & Generative Model

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CS 6789: Foundations of Reinforcement Learning

Recap: Linear Bellman Completion

Given feature ϕ , take any linear function $w^\top \phi(s, a)$:

$$\forall h, \exists \theta \in \mathbb{R}^d, s.t., \theta^\top \phi(s, a) = r(s, a) + \mathbb{E}_{s' \sim P_h(s, a)} \max_{a'} w^\top \phi(s', a'), \forall s, a$$

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But adding additional elements may just break the condition

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BC always ensures linear regression is realizable:

i.e., our regression target $r(s, a) + \mathbb{E}_{s' \sim P_h(s, a)} \max_{a'} \theta_{h+1}^T \phi(s', a')$ is always linear:

Outline for Today

1. Proof Sketch of LSVI

2. LSVI in Offline RL

Theorem

Theorem: There exists a way to construct datasets $\{\mathcal{D}_h\}_{h=0}^{H-1}$, such that with probability at least $1 - \delta$, we have:

$$V^{\hat{\pi}} - V^{\star} \leq \epsilon$$

w/ total number of samples in these datasets scaling $\tilde{\mathcal{O}}(d^2 + H^6 d^2 / \epsilon^2)$

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2. Show that our estimators are near-bellman consistent: $\|\theta_h^\top \phi - \mathcal{T}_h(\theta_{h+1}^\top \phi)\|_\infty$ is small
3. Near-Bellman consistency implies near optimal performance (s.t. H error amplification)

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$$\max_{y \in \mathcal{X}} y^\top \left[\mathbb{E}_{x \sim \rho^*} [xx^\top] \right]^{-1} y \leq d$$

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$$\left| (\hat{\theta} - \theta^*)^\top x \right| \leq \left\| \Lambda^{1/2}(\hat{\theta} - \theta^*) \right\|_2 \left\| \Lambda^{-1/2}x \right\|_2$$

Summary so far on OLS & D-optimal Design

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D-optimal design allows us to **actively** construct a dataset $\mathcal{D} = \{x, y\}$, such that OLS solution is **POINT-WISE** accurate:

$$\max_{x \in \mathcal{X}} \left| \langle \hat{\theta} - \theta^*, x \rangle \right| \leq \frac{\sigma d \ln(1/\delta)}{\sqrt{N}}$$

Using D-optimal design to construct \mathcal{D}_h in LSVI

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$$\Rightarrow V^\star - V^{\hat{\pi}} \leq \widetilde{O}(H^3d/\sqrt{N})$$

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2. LSVI in **Offline RL**

Offline Reinforcement Learning

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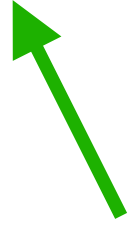
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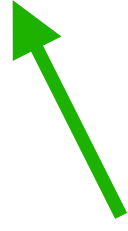


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Offline RL is promising for safety critical applications
(i.e., learning from logged data for health applications...)

Recall Least-Square Value Iteration

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LSVI directly can directly
operate in offline model!

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Assumptions

1. Full offline data coverage: $\sigma_{\min} \left(\mathbb{E}_{s, a \sim \nu} \phi(s, a) \phi(s, a)^\top \right) \geq \kappa$
2. Linear Bellman completion

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Then, with probability at least $1 - \delta$, LSVI return $\hat{\pi}$ with $V^\star - V^{\hat{\pi}} \leq \epsilon$, using at most $\text{poly} \left(H, 1/\epsilon, 1/\kappa, d, \ln(1/\delta) \right)$

The proof for the offline set is almost identical

Key step:

Linear Bellman completion + Linear Regression w/ full data coverage

=> Near-Bellman consistency, i.e., $\|Q_h - \mathcal{T}_h Q_{h+1}\|_\infty$ is small

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e.g., with N training examples where $(s, a) \sim \nu$, and $r = r(s, a)$, $s' \sim P_h(\cdot | s, a)$, we have

$$\mathbb{E}_{s,a \sim \nu} \left(\theta_h^\top \phi(s, a) - \mathcal{T}_h(\theta_{h+1})^\top \phi(s, a) \right)^2 \leq \text{poly}(H, d, 1/N)$$

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(we will give a HW question on a related topic)

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2. Least square value iteration: integrate Linear regression into DP, i.e., $Q_h := \theta_h^\top \phi \approx Q_h^\star$ via

$$\phi(s, a) \mapsto r(s, a) + \max_{a'} \theta_{h+1}^\top \phi(s', a')$$

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$$\phi(s, a) \mapsto r(s, a) + \max_{a'} \theta_{h+1}^\top \phi(s', a')$$

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4. Near-Bellman consistency implies small approximation error of Q_h (holds in general)

Next week

Exploration: Multi-armed Bandits and online learning in Tabular MDP