Learning with Linear Bellman Completion & Generative Model

Wen Sun

CS 6789: Foundations of Reinforcement Learning

Announcements

1. HW1 is out.

2. Please sign up reading materials (see course website for the link)

3. Wen’s office hour: every Friday 2-3 pm

Recap: Generative model + Tabular

1. Generative model assumption:

At any (s, a), we can sample $s' \sim P(\cdot \mid s, a)$

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Q: why this could be a strong assumption in practice?

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3. Find optimal policy under $\hat{P}$, i.e., $\hat{\pi}^\star = \mathrm{PI}(\hat{P}, r)$

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Result:

When $N \geq \dfrac{\ln(SA/\delta)}{\epsilon^2(1-\gamma)^6}$, then w/ prob $1-\delta$, we will learn a $\hat{\pi}^\star$, such that $\|Q^\star - Q^{\hat{\pi}^\star}\|_\infty \leq \epsilon$

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1. Horizon factor is not tight at all (Ch2 in AJKS optimizes it to $1/(1-\gamma)^5$)

2. Remarkably, our learned model $\hat{P}$ in this case is not necessarily accurate at all

Today: Generative model + linear function approximation

Key question: what happens when state-action space is large or even continuous?

Outline:

1. The Linear Bellman Completion Condition

2. The Least Square Value Iteration Algorithm

3. Guarantee and the proof sketch

Finite Horizon MDPs and DP

$$\mathcal{M} = \{S, A, P_h, r, H\}$$

$$P_h : S \times A \mapsto \Delta(S), \quad r : S \times A \to [0,1]$$

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2. At h, set $Q^\star_h(s,a) = r(s,a) + \mathbb{E}_{s' \sim P_h(\cdot|s,a)} V^\star_{h+1}(s')$,
$\pi^\star_h(s) = \arg\max_a Q^\star_h(s,a)$, $V^\star_h(s) = \max_a Q^\star_h(s,a)$

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$\|Q - Q^\star\|_\infty \leq \epsilon/(1-\gamma)$, => $V^\star - V^{\hat{\pi}} \leq \epsilon/(1-\gamma)^2$

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Similar results hold in finite horizon, with the effective horizon $1/(1-\gamma)$ being replaced by H

Linear Bellman Completion

Given feature ϕ, take any linear function $w^\top \phi(s,a)$:

$$\forall h, \exists \theta \in \mathbb{R}^d, s.t., \theta^\top \phi(s,a) = r(s,a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} w^\top \phi(s',a'), \forall s, a$$

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Notation: we will denote such $\theta := \mathcal{T}_h(w)$, where $\mathcal{T}_h : \mathbb{R}^d \mapsto \mathbb{R}^d$

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Why?

reward $r(s, a)$ is linear in ϕ, i.e., $Q^\star_{H-1}(s, a)$ is linear,

now recursively show that $Q^\star_h$ is linear

Why this is a reasonable assumption?

It captures at least two special cases: tabular MDP and linear dynamical systems

1. Tabular MDP:

Set $\phi(s, a)$ to be a one-hot encoding vector in $\mathbb{R}^{SA}$, i.e., $\phi(s, a) = [0, \ldots, 0, 1, 0, \ldots 0]^\top$

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(we will see the details when we get to the LQR lectures)

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This is counter-intuitive: in SL (e.g., linear regression), adding elements to features is ok!

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(We will work on a slightly different result later when we talk about online learning in MDPs)

What we will show today:

1. Generative Model

(i.e., we can reset system to any (s, a), query $r(s, a), s' \sim P(\,.\,|\,s, a)$)

$$+$$

2. Linear Bellman Completion

$$=$$

Sample efficient Learning

(poly time)

Outline:

✓ 1. The Linear Bellman Completion Condition

2. Learning: The Least Square Value Iteration Algorithm

3. Guarantee and the proof sketch

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$$\theta_h = \arg\min_\theta \sum_{\mathcal{D}_h} \left(\theta^T \phi(s,a) - \left(r + V_{h+1}(s') \right) \right)^2$$

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If $V_{h+1} \approx V_{h+1}^\star$, and linear regression succeeds (e.g., $\theta_h \approx \mathcal{T}_h(\theta_{h+1})$),

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$$\theta_h = \arg\min_\theta \sum_{\mathcal{D}_h} \left(\theta^T \phi(s,a) - \left(r + V_{h+1}(s') \right) \right)^2$$

Set $V_h(s) := \max_a \theta_h^\top \phi(s,a), \forall s$

Return $\hat{\pi}_h(s) = \arg\max_a \theta_h^\top \phi(s,a), \forall h$

If $V_{h+1} \approx V_{h+1}^\star$, and linear regression succeeds (e.g., $\theta_h \approx \mathcal{T}_h(\theta_{h+1})$),

Then we should hope $\theta_h^\top \phi(s,a) \approx Q_h^\star(s,a)$

Outline:

1. The Linear Bellman Completion Condition

2. Learning: The Least Square Value Iteration Algorithm

3. Guarantee and the proof sketch

Sample complexity of LSVI

Theorem: There exists a way to construct datasets $\{\mathscr{D}_h\}_{h=0}^{H-1}$, such that with probability at least $1-\delta$, we have:

$$V^{\hat{\pi}} - V^{\star} \leq \epsilon$$

w/ total number of samples in these datasets scaling $\widetilde{O}\left(d^2 + H^6 d^2/\epsilon^2\right)$

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Plans: (1) OLS and D-optimal design; (2) construct $\mathcal{D}_h$ using D-optimal design; (3) transfer regression error to $\|\theta_h^\top \phi - Q_h^\star\|_\infty$

Detour: Ordinary Linear Squares

Consider a dataset $\{x_i, y_i\}_{i=1}^N$, where $y_i = (\theta^\star)^\top x_i + \epsilon_i, \quad \mathbb{E}[\epsilon_i \,|\, x_i] = 0$, ϵ_i are independent with $|\epsilon_i| \leq \sigma$, assume $\Lambda = \sum_{i=1}^N x_i x_i^\top / N$ is full rank;

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Standard OLS guarantee: with probability at least $1 - \delta$, we have:

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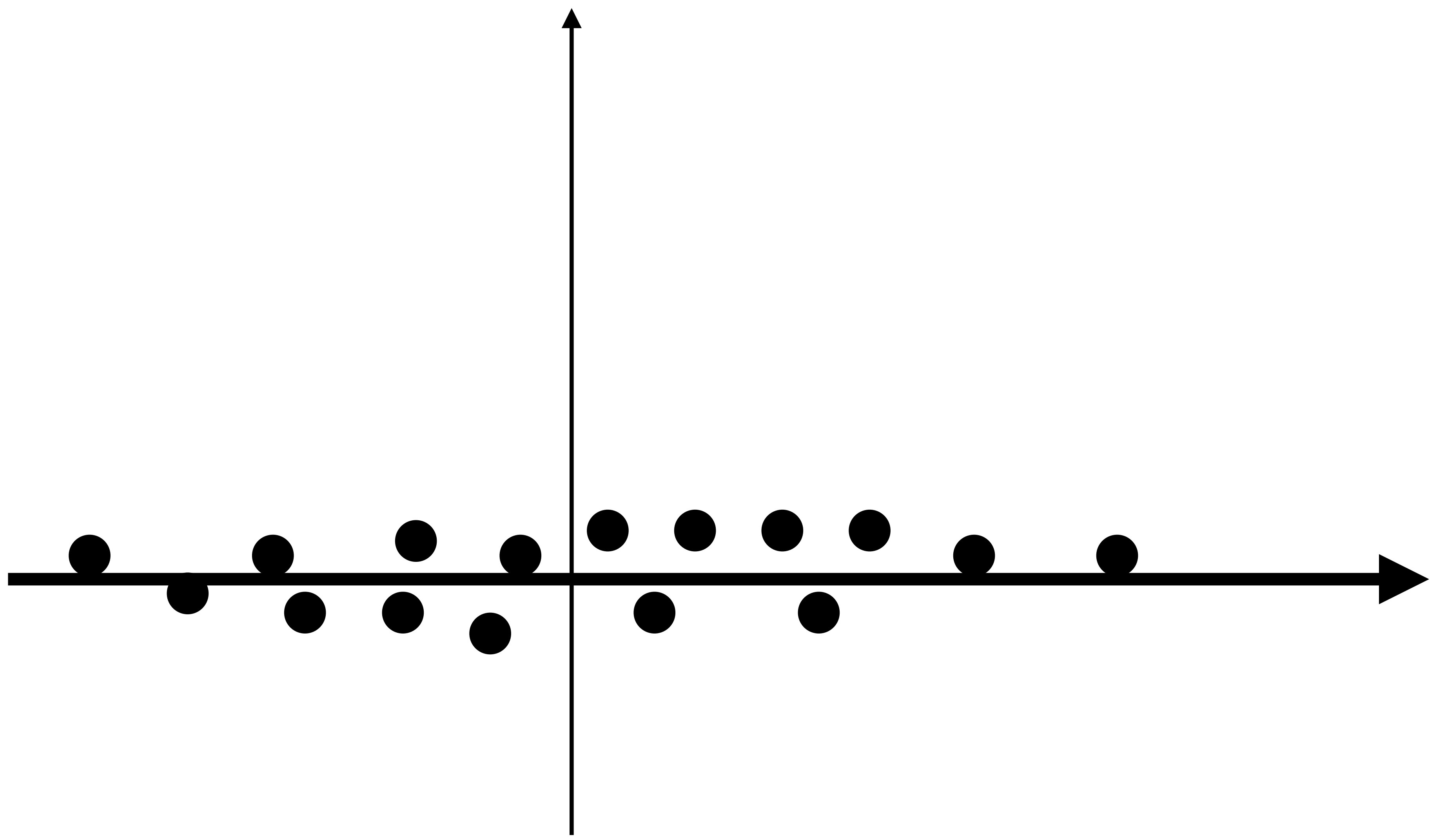

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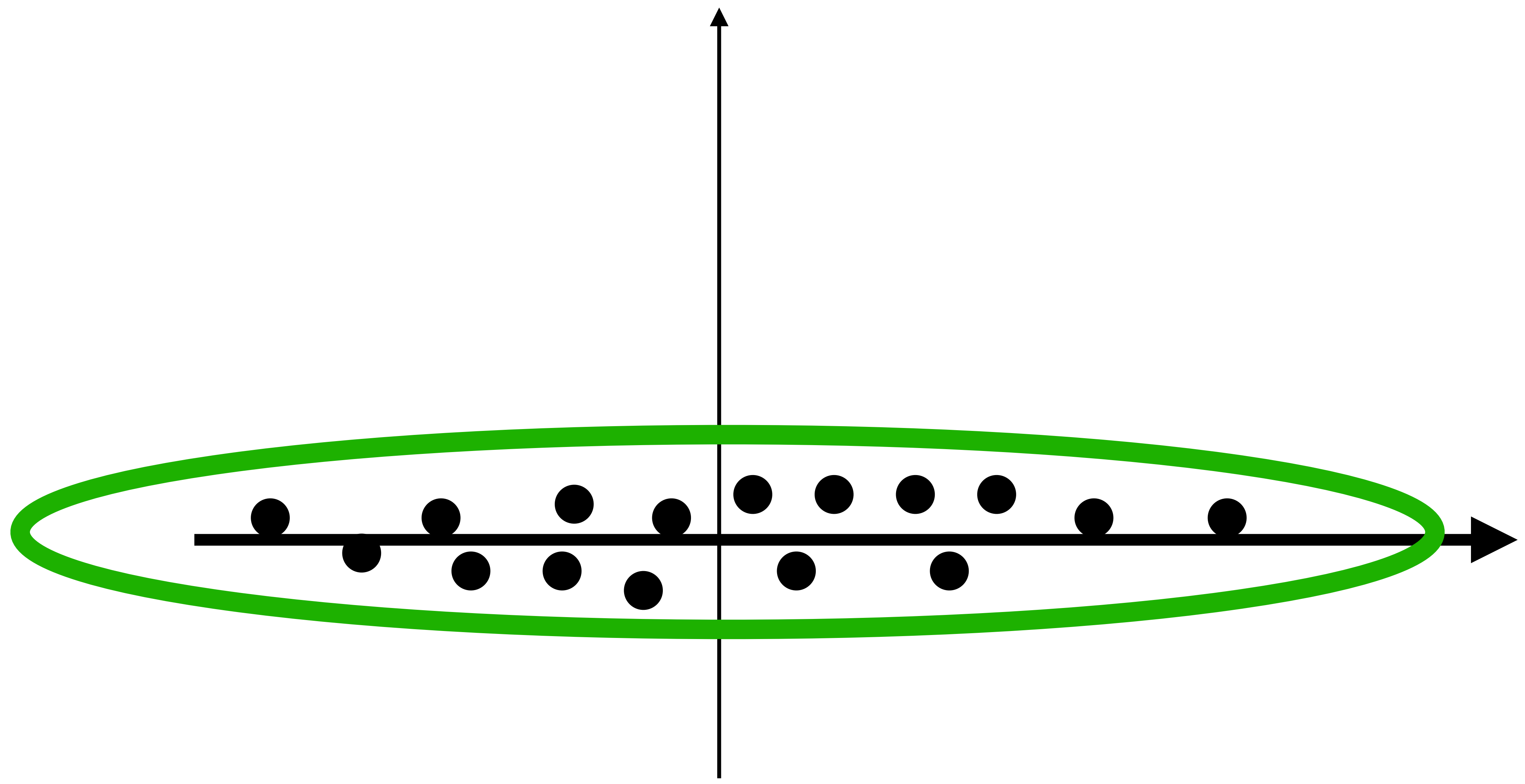

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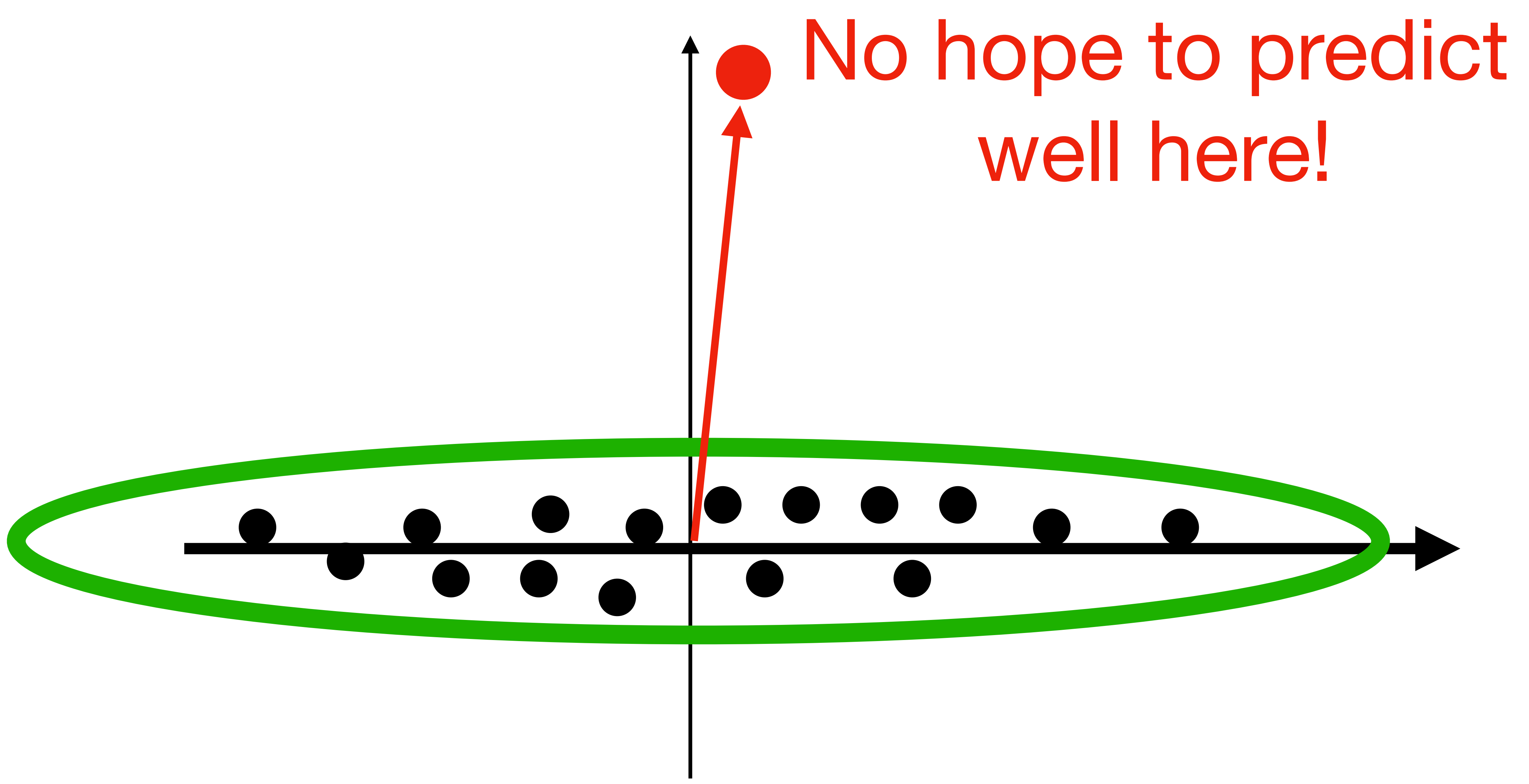

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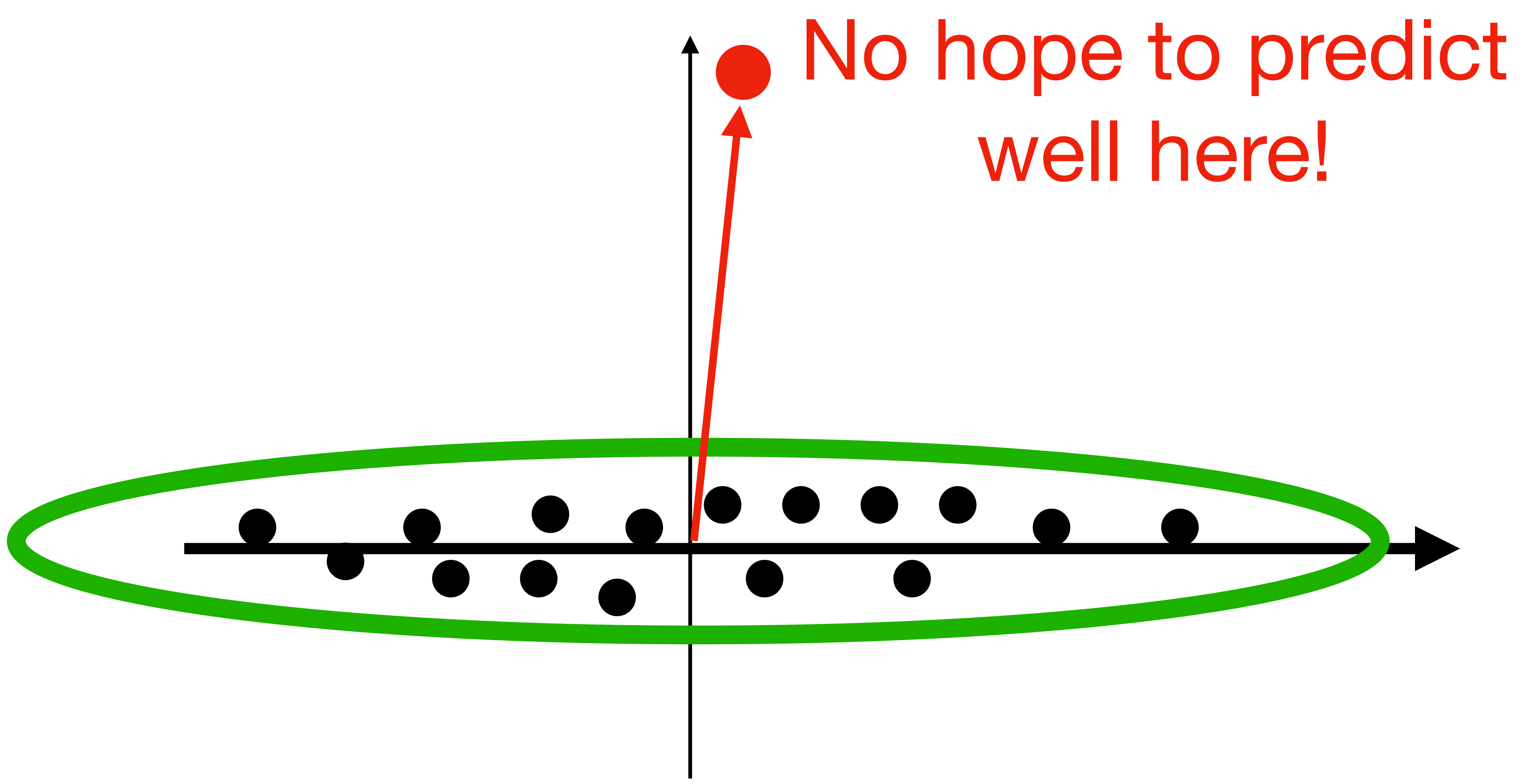

Let's actively design a diverse dataset !
(D-optimal Design)

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support$(\rho^\star) \leq d(d+1)/2$

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$$\max_{y\in\mathscr{X}} y^\top \left[\mathbb{E}_{x\sim\rho^\star} xx^\top\right]^{-1} y \leq d$$

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The OLS solution $\hat\theta$ on $\mathscr{D}$ has the following point-wise guarantee: w/ prob $1-\delta$

$$\max_{x\in\mathscr{X}} \left| \langle \hat\theta - \theta^\star, x\rangle \right| \leq \frac{\sigma d \ln(1/\delta)}{\sqrt{N}}$$

Summary so far on OLS & D-optimal Design

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D-optimal design allows us to **actively** construct a dataset $\mathscr{D} = \{x, y\}$, such that OLS solution is **POINT-WISE** accurate:

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