Learning with Linear Bellman Completion & Generative Model

Wen Sun

CS 6789: Foundations of Reinforcement Learning

Announcements

1. HW1 is out.

2. Please sign up reading materials (see course website for the link)

3. Wen’s office hour: every Friday 2-3 pm

Recap: Generative model + Tabular

1. Generative model assumption:

At any (s, a), we can sample $s' \sim P(\cdot \mid s, a)$

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Q: why this could be a strong assumption in practice?

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3. Find optimal policy under $\hat{P}$, i.e., $\hat{\pi}^\star = \mathrm{PI}(\hat{P}, r)$

Recap: Generative model + Tabular

Result:

When $N \geq \frac{\ln(SA/\delta)}{\epsilon^2(1-\gamma)^6}$, then w/ prob $1-\delta$, we will learn a $\hat{\pi}^\star$, such that $\|Q^\star - Q^{\hat{\pi}^\star}\|_\infty \leq \epsilon$

$\hat{P}(\cdot|sa) \in \mathbb{R}^{|S|}$

Remarks:

$\| \hat{P}(\cdot|sa) - P^\star(\cdot|sa) \|_{TV}$

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Remarks:

1. Horizon factor is not tight at all (Ch2 in AJKS optimizes it to $1/(1-\gamma)^5$)

2. Remarkably, our learned model $\hat{P}$ in this case is not necessarily accurate at all

Today: Generative model + linear function approximation

Key question: what happens when state-action space is large or even continuous?

Outline:

1. The Linear Bellman Completion Condition

2. The Least Square Value Iteration Algorithm

3. Guarantee and the proof sketch

Finite Horizon MDPs and DP

$$\mathcal{M} = \{S, A, P_h, r, H\}$$

$H \in \mathbb{Z}^+$

$$P_h : S \times A \mapsto \Delta(S), \quad r : S \times A \to [0,1]$$

Compute $\pi^\star$ via DP (backward in time):

$h=0$ $h=H-1$

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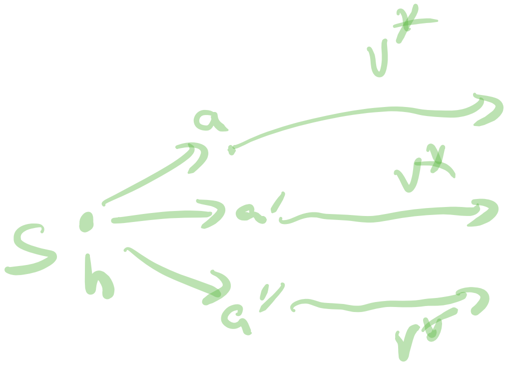
1. set $Q^\star_{H-1}(s,a) = r(s,a)$, $\pi^\star_{H-1}(s) = \arg\max_a Q^\star_{H-1}(s,a)$, $V^\star_{H-1}(s) = \max_a Q^\star_{H-1}(s,a)$

$V^\star_{h+1}(s)$



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2. At h, set $Q^\star_h(s,a) = r(s,a) + \mathbb{E}_{s' \sim P_h(\cdot|s,a)} V^\star_{h+1}(s')$,
$\pi^\star_h(s) = \arg\max_a Q^\star_h(s,a)$, $V^\star_h(s) = \max_a Q^\star_h(s,a)$

Recall Error amplification

1. **Bellman optimality:** $\|Q - \mathscr{T}Q\|_\infty = 0$, then $Q = Q^\star$

Bell-operator

$$(\mathcal{T}Q)(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim p(s,a)} \max_{a'} Q(s',a')$$

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Then we have error amplification:

$\|Q - Q^\star\|_\infty \leq \epsilon/(1-\gamma)$, => $V^\star - V^{\hat{\pi}} \leq \epsilon/(1-\gamma)^2$

$\hat{\pi} = \arg\max_a Q(s,a)$

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Similar results hold in finite horizon, with the effective horizon $1/(1-\gamma)$ being replaced by H

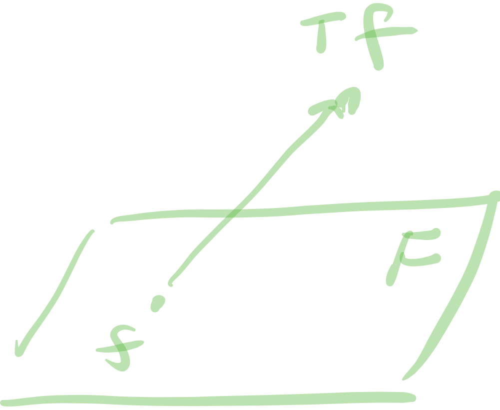
Linear Bellman Completion

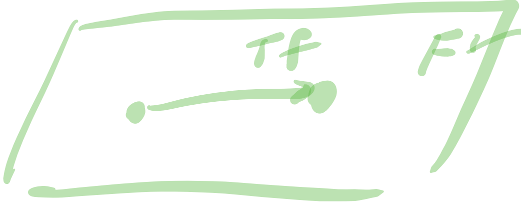
$\phi : S \times A \mapsto \mathbb{R}^d$

Given feature ϕ, take any linear function $w^\top \phi(s,a)$:

$$\forall h, \exists \theta \in \mathbb{R}^d, s.t., \theta^\top \phi(s,a) = r(s,a) + \underbrace{\mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} w^\top \phi(s',a')}_{:= T(w^\top \phi)(s,a)}, \forall s, a$$

$f := w^\top \phi(s,a)$





$F = \{ w^\top \phi : w \in \mathbb{R}^d \}$

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Notation: we will denote such $\theta := \mathcal{T}_h(w)$, where $\mathcal{T}_h : \mathbb{R}^d \mapsto \mathbb{R}^d$

$w^\top \phi(s,a)$

What does Linear Bellman completion imply

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$= 0$

It implies that $Q^\star_h$ is linear in ϕ:

$w = \vec{0}$

$$Q^\star_h = (\theta^\star)^\top \phi, \forall h$$

Why?

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Given feature ϕ, take any linear function $w^\top \phi(s,a)$:

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Why?

$= r(s,a)$

reward $r(s,a)$ is linear in ϕ, i.e., $Q_{H-1}^\star(s,a)$ is linear,

now recursively show that $Q_h^\star$ is linear

$Q_h^\star = T_h Q_{h+1}^\star$

Why this is a reasonable assumption?

It captures at least two special cases: tabular MDP and linear dynamical systems

1. Tabular MDP:

Set $\phi(s, a)$ to be a one-hot encoding vector in $\mathbb{R}^{SA}$, i.e., $\phi(s, a) = [0, \ldots, 0, 1, 0, \ldots 0]^\top$

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2. Linear System with Quadratic feature ϕ

$$s \in \mathbb{R}^2, a \in \mathbb{R}, P_h(\cdot \mid s, a) = \mathcal{N}\left(As + ba, \sigma^2 I\right)$$

$s, a \longrightarrow As + ba + \varepsilon$

$\varepsilon \sim N(0, \sigma^2 I)$

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(we will see the details when we get to the LQR lectures)

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Assume the given feature ϕ has linear Bellman completion, i.e.,

$$\forall h, \exists \theta \in \mathbb{R}^d, s.t., \theta^\top \phi(s,a) = r(s,a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} w^\top \phi(s',a'), \forall s, a$$

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Linear Bellman completion breaks!

This is counter-intuitive: in SL (e.g., linear regression),
adding elements to features is ok!

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$\phi(s,a) \in \mathbb{R}^d$

$\mathrm{poly}(d, H, \frac{1}{\varepsilon})$

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(We will work on a slightly different result later when we talk about online learning in MDPs)

What we will show today:

1. Generative Model

(i.e., we can reset system to any (s, a), query $r(s, a), s' \sim P(\,.\,|\,s, a)$)

$+$

SL: linear Regression

$y = \theta^{*T} x + \epsilon$

2. Linear Bellman Completion

$=$

poly$(d, \frac{1}{\epsilon})$

Sample efficient Learning
(poly time)

Poly$(d, H, \frac{1}{\epsilon})$

Outline:

1. The Linear Bellman Completion Condition

2. Learning: The Least Square Value Iteration Algorithm

3. Guarantee and the proof sketch

LSVI: Least-Square Value Iteration

Recall linear bellman-completion implies $Q_h^\star(s,a) = (\theta_h^\star)^\top \phi(s,a), \forall s, a, h$

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Given datasets $\mathcal{D}_0, \ldots, \mathcal{D}_{H-1}$, w/
$\mathcal{D}_h = \{s, a, r, s'\}, r = r(s,a), s' \sim P_h(\cdot \mid s, a)$

$r = r(s, a)$

$s' \sim P_h(\cdot / s a)$

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Set $V_H(s) = 0, \forall s$

Given datasets $\mathscr{D}_0, \ldots, \mathscr{D}_{H-1}$, w/
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Backward

For h = H-1 to 0:

$$\theta_h = \arg\min_{\theta} \sum_{\mathcal{D}_h} \Big(\underbrace{\theta^T \phi(s,a)}_{x} - \underbrace{\big(r + V_{h+1}(s')\big)}_{y} \Big)^2$$

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$\approx V^\star_{h+1}(s')$

Set $V_h(s) := \max_a \theta_h^\top \phi(s,a), \forall s$

$\approx Q^\star_h(s,a)$

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Set $V_h(s) := \max_a \theta_h^\top \phi(s,a), \forall s$

Return $\hat{\pi}_h(s) = \arg\max_a \theta_h^\top \phi(s,a), \forall h$

$\approx Q_h^\star(s,a)$

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Let's simulate the DP process w/ linear function to approximate $Q^\star$

Why LSVI may work?

When we do linear regression at step h:

$$\underbrace{x := \phi(s,a)}_{\text{feature}}, \quad \underbrace{y := r + V_{h+1}(s')}_{\text{regression target}}$$

Set $V_H(s) = 0, \forall s$

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When we do linear regression at step h:

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We note that:

$$\mathbb{E}[y \mid x] = \underbrace{r(s,a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} \theta_{h+1}^\top \phi(s',a')}_{\mathcal{T}_h(\theta_{h+1})^\top \phi(s,a) \text{ due to Linear BC}}$$

$\mathcal{T}(\theta^*)(s,a)$

Bayes opt

$\Rightarrow$ linear

Set $V_H(s) = 0, \forall s$

For h = H-1 to 0:

$$\theta_h = \arg\min_{\theta} \sum_{\mathcal{D}_h} \left(\theta^T \phi(s,a) - \left(r + V_{h+1}(s')\right)\right)^2$$

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$$\mathbb{E}[y \mid x] = \underbrace{r(s,a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} \theta_{h+1}^\top \phi(s',a')}_{\mathcal{T}_h(\theta_{h+1})^\top \phi(s,a) \text{ due to Linear BC}}$$

i.e., our regression target is indeed linear in ϕ, and it is close to $Q_h^\star$ if

$$V_{h+1} \approx V_{h+1}^\star$$

Set $V_H(s) = 0, \forall s$

For h = H-1 to 0:

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Set $V_h(s) := \max_a \theta_h^\top \phi(s,a), \forall s$

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At h:

$\Rightarrow E(y|x) \approx Q_h^\star(s,a)$

$T V_{h+1} \Leftarrow\Rightarrow T V_{h+1}^\star \Rightarrow Q_h^\star$

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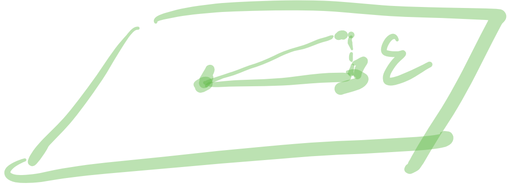
If $V_{h+1} \approx V_{h+1}^\star$, and linear regression succeeds (e.g., $\theta_h \approx \mathcal{T}_h(\theta_{h+1})$),

Then we should hope $\theta_h^\top \phi(s,a) \approx Q_h^\star(s,a)$



Outline:

1. The Linear Bellman Completion Condition

2. Learning: The Least Square Value Iteration Algorithm

3. Guarantee and the proof sketch

Sample complexity of LSVI

Theorem: There exists a way to construct datasets $\{\mathscr{D}_h\}_{h=0}^{H-1}$, such that with probability at least $1-\delta$, we have:

$$V^{\hat{\pi}} - V^{\star} \leq \epsilon$$

w/ total number of samples in these datasets scaling $\widetilde{O}\left(d^2 + H^6 d^2/\epsilon^2\right)$

$\mathrm{Poly}\left(d, H, \frac{1}{\epsilon}\right)$

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Plans: (1) OLS and D-optimal design; (2) construct $\mathscr{D}_h$ using D-optimal design; (3) transfer regression error to $\|\theta_h^\top \phi - Q_h^\star\|_\infty$

Detour: Ordinary Linear Squares

Consider a dataset $\{x_i, y_i\}_{i=1}^{N}$, where $y_i = (\theta^\star)^\top x_i + \epsilon_i$, $\mathbb{E}[\epsilon_i \mid x_i] = 0$, ϵ_i are independent with $|\epsilon_i| \leq \sigma$, assume $\Lambda = \sum_{i=1}^{N} x_i x_i^\top / N$ is full rank;

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$\theta^{\star\top} x$

Bayes opt

Standard OLS guarantee: with probability at least $1 - \delta$, we have:

$$\underbrace{(\hat{\theta} - \theta^\star)^\top \Lambda (\hat{\theta} - \theta^\star)}_{= \frac{1}{N} \sum_{i=1}^N \left((\hat{\theta} - \theta^\star)^\top x_i \right)^2} \leq O\left(\frac{\sigma^2 d \ln(1/\delta)}{N} \right)$$

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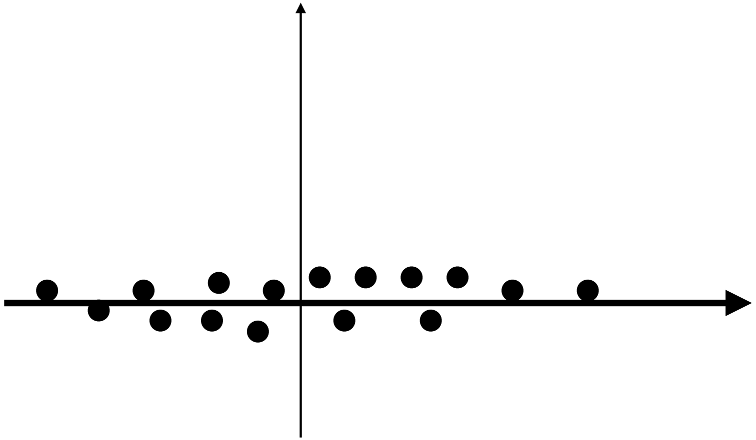

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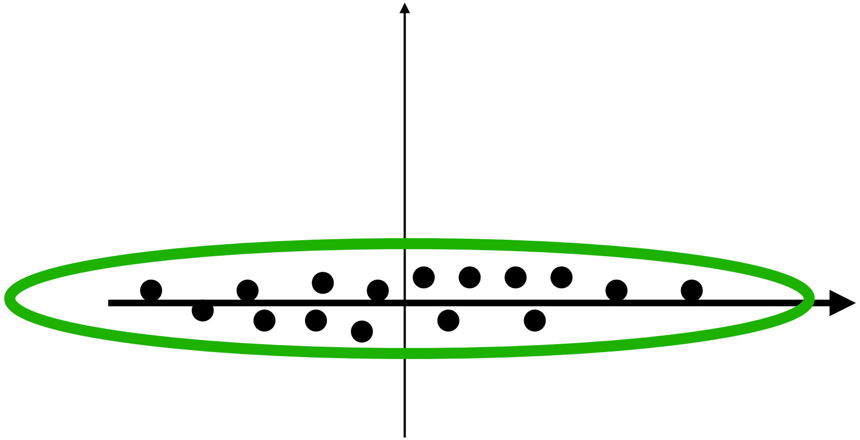

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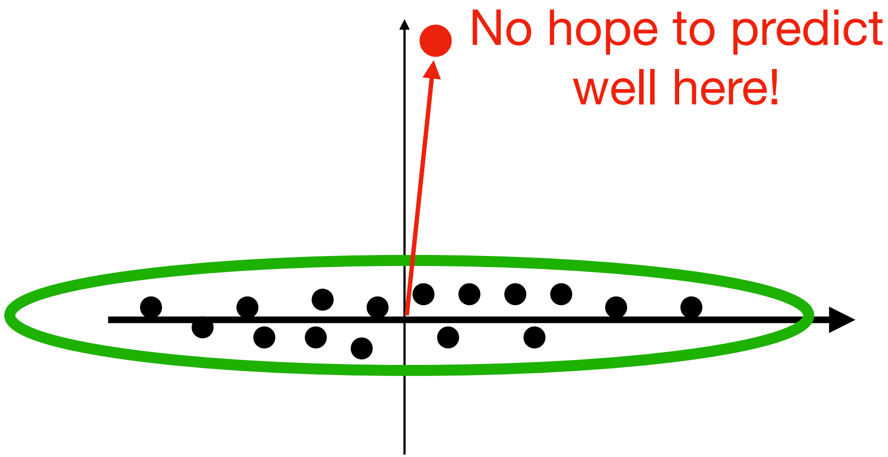

$\Lambda = \sum_{i=1}^{d} \sigma_i u_i u_i^\top$

$x^\top \Lambda^{-1} x$ large

$= \sum_{i=1}^{d} \frac{1}{\sigma_i} (u_i^\top x)^2$

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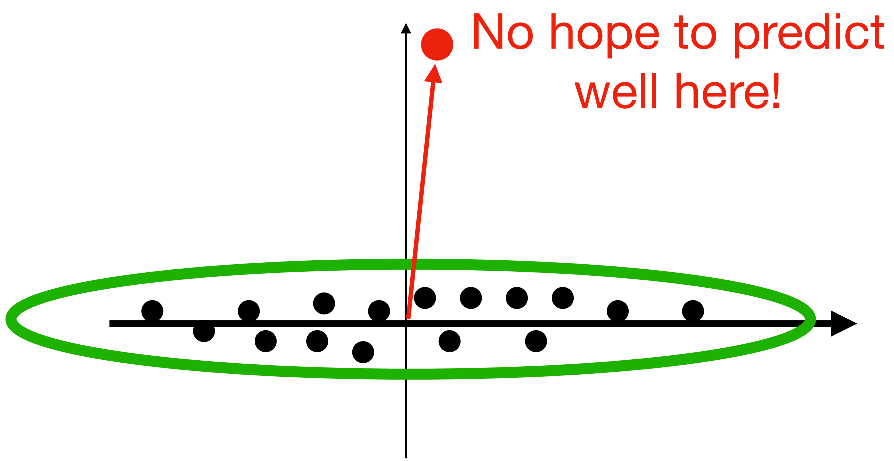


Let's actively design a diverse dataset !
(D-optimal Design)

Detour: D-optimal Design

Consider a compact space $\mathscr{X} \subset \mathbb{R}^d$ (without loss of generality, assume span($\mathscr{X}) = \mathbb{R}^d$)

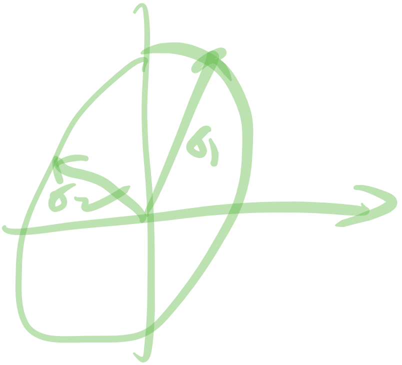
Detour: D-optimal Design

Concave

Consider a compact space $\mathscr{X} \subset \mathbb{R}^d$ (without loss of generality, assume span$(\mathscr{X}) = \mathbb{R}^d$)

D-optimal Design $\rho^\star \in \Delta(\mathscr{X})$**:** $\rho^\star = \arg\max_{\rho \in \Delta(\mathscr{X})} \ln\det\left(\mathbb{E}_{x\sim\rho}\left[xx^\top\right]\right)$

$\det(\Sigma)$



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Properties of the D-optimal Design:

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$$\max_{y \in \mathscr{X}} y^\top \left[\underbrace{\mathbb{E}_{x\sim\rho^\star} xx^\top}_{\Sigma}\right]^{-1} y \leq d$$

$$y^\top \Sigma^{-1} y = \sum_{i=1}^{d} \frac{1}{\sigma_i}\left(u_i^\top y\right)^2$$