Approximate Policy Iteration
Recall Policy Iteration (PI) for the setting where $P$ and $r$ are known:

We compute $Q^\pi(s, a)$ exactly for all $s, a$, PI updates policy as:

$$\pi'(s) = \arg\max_a Q^\pi(s, a)$$

i.e., be greedy with respect to $\pi$ at every state $s$,

Monotonic improvement of PI: $Q^{\pi'}(s, a) \geq Q^\pi(s, a)$, $\forall s, a$
Recap: Policy Iteration

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What if $P$ & $r$ are unknown, and MDP is large (e.g., infinitely many states)?
Recap: Model-based RL

Simulation Lemma:

\[
\hat{V}_\pi(s_0) - V_\pi(s_0) = \gamma \mathbb{E}_{s,a \sim d_\pi^s} \left[ \mathbb{E}_{s' \sim \hat{P}(s,a)} \hat{V}_\pi(s') - \mathbb{E}_{s' \sim P(s,a)} \hat{V}_\pi(s') \right]
\]

\[
\leq \gamma \frac{1}{(1 - \gamma)^2} \mathbb{E}_{s,a \sim d_\pi^s} \left\| \hat{P} \left( \cdot \mid s, a \right) - P \left( \cdot \mid s, a \right) \right\|_1
\]

\[
\text{state-action \ Di's of } \pi \text{ under } P
\]
Recap: Model-based RL

An Algorithm under Generative Model Setting for (small) discrete MDP:
Recap: Model-based RL

An Algorithm under Generative Model Setting for (small) discrete MDP:

1. **Model fitting:**
   \[ \forall s, a: \text{collect } N \text{ next states, } s'_i \sim P(\cdot | s, a), i \in [N]; \]
   \[ \text{set } \widehat{P}(s' | s, a) = \frac{\sum_{i=1}^{N} 1\{s'_i = s'\}}{N}; \]
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2. Planning w/ the learned model:
   \[ \hat{\pi}^* = \text{PI}\left(\hat{P}, r\right) \]
   \[ \text{optimal policy for } \text{MDP}(\hat{P}, r) \]
We are moving on to large scale MDPs

When we face extremely large state space or continuous state space:

Enumerate over all state-action pairs is not possible in both computation, space, and statistics;

What should we do?
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What should we do?

Answer: generalization via function approximation (e.g., linear, decision tree, SVM, GP, neural nets)
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What should we do?

Answer: generalization via function approximation (e.g., linear, decision tree, SVM, GP, neural nets)

Indeed, in LQR, we are using quadratic function to represent $Q$ & $V$

\[ V(x) = x^T P x + \Delta \]
Another example: Video games

State $s$: RGB image
Another example: Video games

State $s$: RGB image

We can try to capture $Q^*(s, a)$ via deep nets:
Another example: Video games

State $s$: RGB image

We can try to capture $Q^*(s, a)$ via deep nets:

Game action $a$
Question for Today (and the next a few lectures):

How to (approximately) learn $\pi^*$ using function approximation for large scale MDPs? (i.e., numeration over state-action is not feasible)
Outline:

1. Quick recap on supervised learning’s performance guarantee (classification & regression)

2. Approximate Policy Iteration (relies regression oracle)
Recap on Supervised Learning: Classification
Recap on Supervised Learning: Classification

Given i.i.d examples at training:

( ,cat ) ( ,cat ) ( ,dog )
Recap on Supervised Learning: Classification

Given i.i.d examples at training:
Recap on Supervised Learning: Classification

Given i.i.d examples at training:

Using function approximator, we are able to predict on cats/dogs that we never see before (i.e., we generalize)
Recap on Supervised Learning: Regression

$X$: distance to whole foods

$Y$: value of a house

Diagram showing a downward trend of points representing the relationship between $X$ and $Y$. The points decrease as $X$ increases, indicating a negative correlation.
Recap on Supervised Learning: Regression

Y: value of a house

X: distance to whole foods
Recap on Supervised Learning: Regression

$X$: distance to Whole Foods

$Y$: value of a house

Using function approximator, we are able to predict on the value of some house not from the training data.
Recap on Supervised Learning: regression

We have a data distribution $\mathcal{D}$, $x_i \sim \mathcal{D}$, $y_i = f^*(x_i) + \epsilon_i$, where noise $\mathbb{E}[\epsilon_i] = 0, |\epsilon_i| \leq c$. 

\[ \Delta \]
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Q: quality of ERM $\hat{f}$?
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\[\mathbb{E}_{x \sim \mathcal{D}} \left( \hat{f}(x) - f^*(x) \right)^2 \leq \delta\]
Supervise Learning can fail if there is train-test distribution mismatch

However, for some $\mathcal{D}' \neq \mathcal{D}$, $\mathbb{E}_{x \sim \mathcal{D}'} (f(x) - f^*(x))^2$ might be arbitrarily large
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Deeper neural nets or larger dataset do not help if there is distribution shift;
(ML is not black magic)
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$$\mathbb{E}_{x \sim \mathcal{D}} \left( \hat{f}(x) - f^*(x) \right)^2 \leq \delta$$

( where $\delta \approx \sqrt{1/N}$ (sometime it could be $1/N$) )
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Setting and Notation

Discounted infinite horizon MDP:

\[ \mathcal{M} = \{ S, A, \gamma, r, P, \mu_0 \} \]

\[ s_0 \sim \mu_0 \]
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State visitation:

\[ d_{\mu_0}^\pi(s) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h^\pi(s; \mu_0) \]
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As we will consider large scale unknown MDP here, we start with a (restricted) function class \( \mathcal{Q} \):

\[ \mathcal{Q} = \{ Q : S \times A \mapsto [0,1/(1 - \gamma)] \} \]
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We can only reset according to \( s_0 \sim \mu_0 \)
Approximate Policy Iteration

Like Policy Iteration, we iterate between two steps:
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a. How to get training data?
b. Quality of the learned \( \hat{Q}^t \)?
Policy Evaluation: Dataset Generation

Q1: how do we sample a state-action pair \((s, a) \sim d_{\mu_0}^\pi\)?
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1. sample time step \(h\) with probability \(\gamma^h / (1 - \gamma)\)

\[
\forall \ (s, a) \sim \prod_{h}^\pi P_h ( \cdot, \cdot ; \mu_0 )
\]
Policy Evaluation: Dataset Generation

Q1: how do we sample a state-action pair \((s, a) \sim d_{\mu_0}^\pi\) ?

1. sample time step \(h\) with probability \(\gamma^h/(1 - \gamma)\)

2. Roll-in \(\pi\) to time step \(h\), and return \((s_h, a_h)\)
(i.e., we sample \((s, a) \sim P_{\pi}^\mu(\cdot, \cdot ; \mu_0)\))
Q1: how do we sample a state-action pair \((s, a) \sim d^{\pi}_{\mu_0}\)?

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Q2: Given that we are at \((s, a)\), how do we get an unbiased estimate of \(Q^\pi(s, a)\)?

\[
Q^\pi(s, a) = \mathbb{E}_\pi \left[ \sum_{h=0}^{\infty} \gamma^h R^{(s, a)} | s_0, a_0 = (s, a) \right]
\]
Policy Evaluation: Dataset Generation

Q2: Given that we are at \((s, a)\), how do we get an unbiased estimate of \(Q^\pi(s, a)\)?

Denote \((s_0, a_0) = (s, a)\)

For \(h = 0, \ldots, \)
Policy Evaluation: Dataset Generation

Q2: Given that we are at $(s, a)$, how do we get an unbiased estimate of $Q^\pi(s, a)$?

Denote $(s_0, a_0) = (s, a)$

For $h = 0, \ldots,$

Receive $r_h = r(s_h, a_h)$
Policy Evaluation: Dataset Generation

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With probability \(1 - \gamma\): Break and Return \(\sum_{i=0}^{t} r_i\)
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Transition: \(s_{h+1} \sim P(s_h, a_h), a_{h+1} \sim \pi(\cdot | s_{h+1})\)
Policy Evaluation: Dataset Generation

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A Roll-out process
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\[ h = 0 (s, a), r_0 = r(s, a) \]

\[ s_1 \sim P(\cdot | s, a), \quad a_1 \sim \pi(\cdot | s_1) \]

\[ r_1 = r(s_1, a_1) \]

If terminate (w/ p \(1 - \gamma\)), we return \(\sum_{i=0}^{t} r_i\)

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\[ y := \sum_{i=0}^{t} r_i \]

Claim: A roll-out from \((s, a)\) gives an unbiased estimate of \(Q^\pi(s, a)\)

\[ \mathbb{E}[y] = Q^\pi(s, a) \]
Policy Evaluation: Dataset Generation

Claim: A roll-out from $(s, a)$ gives an unbiased estimate of $Q^\pi(s, a)$

$$\mathbb{E} [y] = Q^\pi(s, a)$$

Proof sketch (full proof is left as an exercise):

- $h = 0$ $(s, a), r_0 = r(s, a)$
- $s_1 \sim P(\cdot | s, a)$,
- $a_1 \sim \pi(\cdot | s_1)$
- $r_1 = r(s_1, a_1)$
- $(s_h, a_h), r_h$
- If terminate (w/ p $1 - \gamma$), we return
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and what’s the probability of returning \( y = r_0 + r_1 \)?

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\[ (1 - \gamma)r_0 + \gamma(1 - \gamma)(r_0 + r_1) + \gamma^2(1 - \gamma)(r_0 + r_1 + r_2) + \ldots = \sum_{h=0}^{\infty} \gamma^h r_h \]
Summary of the dataset generation process:

Given $\pi^t$:

1. we roll-in to generate $(s, a) \sim d_{\mu_0}^{\pi}$
2. At $(s, a)$, we roll-out w/ $\pi$ to generate an unbiased estimate of $Q^\pi(s, a)$: $y$

In other words, one roll-in & roll-out gives us a triple $(s, a, y)$
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Given $\pi$, repeat N times of the roll-in & roll-out process, we get a training dataset of N samples:
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Given $\pi$, repeat N times of the roll-in & roll-out process, we get a training dataset of N samples:

$$D^\pi = \{s^i, a^i, y^i\}_{i=1}^{N} \leftarrow \text{Regression Dataset}$$

$s^i, a^i \sim d^\pi_{\mu_0}$

$E[y^i] = Q^\pi(s^i, a^i)$
Estimating the function $Q^\pi(s, a)$ using Least Square Regression

Given $\pi$, repeat $N$ times of the roll-in & roll-out process, we get a training dataset of $N$ samples:

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Estimating the function $Q^\pi(s, a)$ using Least Square Regression

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$$\mathcal{D}^\pi = \{s^i, a^i, y^i\}_{i=1}^N$$

Least square regression:

$$\widehat{Q}^\pi \in \arg\min_{Q \in \Theta} \sum_{i=1}^N \left( Q(s^i, a^i) - y^i \right)^2$$
Estimating the function $Q^\pi(s, a)$ using Least Square Regression

Given $\pi$, repeat $N$ times of the roll-in & roll-out process, we get a training dataset of $N$ samples:

$$\mathcal{D}^\pi = \{s^i, a^i, y^i\}_{i=1}^N$$

Least square regression:

$$\hat{Q}^\pi \in \arg \min_{Q \in \mathcal{Q}} \sum_{i=1}^N (Q(s^i, a^i) - y^i)^2$$

Assume successful supervise learning, we have:

$$\mathbb{E}_{s,a \sim d_\mu^\pi} \left( \hat{Q}^\pi(s, a) - Q^\pi(s, a) \right)^2 \leq \delta,$$

where $\delta$ being some small number (e.g., $1/\sqrt{N}$)
Put things together: Algorithm of Approximate Policy Iteration
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Initialize $\hat{Q}^0 \in \mathbb{Q}$, set $\pi^0(s) = \arg \max_a \hat{Q}^0(s, a)$
Put things together: Algorithm of Approximate Policy Iteration

Initialize $\hat{Q}^0 \in Q$, set $\pi^0(s) = \arg\max_a \hat{Q}^0(s, a)$
Put things together: Algorithm of Approximate Policy Iteration

Initialize $\hat{Q}^0 \in Q$, set $\pi^0(s) = \arg \max_a \hat{Q}^0(s, a)$

$\pi^t$ -> Data generalization Process (roll-in & roll-out)

$\mathcal{D}^{\pi^t} = \{s^i, a^i, y^i\}_{i=1}^N$

$\mathbb{E}[y_i] = Q^{\pi^t}(s_i, a_i)$

$s^i, a^i \sim d^{\pi}_{\mu_0}$
Put things together: Algorithm of Approximate Policy Iteration

Initialize $\hat{Q}^0 \in \mathcal{Q}$, set $\pi^0(s) = \arg \max_a \hat{Q}^0(s, a)$

$\pi^t$ → Data generalization Process (roll-in & roll-out) → $\mathcal{D}^{\pi^t} = \{s^i, a^i, y^i\}_{i=1}^N$

$\mathbb{E}[y_i] = Q^{\pi^t}(s_i, a_i)$

$s^i, a^i \sim d_{\mu_0}^{\pi}$ → Least Square Regression oracle
Put things together: Algorithm of Approximate Policy Iteration

Initialize $\hat{Q}^0 \in \mathcal{Q}$, set $\pi^0(s) = \arg \max_a \hat{Q}^0(s, a)$

Data generalization Process (roll-in & roll-out)

$D^{\pi^t} = \{s^i, a^i, y^i\}_{i=1}^N$

$\mathbb{E}[y_i] = Q^{\pi^t}(s_i, a_i)$

$s^i, a^i \sim d^{\pi^t}_{\mu_0}$

Least Square Regression oracle

$\hat{Q}^t \in \arg \min_{Q \in \mathcal{Q}} \sum_{i=1}^N (Q(s^i, a^i) - y^i)^2$
Put things together: Algorithm of Approximate Policy Iteration

Initialize $\hat{Q}^0 \in \mathcal{Q}$, set $\pi^0(s) = \arg \max_a \hat{Q}^0(s, a)$

$\pi^t$ → Data generalization Process (roll-in & roll-out)

$D^{\pi^t} = \{s^i, a^i, y^i\}_{i=1}^N$

$E[y^i] = Q^{\pi^t}(s^i, a^i)$

$s^i, a^i \sim d^{\pi^t}_{\mu_0}$

Least Square Regression oracle

$\hat{Q}^t \in \arg \min_{Q \in \mathcal{Q}} \sum_{i=1}^N (Q(s^i, a^i) - y^i)^2$

$\pi^{t+1}(s) = \arg \max_a \hat{Q}^t(s, a)$
Put things together: Algorithm of Approximate Policy Iteration

Initialize $\hat{Q}^0 \in \mathcal{Q}$, set $\pi^0(s) = \arg\max_a \hat{Q}^0(s, a)$

$\mathcal{D}^{\pi^t} = \{s^i, a^i, y^i\}_{i=1}^N$

$\mathbb{E}[y_i] = Q^{\pi^t}(s_i, a_i)$

$s^i, a^i \sim d^\pi_{\mu_0}$

$\pi^{t+1}(s) = \arg\max_a \hat{Q}^t(s, a)$

$\hat{Q}^t \in \arg\min_{Q \in \mathcal{Q}} \sum_{i=1}^N (Q(s^i, a^i) - y^i)^2$
Put things together: Algorithm of Approximate Policy Iteration

Initialize $\widehat{Q}^0 \in \mathbb{Q}$, set $\pi^0(s) = \arg\min_a \widehat{Q}^0(s, a)$

For $t = 0, \ldots$

Repeat N roll-in & roll-out w/ $\pi^t$; get N training points $\{s^i, a^i, y^i\}_{i=1}^N$

Least Square Minimization: $\widehat{Q}^t \in \arg\min_{Q \in \mathbb{Q}} \sum_{i=1}^N (Q(s^i, a^i) - y^i)^2$

Policy Improvement $\pi^{t+1}(s) = \arg\max_a \widehat{Q}^t(s, a)$
Summary
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2. A data generation process: given $\pi$, we roll-in & roll-out to get $(s, a, y)$, where $(s, a) \sim d^\pi$, $\mathbb{E}[y] = Q^\pi(s, a)$.

If terminate (w/ $p \ 1 - \gamma$), we return $y := \sum_{i=h}^{t} r_i$. 

$h = 0$

$s_0 \sim \mu_0, a_0 \sim \pi(\cdot | s_0)$

$(s_h, a_h), r_h$

$h \propto \gamma^h$

$(s_t, a_t), r_t$
Summary

1. Supervised Learning works (in both theory and practice) if there is no train-test mismatch

2. A data generation process: given $\pi$, we roll-in & roll-out to get $(s, a, y)$, where $(s, a) \sim d^\pi$, $\mathbb{E}[y] = Q^\pi(s, a)$

3. API Algorithm: Iterate between:
   (1) estimate $Q^{\pi^t}$ using Least Square Regression; (2) update policy $\pi^{t+1}(s) = \arg\max_a \hat{Q}^t(s, a)$