Approximate Policy Iteration
And Performance Difference Lemma

Recap: Supervised Learning and Data Generation Process

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1. Supervised Learning works (in both theory and practice) if there is no train-test mismatch

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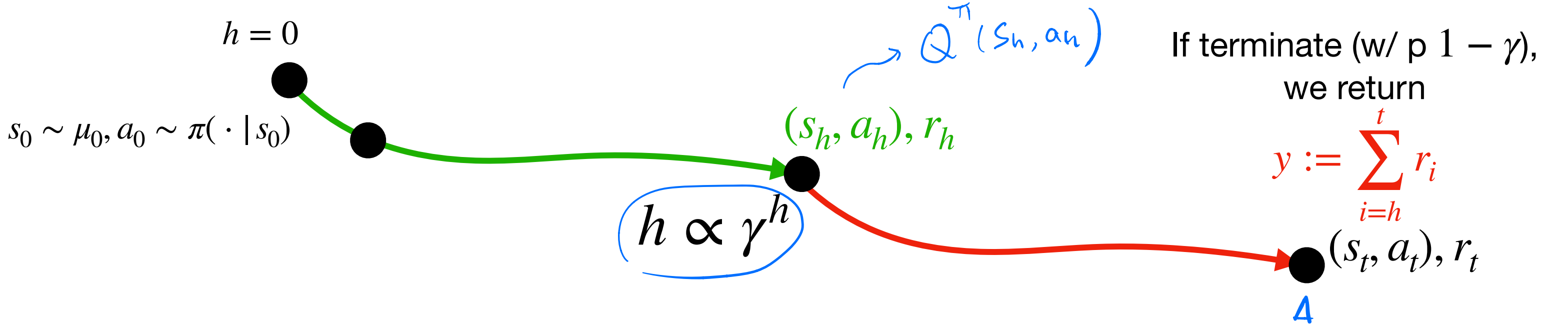
2. A **data generation process**: given π, we roll-in & roll-out to get (s, a, y),
where $(s, a) \sim d^{\pi}_{\mu_0}$, $\mathbb{E}[y] = Q^{\pi}(s, a)$



Plans for Today

1. Algorithm: Approximate Policy Iteration

2. When does API could make monotonic improvement?

3. Performance Difference Lemma (Another important lemma)

Estimating the function $Q^{\pi}(s, a)$ using Least Square Regression

Given π, repeat N times of the roll-in & roll-out process, we get a training dataset of N samples:

$$\mathscr{D}^{\pi} = \{s^i, a^i, y^i\}_{i=1}^{N}$$

$s^i, a^i \sim d^{\pi}_{\mu_0}$

$E[y^i] = Q^{\pi}(s^i, a^i)$

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Least square regression:

$$\widehat{Q}^\pi \in \arg\min_{Q \in \mathcal{Q}} \sum_{i=1}^N \left(Q(s^i, a^i) - y^i\right)^2$$

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$$\mathscr{D}^\pi = \left\{s^i, a^i, y^i\right\}_{i=1}^N$$

$\phi(s,a) \in \mathbb{R}^d$

$\mathcal{Q} = \left\{ \theta^T \phi(s,a) : \|\theta\|_2 \le 1 \right\}$

Least square regression:

$$\widehat{Q}^\pi \in \arg\min_{Q\in\mathcal{Q}} \sum_{i=1}^N \left(Q(s^i, a^i) - y^i\right)^2$$

$$\min_{\theta : \|\theta\|_2 \le 1} \sum_{i=1}^N \left(\theta^T \phi(s^i,a^i) - y^i\right)^2$$

Assume successful supervise learning, we have:

$s^i, a^i \sim d^\pi_{\mu_0}$

$$\mathbb{E}_{s,a\sim d^\pi_{\mu_0}}\left(\widehat{Q}^\pi(s, a) - Q^\pi(s, a)\right)^2 \le \delta,$$

where δ being some small number (e.g., $1/\sqrt{N}$)

Put things together: Algorithm of Approximate Policy Iteration

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Inside Iteration t:

$\pi^t \longrightarrow$ **Data generalization Process (roll-in & roll-out)**

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$\pi^t \longrightarrow$ **Data generalization Process (roll-in & roll-out)** $\longrightarrow$

$$\mathcal{D}^{\pi^t} = \{s^i, a^i, y^i\}_{i=1}^N$$

$$\mathbb{E}[y_i] = Q^{\pi^t}(s_i, a_i)$$

$$\underline{s^i, a^i \sim d_{\mu_0}^{\pi^t}}$$

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$\longrightarrow$ **Least Square Regression oracle** $\longrightarrow$

$$\widehat{Q}^t \in \arg\min_{Q \in \mathcal{Q}} \sum_{i=1}^N (Q(s^i, a^i) - y^i)^2$$

$$\widehat{Q}^t \approx Q^{\pi^t}$$

Put things together: Algorithm of Approximate Policy Iteration

Initialize $\widehat{Q}^0 \in \mathcal{Q}$, set $\pi^0(s) = \arg\max_a \widehat{Q}^0(s,a)$

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$\rightarrow$ **Least Square Regression oracle** $\downarrow$

$$\widehat{Q}^t \in \arg\min_{Q\in\mathcal{Q}} \sum_{i=1}^N (Q(s^i, a^i) - y^i)^2$$

$\leftarrow$ **Policy Improvement**

$$\pi^{t+1}(s) = \arg\max_a \widehat{Q}^t(s,a)$$

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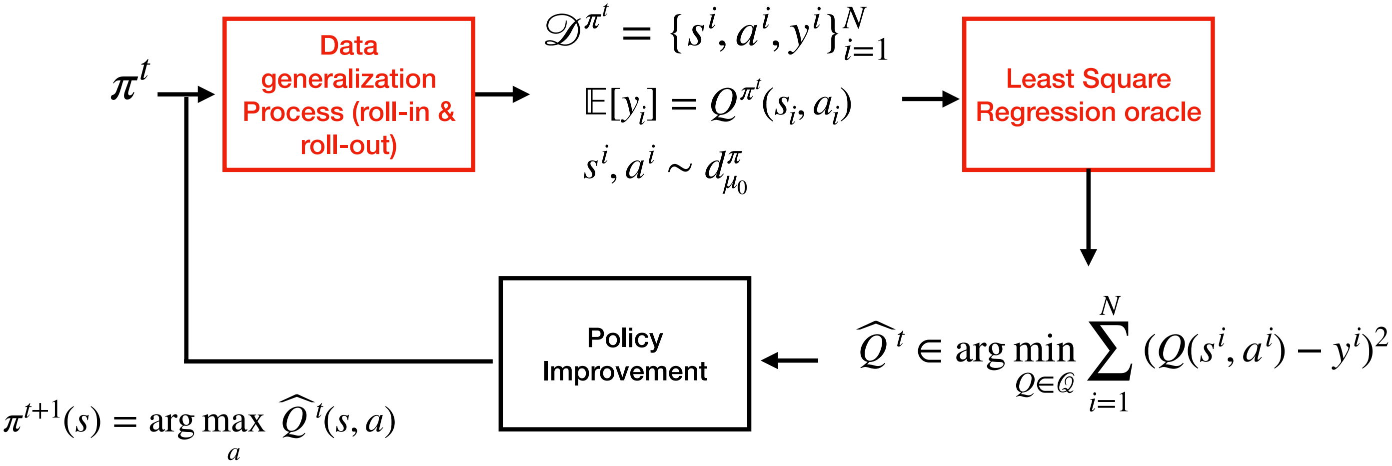


Put things together: Algorithm of Approximate Policy Iteration

Initialize $\widehat{Q}^0 \in \mathcal{Q}$, set $\pi^0(s) = \arg\min_a \widehat{Q}^0(s,a)$

For $t = 0, \ldots,$ start μ_0

Repeat N roll-in & roll-out w/ π^t; get N training points $\{s^i, a^i, y^i\}_{i=1}^N$

Least Square Minimization: $\widehat{Q}^t \in \arg\min_{Q \in \mathcal{Q}} \sum_{i=1}^N \left(Q(s^i, a^i) - y^i\right)^2$ → Approximating $Q^{\pi^t}(s,a)$

Policy Improvement $\pi^{t+1}(s) = \arg\max_a \widehat{Q}^t(s,a)$

Plans for Today

1. Algorithm: Approximate Policy Iteration

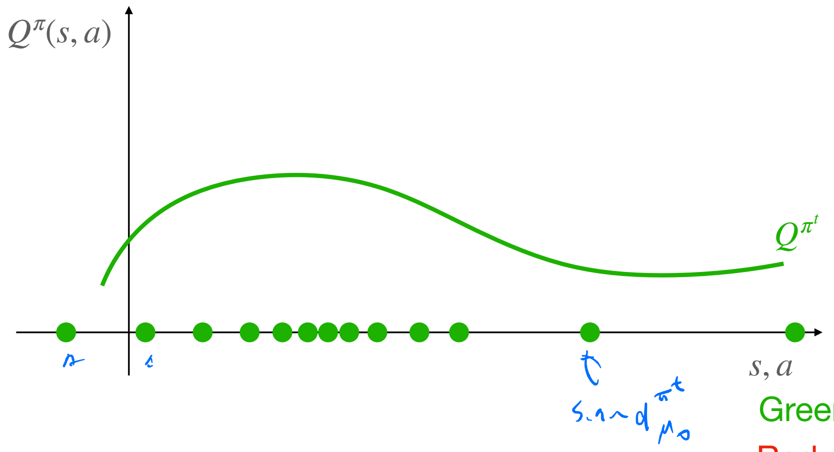
2. When does API could make monotonic improvement?

3. Performance Difference Lemma (Another important lemma)

The Oscillation of API from Abrupt Distribution Change

Recall that Policy Iteration w/ known (P, r) makes monotonic improvement;

But API cannot guarantee to make monotonic improvement:



Green dots: (s, a) from π^t

Red dots: (s, a) from π^{t+1}

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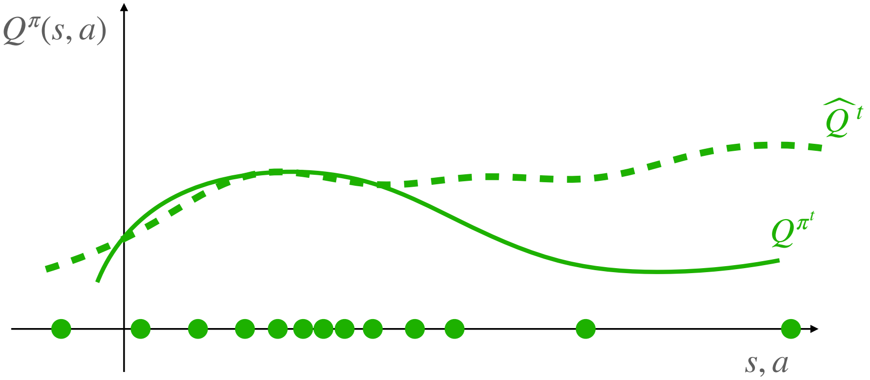

$\pi^{t+1} = \arg\max_a \widehat{Q}^t$

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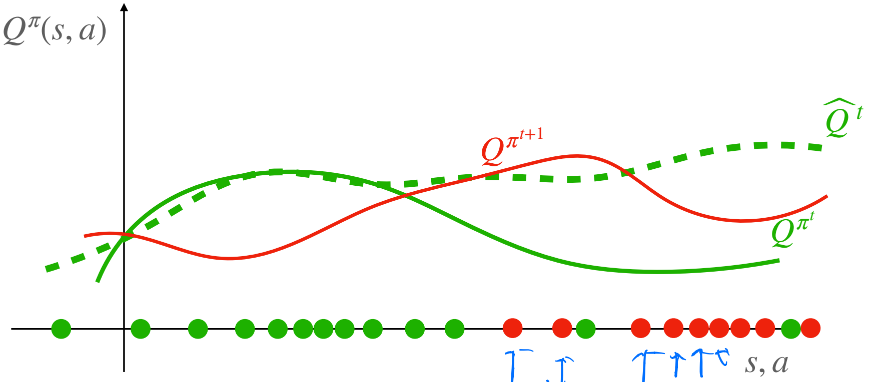

$\pi^{t+1} = \arg\max \widehat{Q}^{t}$

$\bullet \sim d^{\pi^{t+1}}_{\mu_0}$

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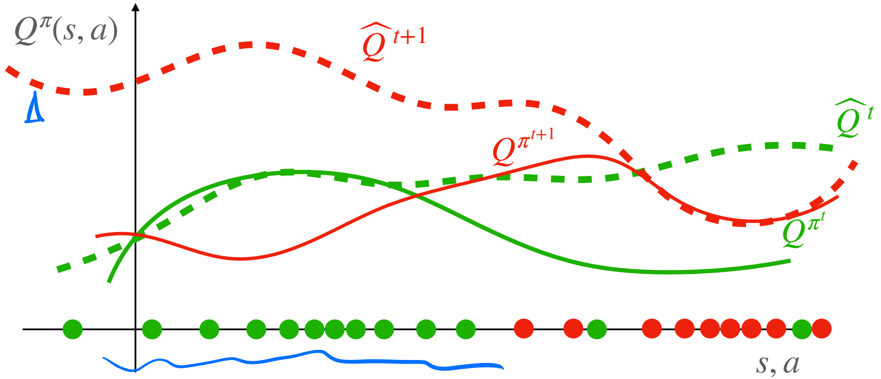
$\pi^{t+2} = \arg\max_a \widehat{Q}^{t+1}$



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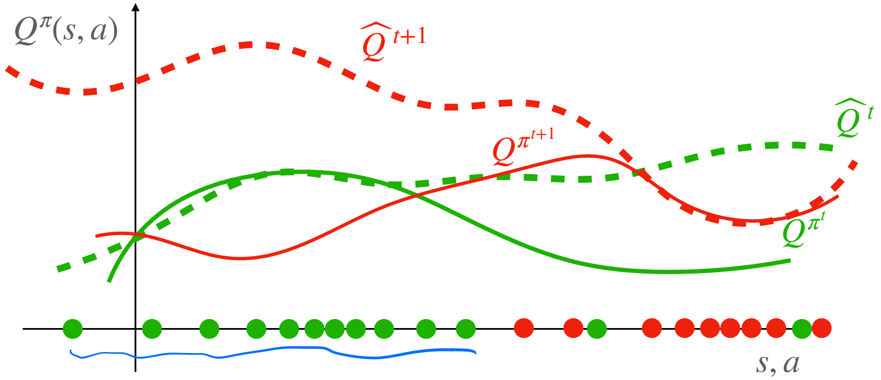

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Oscillation between two updates:

No monotonic improvement

Key Issue: Abrupt Policy Change, i.e., $d_{\mu_0}^{\pi^{t+1}}$ and $d_{\mu_0}^{\pi^t}$ could be widely different

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Our estimator $\widehat{Q}^t$ is only good under $d_{\mu_0}^{\pi^t}$,
i.e. $\mathbb{E}_{s\sim d_{\mu_0}^{\pi^t}}(\widehat{Q}^t(s,a) - Q^{\pi^t}(s,a))^2$ small,

Key Issue: Abrupt Policy Change, i.e., $d_{\mu_0}^{\pi^{t+1}}$ and $d_{\mu_0}^{\pi^t}$ could be widely different

Trained under $s, a \sim d_{\mu_0}^{\pi^t}$,

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To make API to make monotonic improvement, we need a strong coverage assumption:

A strong Concentrability Coefficient: $\underbrace{\max_{\pi} \max_{s} \frac{d^{\pi}_{\mu_0}(s)}{\mu_0(s)}}\leq C < \infty$

$\forall \pi,\ \forall s,\ \frac{d^{\pi}_{\mu_0}(s)}{\mu_0(s)} \leq C < \infty$

Key Issue: Abrupt Policy Change, i.e., $d_{\mu_0}^{\pi^{t+1}}$ and $d_{\mu_0}^{\pi^t}$ could be widely different

$\mu_0 \in \Delta(S)$

Our estimator $\widehat{Q}^t$ is only good under $d_{\mu_0}^{\pi^t}$,

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$\forall \pi,\ \forall s,\ \frac{d_{\mu_0}^\pi(s)}{\mu_0(s)}$ is finite

If $C < \infty$, i.e., μ covers **all** $d_{\mu_0}^\pi$,

then we can expect $\widehat{Q}^t$ can approximate Q^{π^t} almost everywhere

$s, a \sim d_{\mu_0}^\pi$ $\quad d_{\mu_0}^\pi = (1-\gamma)\mu_0 + (1-\gamma)\gamma \mathbb{P}_1^\pi \cdots$

Outline for Today

1. API could fail to make improvement?

2. When does API could make steady improvement?
(Next a few lectures, we will talk about **incremental** algorithms that **forces** π^{t+1} **to be close to** π^t

3. Performance Difference Lemma (Another important lemma)

Motivation (or the key question) behind the Performance Difference Lemma (PDL)

Let's recall simulation lemma, given two MDPs, $\widehat{P}, P$, and a policy π,

$$\left| \widehat{V}^{\pi}(s_0) - V^{\pi}(s_0) \right| \leq \frac{\gamma}{(1-\gamma)^2} \mathbb{E}_{s,a\sim d^{\pi}_{s_0}} \left\| \widehat{P}(s,a) - P(s,a) \right\|_1$$

$$= \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r_h \,\middle|\, \pi, \widehat{P}\right]$$

i.e., we can upper bound value difference by model disagreement (average over real traces)

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Given an infinite horizon MDP, and two policies π and π', what is the performance difference: $V^{\pi}(s_0) - V^{\pi'}(s_0) = ??$

By the difference between π, π'

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Given an infinite horizon MDP, and two policies π and π',
what is the performance difference: $V^{\pi}(s_0) - V^{\pi'}(s_0) = ??$

(Diff in performances $\Leftarrow \Rightarrow$ Diff in policies?)

Setting and Notation

Discounted infinite horizon MDP:

$$\mathcal{M} = \{S, A, \gamma, r, P\}$$

State visitation: $d_{s_0}^{\pi}(s) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h^{\pi}(s; s_0)$

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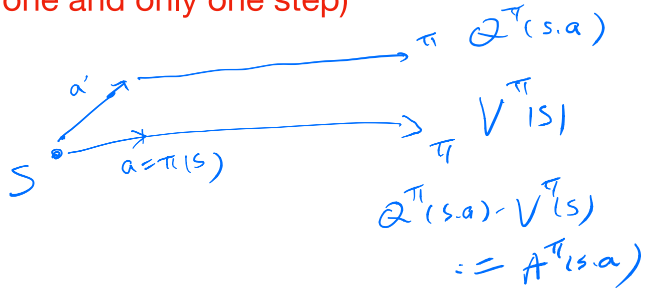
A new definition: Advantage $A^{\pi}(s,a) := Q^{\pi}(s,a) - V^{\pi}(s)$

(The "advantage" of deviating from π for one and only one step)



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(Quick sanity check: $A^{\pi}(s, \pi(s)) = 0$)

$A^{\pi}(s, \pi(s))$

$= Q^{\pi}(s, \pi(s)) - V^{\pi}(s)$

Setting and Notation

Discounted infinite horizon MDP:

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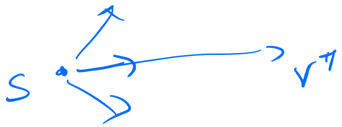

Recall PI:

$$\arg\max_a Q^{\pi}(s,a) \underset{\Delta}{=} \arg\max_a A^{\pi}(s,a),$$

i.e., Policy-improve step seeks the action that has the **largest adv**

PDL:

Given two policies $\pi : S \mapsto \Delta(A)$, $\pi' : S \mapsto \Delta(A)$, recall $V^{\pi}(s_0) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \,|\, \pi\right]$

PDL:

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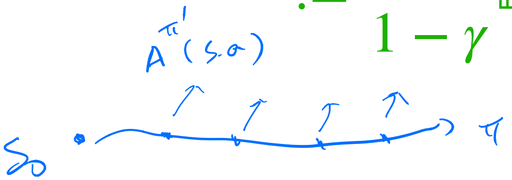
Performance Difference Lemma (PDL):

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}Q^{\pi'}(s,a) - V^{\pi'}(s)\right]$$

$$:= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]$$

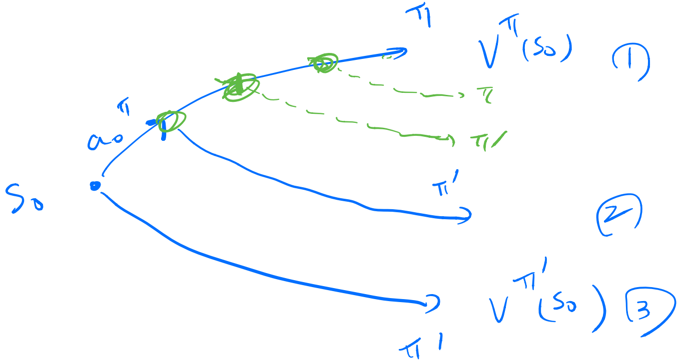


PDL Explanation

$$d^{\pi}_{s_0}(s,a) = d^{\pi}_{s_0}(s)\cdot\pi(a|s)$$

$$= \mathbb{E}_{s,a\sim d^{\pi}_{s_0}}\left[A^{\pi'}(s,a)\right]$$

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]$$



① − ③

$= \underbrace{① - ②}_{\text{Recursion}} + ② - ③$

$$A^{\pi'}(s_0,a_0) = Q^{\pi'}(s_0,a_0) - V^{\pi'}(s_0) = ② - ③$$

PDL Proof

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]$$

Proof of Sketch (see reading material for detailed steps)

PDL Proof

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]$$

Proof of Sketch (see reading material for detailed steps)

$$V^{\pi}(s_0) - V^{\pi'}(s_0)$$

$$= V^{\pi}(s_0) - \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[\cancel{r(s_0,a_0)} + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right] + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[r(s_0,a_0) + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right] - V^{\pi'}(s_0)$$

① ② ② ③

BE

$$= \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[\cancel{r(s_0,a_0)} + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi}(s')\right]$$

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$\mathbb{E}_{y} f(x) = f(x)$

Proof of Sketch (see reading material for detailed steps)

$$V^{\pi}(s_0) - V^{\pi'}(s_0)$$

$$= V^{\pi}(s_0) - \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[r(s_0,a_0) + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right] + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[r(s_0,a_0) + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right] - V^{\pi'}(s_0)$$

$$= \gamma\mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\mathbb{E}_{s_1\sim P(s_0,a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\underbrace{\left[r(s_0,a_0) + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right]}_{Q^{\pi'}(s_0,a_0)} - \underbrace{V^{\pi'}(s_0)}_{=\mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[V^{\pi'}(s_0)\right]}$$

Recursion

PDL Proof

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]$$

Proof of Sketch (see reading material for detailed steps)

$$
\begin{aligned}
& V^{\pi}(s_0) - V^{\pi'}(s_0) \\
&= V^{\pi}(s_0) - \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[r(s_0,a_0) + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right] + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[r(s_0,a_0) + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right] - V^{\pi'}(s_0) \\
&= \gamma\mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\mathbb{E}_{s_1\sim P(s_0,a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[r(s_0,a_0) + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right] - V^{\pi'}(s_0) \\
&= \gamma\mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\mathbb{E}_{s_1\sim P(s_0,a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[\underbrace{Q^{\pi'}(s_0,a_0) - V^{\pi'}(s_0)}_{A^{\pi'}(s_0,a_0)}\right]
\end{aligned}
$$

PDL Proof

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]$$

Proof of Sketch (see reading material for detailed steps)

$$V^{\pi}(s_0) - V^{\pi'}(s_0)$$

$$= V^{\pi}(s_0) - \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[r(s_0,a_0) + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right] + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[r(s_0,a_0) + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right] - V^{\pi'}(s_0)$$

$$= \gamma\mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\mathbb{E}_{s_1\sim P(s_0,a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[r(s_0,a_0) + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right] - V^{\pi'}(s_0)$$

$$= \gamma\mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\mathbb{E}_{s_1\sim P(s_0,a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[Q^{\pi'}(s_0,a_0) - V^{\pi'}(s_0)\right]$$

$$= \gamma\mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\mathbb{E}_{s_1\sim P(s_0,a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[A^{\pi'}(s_0,a_0)\right]$$

Summary of PDL:

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}Q^{\pi'}(s,a) - V^{\pi'}(s)\right]$$

$$:= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]$$

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(Use the fact that $Q^{\pi}(s,a) \in [0, 1/(1-\gamma)]$)

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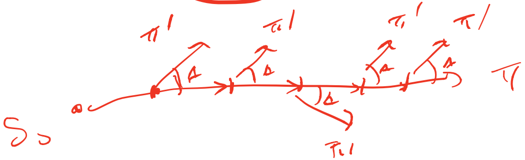
(Use the fact that $Q^{\pi}(s,a) \in [0,1/(1-\gamma)]$)

$\pi(\cdot|s) \in \Delta(A)$

$$\left|V^{\pi}(s_0) - V^{\pi'}(s_0)\right| \leq \frac{1}{(1-\gamma)^2}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\left\|\pi(\cdot|s) - \pi'(\cdot|s)\right\|_1\right]$$

$$= \sum_{a\in A}\left|\pi(a|s) - \pi'(a|s)\right|$$

$$= 2\left\|\pi(\cdot|s) - \pi'(\cdot|s)\right\|_{TV}$$



$P, Q \quad f(x)$

$\left| \mathbb{E}_{x\sim P}(f(x)) - \mathbb{E}_{x\sim Q} f(x) \right|$

Summary of PDL:

$\left| \mathbb{E}_{a\sim\pi(\cdot|s)} Q^{\pi'}(s,a) - \mathbb{E}_{a\sim\pi'(\cdot|s)} Q^{\pi'}(s,a) \right|$

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\underline{\mathbb{E}_{a\sim\pi(\cdot|s)}Q^{\pi'}(s,a) - V^{\pi'}(s)}\right]$$

$\le \max_a |f(x)| \left[\sum_x |P(x) - Q(x)| \right]$

$$:= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]$$

$:= \max_a |f(x)| \, \|P - Q\|_1$

$\le \max_{s,a} Q^{\pi}(s,a) \cdot \|\pi(\cdot|s) - \pi'(\cdot|s)\|_1$

(Use the fact that $Q^{\pi}(s,a) \in [0, 1/(1-\gamma)]$)

$$\left|V^{\pi}(s_0) - V^{\pi'}(s_0)\right| \le \frac{1}{(1-\gamma)^2}\underline{\mathbb{E}_{s\sim d^{\pi}_{s_0}}}\left[\left\| \pi(\cdot|s) - \pi'(\cdot|s) \right\|_1\right]$$

Policy disagreement (ℓ_1) averaged over one policy's traces

An Application of PDL in Policy Iteration

Recall that $\pi^{t+1}(s) = \arg\max_a A^{\pi^t}(s, a)$ $= \arg\max_a Q^{\pi^t}(s,a)$

Show monotonic improvement using PDL:

An Application of PDL in Policy Iteration

Recall that $\pi^{t+1}(s) = \arg\max_a A^{\pi^t}(s, a)$

Show monotonic improvement using PDL:

PDL

$$V^{\pi^{t+1}}(s_0) - V^{\pi^t}(s_0) = \frac{1}{1-\gamma}\underline{\mathbb{E}_{s \sim d_{s_0}^{\pi^{t+1}}} \mathbb{E}_{a \sim \pi^{t+1}(\cdot|s)}} A^{\pi^t}(s, a)$$

An Application of PDL in Policy Iteration

Recall that $\pi^{t+1}(s) = \arg\max_a A^{\pi^t}(s, a)$

Show monotonic improvement using PDL:

$$V^{\pi^{t+1}}(s_0) - V^{\pi^t}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_{s_0}^{\pi^{t+1}}}\mathbb{E}_{a\sim\pi^{t+1}(\cdot|s)}A^{\pi^t}(s, a)$$

$$= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_{s_0}^{\pi^{t+1}}}A^{\pi^t}(s, \pi^{t+1}(s))$$

An Application of PDL in Policy Iteration

Recall that $\pi^{t+1}(s) = \arg\max_a A^{\pi^t}(s,a)$

Show monotonic improvement using PDL:

$$V^{\pi^{t+1}}(s_0) - V^{\pi^t}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_{s_0}^{\pi^{t+1}}}\mathbb{E}_{a\sim\pi^{t+1}(\cdot|s)}A^{\pi^t}(s,a)$$

$$= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_{s_0}^{\pi^{t+1}}}A^{\pi^t}(s,\pi^{t+1}(s))$$

$\leftarrow \arg\max_a A^{\pi^t}(s,a)$

$$\geq \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_{s_0}^{\pi^{t+1}}}A^{\pi^t}(s,\pi^t(s)) = 0$$

An Application of PDL in Policy Iteration

Recall that $\pi^{t+1}(s) = \arg\max_a A^{\pi^t}(s,a)$

← Data-Driven App to estimate A^{π^t},

Show monotonic improvement using PDL:

$$V^{\pi^{t+1}}(s_0) - V^{\pi^t}(s_0) \underset{\text{PDL}}{=} \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_{s_0}^{\pi^{t+1}}}\mathbb{E}_{a\sim\pi^{t+1}(\cdot|s)}A^{\pi^t}(s,a)$$

$$= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_{s_0}^{\pi^{t+1}}}A^{\pi^t}(s,\pi^{t+1}(s))$$

$$\geq \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_{s_0}^{\pi^{t+1}}}A^{\pi^t}(s,\pi^t(s)) \;\; = 0$$

$\| d_{\mu_0}^{\pi^{t+1}} - d_{\mu_0}^{\pi^t} \|_1 \leq \delta$

Train A^{π^t} under $d_{\mu_0}^{\pi^t}$

if $\hat{A}^t$ is good under $d_{\mu_0}^{\pi^t}$

then, $\hat{A}^t$ is good under $d_{\mu_0}^{\pi^{t+1}}$

Summary for the recent 3 lectures:

Three fundamental ingredients in RL and MDPs:

Two Algorithms:

Summary for the recent 3 lectures:

Three fundamental ingredients in RL and MDPs:

1. Simulation Lemma (concerns the perf difference of π under two MDPs)

Two Algorithms:

Summary for the recent 3 lectures:

Three fundamental ingredients in RL and MDPs:

Model-Based RL

1. Simulation Lemma (concerns the perf difference of π under two MDPs)
2. PDL (co~~ncert~~ the perf diff of π & π' under one MDP)

policy improvement

Two Algorithms:

Summary for the recent 3 lectures:

Three fundamental ingredients in RL and MDPs:

1. Simulation Lemma (concerns the perf difference of π under two MDPs)
2. PDL (concert the perf diff of π & π' under one MDP)
3. How to draw samples from $d^\pi_{\mu_0}$, and how to get unbiased estimate of $Q^\pi(s, a)$

Roll-in

Roll-out

Two Algorithms:

Summary for the recent 3 lectures:

Three fundamental ingredients in RL and MDPs:

1. Simulation Lemma (concerns the perf difference of π under two MDPs)
2. PDL (concert the perf diff of π & π' under one MDP)
3. How to draw samples from $d^{\pi}_{\mu_0}$, and how to get unbiased estimate of $Q^{\pi}(s, a)$

Reset(s,a), observe $s' \sim P(\cdot|s,a)$

Two Algorithms:

1. Model-based RL w/ Generative model: fit $\widehat{P}$ (by counting) and run Policy-Iter on $(\widehat{P}, r)$

LQR: $D = \{x^i, u^i, (x^i)'\}$ $\quad \hat{A}, \hat{B} = \underset{A,B}{\arg\min} \sum_{i=1}^{N} \| A x^i + B u^i - (x^i)' \|_2^2 + \lambda (\|A\|_F^2 + \|B\|_F^2)$

Summary for the recent 3 lectures:

Three fundamental ingredients in RL and MDPs:

1. Simulation Lemma (concerns the perf difference of π under two MDPs)
2. PDL (concert the perf diff of π & π' under one MDP)
3. How to draw samples from $d^{\pi}_{\mu_0}$, and how to get unbiased estimate of $Q^{\pi}(s, a)$

Two Algorithms:

1. Model-based RL w/ Generative model: fit $\widehat{P}$ (by counting) and run Policy-Iter on $(\widehat{P}, r)$
2. Approximate Policy Iteration (Alg that uses a Regression oracle)

$\hat{Q}^t \approx Q^{\pi^t}$

Next Week:

We will talk about Incremental Policy Optimization
(Recall the failure case of API; we will force incremental update on policies)

$\pi^{t+1} \approx \pi^{t}$

$$\left| \sum_{x} P(x) f(x) - \sum_{x} Q(x) f(x) \right|$$