Approaches for Nonlinear Control

Recap: The Linear Quadratic Regulator (LQR)

$Q \in \mathbb{R}^{d \times d}$

Given by Designer

$$\min_{\pi_0,\ldots,\pi_{H-1}} E\left[x_H^\top Q x_H + \sum_{h=0}^{H-1} (x_h^\top Q x_h + u_h^\top R u_h) \right]$$

such that $\underline{x_{h+1} = Ax_h + Bu_h + w_h,}\ u_h = \pi_h(x_h) \quad x_0 \sim \mu_0,\ w_h \sim N(0, \sigma^2 I)$,

Markovian

$x_{h+1} \sim N(Ax_h + Bu_h, \sigma^2 I) \Leftrightarrow P(\cdot \mid x_h, u_h)$

Here, $x_h \in \mathbb{R}^d$, $u_h \in \mathbb{R}^k$,

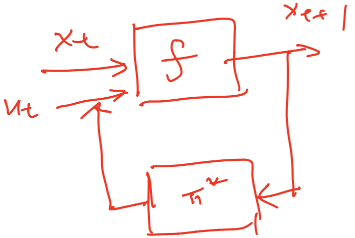
the disturbance $w_t \in \mathbb{R}^d$ is multi-variate normal, with covariance $\sigma^2 I$;

$A \in \mathbb{R}^{d \times d}$ and $B \in \mathbb{R}^{d \times k}$ are referred to as system (or transition) matrices;

$Q \in \mathbb{R}^{d \times d}$ and $R \in \mathbb{R}^{k \times k}$ are psd matrices that parameterize the quadratic costs.

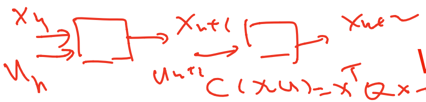
Recap: Optimal Control on LQR:

closed-loop



open-loop

$u_0, \ldots u_h, \ldots u_{H-1}$



$c(x,u) = x^\top Q x + u^\top R u$

$$V^\star_H(x) = x^\top Q x, \text{ define } P_H = Q, p_H = 0,$$

We have shown that $V^\star_h(x) = x^\top P_h x + p_h$, where:

$$P_h = Q + A^\top P_{h+1} A - A^\top P_{h+1} B (R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A,$$

$$p_h = \text{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

$V^\star_h(x) = x^\top P_h x + p_h$

Along the way, we also have shown that $\pi^\star_h(x) = -K^\star_h x$ where:

$$\pi^\star_h(x) = -\underbrace{(R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A}_{:= K^\star_h} x$$

Optimal control has nothing to do with initial distribution, and the noise!

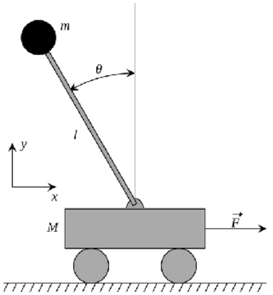
Today's Question:
What about nonlinear and non-quadratic control?

Outline for today:

1. Local Linearization Approach
(We will implement in HW1 for CartPole simulation)

2. Iterative LQR

Setting for Local Linearization Approach:



$x^\star = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$u^\star = \begin{bmatrix} 0 \end{bmatrix}$

Assumptions:

Goal: stabilizing around the goal $(x = x^\star, u = u^\star)$

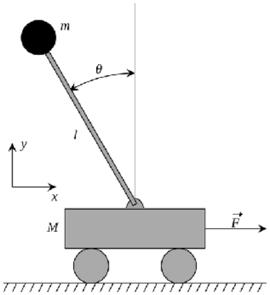
$$c(x_h, u_h) = d(u, u^\star) + d(x_h, x^\star)$$

minimize $\mathbb{E}_\pi \left[\sum_{h=0}^{H-1} c(x_h, u_h) \right]$

such that $x_{h+1} = f(x_h, u_h), u_h = \pi(x_h), x_0 \sim \mu_0$

Δ

Setting for Local Linearization Approach:



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Assumptions:

1. We have black-box access to f & c:

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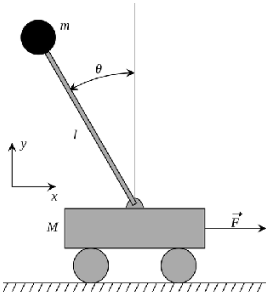

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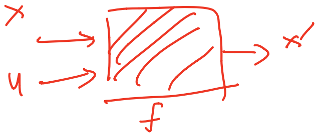
Assumptions:

1. We have black-box access to f & c:

i.e., unknown analytical form,
but can reset to any (x, u),
the black boxes outputs x', c,
where $x' = f(x, u), c = c(x, u)$



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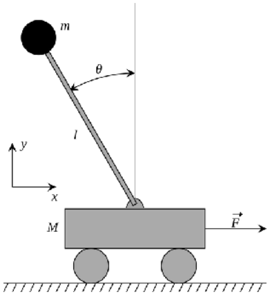

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i.e., unknown analytical form,
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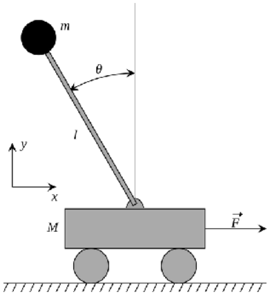
2. f is differentiable and c is double differentiable

Setting for Local Linearization Approach:



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w_h $\quad w_h \sim N(0, \sigma^2 I)$

Assumptions:

1. We have black-box access to f & c:

i.e., unknown analytical form,
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the black boxes outputs x', c,
where $x' = f(x, u), c = c(x, u)$

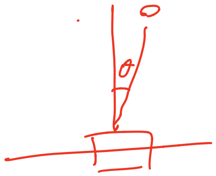
2. f is differentiable and c is double differentiable

$$\nabla_x f(x, u), \nabla_u f(x, u), \nabla_x c(x, u), \nabla_u c(x, u),$$
$$\nabla_x^2 c(x, u), \nabla_u^2 c(x, u), \nabla_{x,u}^2 c(x, u)$$

Local Linearization Approach

Assume that all possible initial states x_0 are close to $(x^\star, u^\star)$

$x_0 \sim N(x^\star, \sigma^2 I)$



Local Linearization Approach

Assume that all possible initial states x_0 are close to $(x^\star, u^\star)$

We can approximate $f(x, u)$ locally with First-order Taylor Expansion:

$$f(x,u) \approx f(x^\star, u^\star) + \nabla_x f(x^\star, u^\star)\left(x - x^\star\right) + \nabla_u f(x^\star, u^\star)(u - u^\star)$$

Local Linearization Approach

$x, u \to f \to x' \in \mathbb{R}^d$

Assume that all possible initial states x_0 are close to $(x^\star, u^\star)$

We can approximate $f(x, u)$ locally with First-order Taylor Expansion:

$$f(x,u) \approx f(x^\star, u^\star) + \boxed{\nabla_x f(x^\star, u^\star)}\left(x - x^\star\right) + \nabla_u f(x^\star, u^\star)(u - u^\star)$$

$x \in \mathbb{R}^d$

where:

$$\nabla_x f(x,u) \in \mathbb{R}^{d\times d}, \nabla_x f(x,u)[i,j] = \frac{\partial f[i]}{\partial x[j]}(x,u), \quad \nabla_u f(x,u) \in \mathbb{R}^{d\times k}, \nabla_u f(x,u)[i,j] = \frac{\partial f[i]}{\partial u[j]}(x,u)$$

$f: \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d$

$x \in \mathbb{R}^d$

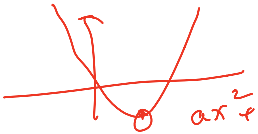
$u \in \mathbb{R}^k$

$u[i]$ = i-th entry of vector u.

Local Linearization Approach

We can approximate $c(x, u)$ locally at $(x^\star, u^\star)$ with second-order Taylor Expansion:

Local Linearization Approach



ax^2+bx+c

$a>0$

We can approximate $c(x,u)$ locally at $(x^\star, u^\star)$ with second-order Taylor Expansion:

0-th

← First-order

$$
\begin{aligned}
c(x,u) \approx\ & c(x^\star, u^\star) + \nabla_x c(x^\star, u^\star)^\top (x - x^\star) + \nabla_u c(x^\star, u^\star)^\top (u - u^\star) \\
& + \frac{1}{2}(x - x^\star)^\top \nabla_x^2 c(x^\star, u^\star)(x - x^\star) + \frac{1}{2}(u - u^\star)^\top \nabla_u^2 c(x^\star, u^\star)(u - u^\star) + (u - u^\star)^\top \nabla_{u,x}^2 c(x,u)(x - x^\star)
\end{aligned}
$$

Local Linearization Approach

$g: \mathbb{R}^2 \to \mathbb{R}$

$\nabla g(a,b) = \begin{bmatrix} g''_a(a,b) \\ g''_b(a,b) \end{bmatrix}$

We can approximate $c(x,u)$ locally at $(x^\star, u^\star)$ with second-order Taylor Expansion:

$$c(x,u) \approx c(x^\star,u^\star) + \nabla_x c(x^\star,u^\star)^\top (x - x^\star) + \nabla_u c(x^\star,u^\star)^\top (u - u^\star)$$
$$+\frac{1}{2}(x-x^\star)^\top \nabla_x^2 c(x^\star,u^\star)(x-x^\star) + \frac{1}{2}(u-u^\star)^\top \nabla_u^2 c(x^\star,u^\star)(u-u^\star) + \underline{(u-u^\star)^\top \nabla_{u,x}^2 c(x,u)(x-x^\star)}$$

$c: \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$

$\underline{\nabla_x c(x,u) \in \mathbb{R}^d}, \quad \nabla_x c(x,u)[i] = \dfrac{\partial c}{\partial x[i]}(x,u),$ (similarly $\nabla_u c(x,u)$)

$\underline{\nabla_x^2 c(x,u) \in \mathbb{R}^{d\times d}}, \quad \underline{\nabla_x^2 c(x,u)[i,j]} = \dfrac{\partial^2 c}{\partial x[i]\partial x[j]}(x,u),$ (similarly $\nabla_u^2 c(x,u)$ $\in \mathbb{R}^{k\times k}$)

$\boxed{\nabla_{u,x}^2 c(x,u) \in \mathbb{R}^{k\times x}}, \quad \nabla_{u,x}^2 c(x,u)[i,j] = \dfrac{\partial^2 c}{\partial u[i]\partial x[j]}(x,u)$

$\nabla^2 c(x,u) = \begin{bmatrix} \nabla_x^2 c(x,u), & \nabla_{x,u} c(x,u) \\ \nabla_{u,x} c(x,u), & \nabla_u^2 c(x,u) \end{bmatrix}$

Define $y = \begin{bmatrix} x \\ u \end{bmatrix}$

$c(y)$

$= \nabla_y^2 c(y)$

Local Linearization Approach

$$
\begin{aligned}
c(x,u) \approx\ & c(x^\star,u^\star) + \nabla_x c(x^\star,u^\star)^\top (x-x^\star) + \nabla_u c(x^\star,u^\star)^\top (u-u^\star) \\
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\end{aligned}
$$

$$
f(x,u) \approx f(x^\star,u^\star) + \nabla_x f(x^\star,u^\star)\left(x-x^\star\right) + \nabla_u f(x^\star,u^\star)(u-u^\star)
$$

Local Linearization Approach

$$c(x,u) \approx c(x^\star,u^\star) + \nabla_x c(x^\star,u^\star)^\top (x - x^\star) + \nabla_u c(x^\star,u^\star)^\top (u - u^\star)$$
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$$f(x,u) \approx f(x^\star,u^\star) + \underbrace{\nabla_x f(x^\star,u^\star)}_{:=A}\left(x-x^\star\right) + \underbrace{\nabla_u f(x^\star,u^\star)}_{:=B}(u-u^\star)$$

$Q := \frac{1}{2}\nabla_x^2 c(x^\star,u^\star)$

Re-arrange terms, we get back to the following formulation:

$R := \frac{1}{2}\nabla_u^2 c(x^\star,u^\star)$

$$\min_{\pi_0,\ldots,\pi_{H-1}} \quad \mathbb{E}\left[\sum_{h=0}^{H-1}\left(x_h^\top Q x_h + u_h^\top R u_h + u_h^\top M x_h + x_h^\top \mathbf{q} + u_h^\top \mathbf{r} + c\right)\right]$$

such that $x_{h+1} = Ax_h + Bu_h + \mathbf{v}, \; u_h = \pi_h(x_h) \quad x_0 \sim \mu_0,$

(HW1 problem)

Summary So far:

For tasks such as balancing on goal state $(x^\star, u^\star)$:
we can perform **first order Taylor expansion on** $f(x, u)$,
and **second order Taylor expansion on** $c(x, u)$ around the balancing point $(x^\star, u^\star)$

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Summary So far:

offline computation

For tasks such as balancing on goal state $(x^\star, u^\star)$:
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such that $x_{h+1} = Ax_h + Bu_h + \mathbf{v},\ u_h = \pi_h(x_h) \quad x_0 \sim \mu_0$

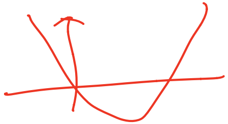
Last step: compute the optimal policy of the above problem, and test on the real system!

$\pi^\star : x \mapsto u$

$\sqrt{x}$ 

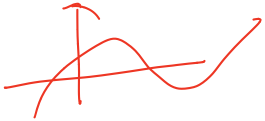
$\|x - x^*\|_2$

Some practical concerns in Local Linearization Approach



Note that $c(x, u)$ might not even be convex;

So, $\nabla_x^2 c(x, u)$ & $\nabla_u^2 c(x, u)$ may not be positive definite



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In practice, we force them to be Positive definite:

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Note that $c(x, u)$ might not even be convex;

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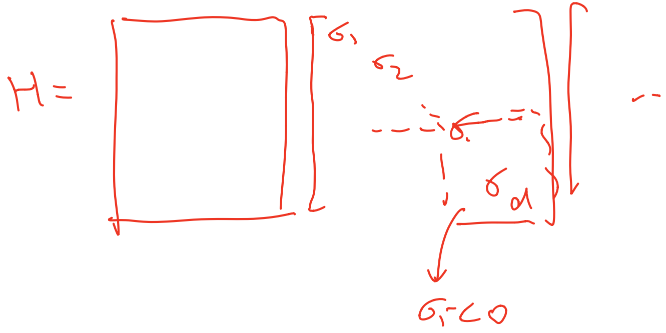
In practice, we force them to be Positive definite:

Given a symmetric matrix $H \in \mathbb{R}^{d \times d}$*:*



we compute the eigen-decomposition $H = \sum_{i=1}^{d} \sigma_i u_i u_i^\top$,

and we approximate it as $H \approx \sum_{i=1}^{d} \mathbf{1}(\sigma_i > 0) \sigma_i u_i u_i^\top + \lambda I$,

where $\lambda \in \mathbb{R}^+$

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Recall our assumption: we only have black-box access to f & c:

i.e., unknown analytical form, but given any (x, u), the black boxes outputs x', c, where

$$x' = f(x, u), c = c(x, u)$$

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Compute gradient using Finite differencing:

$g : \mathbb{R} \to \mathbb{R}$

$$g'(v_0) = \lim_{\delta \to 0} \frac{g(v_0+\delta) - g(v_0 - \delta)}{2\delta}$$

$$\approx \frac{g(v_0+\delta) - g(v_0 - \delta)}{2\delta}$$

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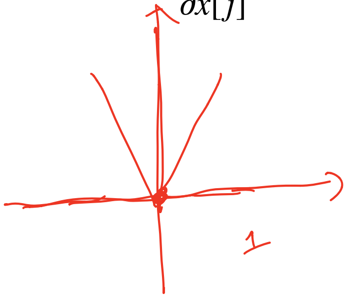
$\nabla_x f(x,u)[i,j] = \frac{\partial f[i]}{\partial x[j]}(x,u)$

Compute gradient using Finite differencing:

$$\frac{\partial f[i]}{\partial x[j]}(x, u) \approx \frac{f(x + \delta_j, u)[i] - f(x - \delta_j, u)[i]}{2\delta}, \text{ where } \delta_j = [0,\ldots,0, \underbrace{\delta}_{j'th \text{ entry}} ,0,\ldots0]^\top$$

→ 2 queries to the ~~box~~ black-box f



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To compute second derivative, i.e., $\dfrac{\partial^2 c}{\partial u[i] \partial x[j]}(x, u)$

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To compute second derivative, i.e., $\frac{\partial^2 c}{\partial u[i] \partial x[j]}(x, u)$

First implement FD procedure for $\partial c / \partial u[i]$,

and then perform another FD wrt $x[j]$ on top of the FD procedure for $\partial c / \partial u[i]$

Summary for local linearization approach

1. we perform first order Taylor expansion on $f(x, u)$,
and second order Taylor expansion on $c(x, u)$ around the balancing point $(x^\star, u^\star)$

2. We force Hessians $\nabla_x^2 c(x, u)$ & $\nabla_u^2 c(x, u)$ to be Positive Definite

3. Leverage Finite difference to approximate Gradients and Hessians

4. The approximation is an LQR from which we compute the optimal policy

Outline for today:

1. Local Linearization Approach
(We will implement it in HW1 for CartPole simulation)

2. Iterative LQR

Iterative LQR

Local Linearization approach could work if x_0 is very close to $(x^\star)$

But when x_0 is far away from $x^\star$, first/second-order Taylor expansion is not accurate anymore

Iterative LQR

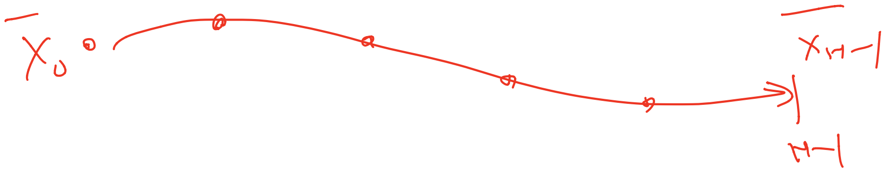
Recall $x_0 \sim \mu_0$; denote $\mathbb{E}_{x_0 \sim \mu_0}[x_0] = \bar{x}_0$

$\mu_0 \in N(\bar{x}_0, \Sigma)$

Initialize $\bar{u}_0^0, \ldots, \bar{u}_{H-1}^0$,

Generate nominal trajectory: $\bar{x}_0, \bar{u}_h, \bar{x}_{h+1} = f(\bar{x}_h, \bar{u}_h) \ldots, \bar{x}_{H-1}, \bar{u}_{H-1}$

$\bar{x}_0 \xrightarrow{\bar{u}_0^0} \bar{x}_1 = f(\bar{x}_0, \bar{u}_0) \cdots \rightarrow \bar{x}_{H-1}, \bar{u}_{H-1}$



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For $t = 0 \ldots,$

Iterative LQR

Recall $x_0 \sim \mu_0$; denote $\mathbb{E}_{x_0 \sim \mu_0}[x_0] = \bar{x}_0$

Initialize $u_0^0, \ldots, u_{H-1}^0$,

Generate nominal trajectory: $\bar{x}_0, \bar{u}_h, \bar{x}_{h+1} = f(\bar{x}_h, \bar{u}_h) \ldots, \bar{x}_{H-1}, \bar{u}_{H-1}$

For $t = 0 \ldots$,

Linearize $f(x, u)$ at $(\bar{x}_h, \bar{u}_h), \forall h$: $f(x, u) \approx f(\bar{x}_h, \bar{u}_h) + \nabla_x f(\bar{x}_h, \bar{u}_h)(x - \bar{x}_h) + \nabla_u f(\bar{x}_h, \bar{u}_h)(u - \bar{x}_h)$

① first-order-Taylor Exp of $f(x,u)$ at $(\bar{x}_h, \bar{u}_h)$

② second ... Expan.. of $c(x,u)$ at $(\bar{x}_h, \bar{u}_h)$

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Set new nominal trajectory: $\underline{\bar{u}_0} = \pi_0(\bar{x}_0), \underline{\bar{u}_h} = \pi_h(\bar{x}_h)$, where $\bar{x}_{h+1} = f(\bar{x}_h, \bar{u}_h)$

Iterative LQR

Consider iteration t, we have computed $\pi_h^t, \forall h$:

$\bar{x}_0 \qquad \bar{u}_0 = \pi_0^t(\bar{x}_0)$

$h:$

$H-1$

$\bar{x}_{H-1}$

$f(x,u) = A_h x + B_h u + v_h$

$A_h = \nabla_x f(\bar{x}_h, \bar{u}_h)$ ✓ , $A_{h+1} = \nabla_x f(\bar{x}_{h+1}, \bar{u}_{h+1})$

$B_h = \nabla_u f(\bar{x}_h, \bar{u}_h)$ ✓

Iterative LQR

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After linearization and qudartization around H waypoints $(\bar{x}_h, \bar{u}_h), \forall h$, re-arrange terms, we get:

$$\min_{\pi_0,\ldots,\pi_{H-1}} \quad E\left[\sum_{h=0}^{H-1} \left(x_h^\top Q_h x_h + u_h^\top R_h u_h + u_h^\top M_h x_h + x_h^\top \mathbf{q}_h + u_h^\top \mathbf{r}_h + c_h\right)\right]$$

such that $\quad x_{h+1} = A_h x_h + B_h u_h + v_h \quad u_h = \pi_h(x_h) \quad x_0 \sim \mu_0;$

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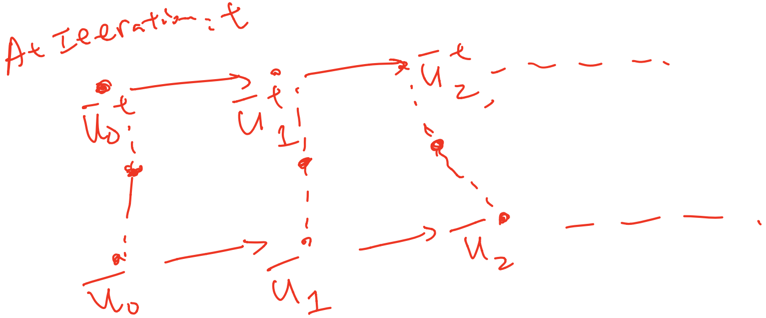

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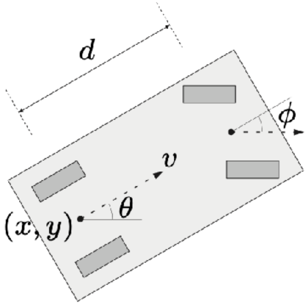
$$\min_{\alpha \in [0,1]} \sum_{h=0}^{H-1} c(x_h, \bar{u}_h^{t+1}),$$

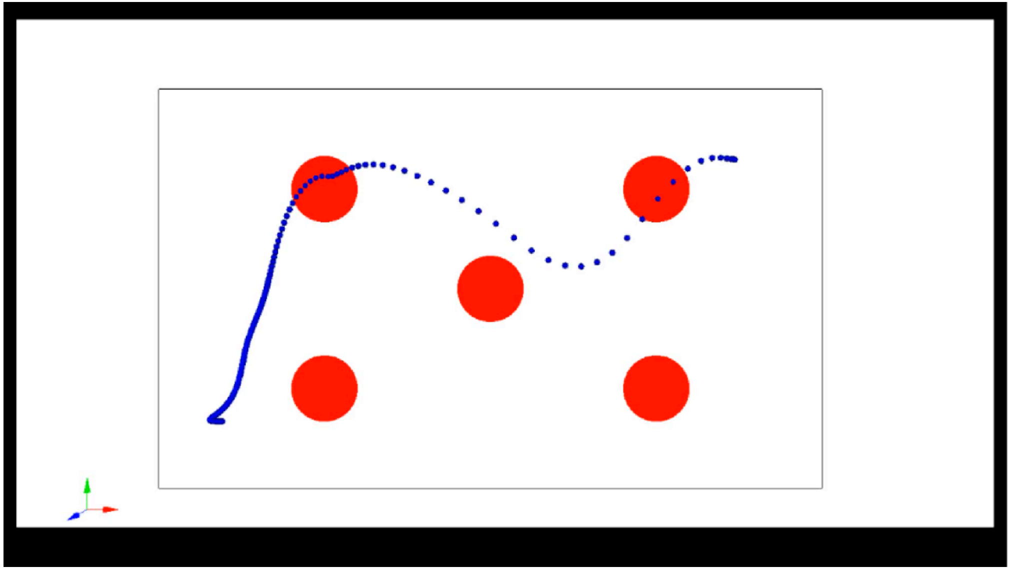
s.t., $x_{h+1} = f(x_h, \bar{u}_h^{t+1}), \bar{u}_h^{t+1} = \alpha \bar{u}_h^t + (1-\alpha)\bar{u}_h, x_0 = \bar{x}_0$

Example:

2-d car navigation

Cost function is designed such that it gets to the goal without colliding w/ obstacles (red)





Summary:

Local Linearization:

Approximate an LQR at the balance (goal) position $(x^\star, u^\star)$;
and then solve the approximated LQR;

Iterative LQR

Iterate between (1) forming an LQR around the current nominal trajectory,
(2)compute a new nominal trajectory using the optimal policy of the LQR;

Starting from next week:

We will move on to data-driven approach for computing approximately optimal policy

1. Model-based RL: certainty equivalence

2. Model-free RL: Fitted Value Iteration

$a \in \{0, 1\}$

$\uparrow$

engine fails

$P(a_n \mid x_n, u_n, a_{n-1})$

$x_{n+1} = f(x_n, u_n, a_n)$

$a_{n+1} \sim P(\cdot \mid x_n, u_n, a_{n-1})$

$g: \mathbb{R} \to \mathbb{R}$

$g''(x)$

$$g'(x) = \frac{g(x+\delta) - g(x-\delta)}{2\delta}$$

$$FD(g, x) = \frac{g(x+\delta) - g(x-\delta)}{2\delta}$$

$$g''(x) = \frac{FD(g, x+\delta) - FD(g, x-\delta)}{2\delta}$$

$FD2(g, x)$

$$\approx \frac{g'(x+\delta) - g'(x-\delta)}{2\delta}$$

$f: \mathbb{R}^2 \to \mathbb{R}$

$f(a,b)$

$$\nabla_{[a,b]} f(a,b) = \begin{bmatrix} f'_a(a,b) \\ f'_b(a,b) \end{bmatrix}$$

$$\nabla f(x)[i] = \frac{\partial f}{\partial x[i]}(x)$$

$i \in \{1,2,3\}$

$j \in \{1,2\}$

$f: \mathbb{R}^3 \to \mathbb{R}^2$

$\boxed{\forall\, i, j}$

$$\frac{\partial f[j]}{\partial x[i]}$$