Optimal Control for
Linear Quadratic Regulators

Recap: Dynamic Programming

$$\pi^\star = \{\underbrace{\pi_0^\star, \pi_1^\star, \ldots, \pi_{H-1}^\star}\}$$

We use Dynamic Programming, and do DP backward in time; start at $H-1$

$$Q_{H-1}^\star(s,a) = r(s,a) \qquad \pi_{H-1}^\star(s) = \arg\max_a Q_{H-1}^\star(s,a)$$

$$V_{H-1}^\star(s) = \max_a Q_{H-1}^\star(s,a) = Q_{H-1}^\star(s, \pi_{H-1}^\star(s))$$

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Now assume that we have already computed $V_{h+1}^\star$, $h \leq H-2$

(i.e., we know how to perform optimally at $h+1$)

$$Q_h^\star(s,a) = r(s,a) + \mathbb{E}_{s' \sim P(\cdot|s,a)} V_{h+1}^\star(s')$$

$$\pi_h^\star(s) = \arg\max_a Q_h^\star(s,a)$$

Recap: The Linear Quadratic Regulator (LQR)

← Finite Horizon MDP

$$\min_{\pi_0,\ldots,\pi_{H-1}} \quad E\left[x_H^\top Q x_H + \sum_{h=0}^{H-1} (x_h^\top Q x_h + u_h^\top R u_h) \right]$$

such that $\underbrace{x_{h+1} = Ax_h + Bu_h + w_h}_{\Delta}, \; u_h = \pi_h(x_h) \quad x_0 \sim \mu_0, \; w_h \sim N(0, \sigma^2 I)\,,$

Here, $x_h \in \mathbb{R}^d$, $u_h \in \mathbb{R}^k$,

the disturbance $w_t \in \mathbb{R}^d$ is Gaussian noise

$A \in \mathbb{R}^{d\times d}$ and $B \in \mathbb{R}^{d\times k}$;

$Q \in \mathbb{R}^{d\times d}$ and $R \in \mathbb{R}^{k\times k}$ are pd matrices

V/Q functions:

- Value function $V_h^\pi : \mathbb{R}^d \to \mathbb{R}$ as

$$V_h^\pi(x) = \mathbb{E}\left[x_H^\top Q x_H + \sum_{t=h}^{H-1} (x_t^\top Q x_t + u_t^\top R u_t) \,\middle|\, \pi, x_h = x\right],$$

- And $Q_h^\pi : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ as:

$$Q_h^\pi(x, u) = \mathbb{E}\left[x_H^\top Q x_H + \sum_{t=h}^{H-1} (x_t^\top Q x_t + u_t^\top R u_t) \,\middle|\, \pi, x_h = x, u_h = u\right],$$

Optimal Value functions:

$$V_h^\star(x) = \min_{\pi_h, \pi_{h+1}, \ldots, \pi_{H-1}} \mathbb{E}\left[x_H^T Q x_H + \sum_{t=h}^{H-1} x_t^\top Q x_t + u_t^\top R u_t \,\middle|\, u_t = \pi_t(x_t), x_h = x \right]$$

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$P \succ 0 \in \mathbb{R}^{d \times d}$ $\quad$ $\mathbb{R}$

Theorem:

$V_h^\star$ is a quadratic function, i.e., $V_h^\star(x) = x^\top P_h x + p_h$,

and optimal policy is linear:

$\pi_h^\star(x) = -K_h^\star x,$ $\quad$ $K_h^\star \in \mathbb{R}^{k \times d}$

and P_h & $K_h^\star$ can be computed exactly

Optimal Value functions:

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Theorem:
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Today: we prove the above theorem and derive optimal policies

Key Steps to Deriving Optimal Control

Again, we will do dynamic programming **backward in time**, i.e., from H to 0

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Show that $V_H^\star(x)$ is quadratic for all $x \in \mathbb{R}^d$

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1. **Base case:**
Show that $V_H^\star(x)$ is quadratic for all $x \in \mathbb{R}^d$

2. **Inductive hypothesis:** Assume $V_{h+1}^\star(x)$ is quadratic $\forall x$:

- show that $Q_h^\star(x, u)$ is quadratic in both (x, u)
- Derive the optimal policy $\pi_h^\star(x) = \arg\min_u Q_h^\star(x, u)$, and show that it's linear

Key Steps to Deriving Optimal Control

Again, we will do dynamic programming **backward in time**, i.e., from H to 0

1. **Base case:**
Show that $V^\star_H(x)$ is quadratic for all $x \in \mathbb{R}^d$

2. **Inductive hypothesis:** Assume $V^\star_{h+1}(x)$ is quadratic $\forall x$:

- show that $Q^\star_h(x, u)$ is quadratic in both (x, u)
- Derive the optimal policy $\pi^\star_h(x) = \arg\min_u Q^\star_h(x, u)$, and show that it's linear

3. **Conclusion:**
show $V^\star_h(x)$ is quadratic for all x;

Base case at H

Recall our cost functions:

$$\min_{\pi_0,\ldots,\pi_{H-1}} \mathbb{E}\left[x_H^\top Q x_H + \sum_{h=0}^{H-1} (x_h^\top Q x_h + u_h^\top R u_h)\right]$$

Base case at H

$Q, R \leftarrow$ PD matrices

Recall our cost functions:

$$\min_{\pi_0,\ldots,\pi_{H-1}} \mathbb{E}\left[x_H^\top Q x_H + \sum_{h=0}^{H-1} (x_h^\top Q x_h + u_h^\top R u_h)\right]$$

So, at time step H, given x, the cost-to-go is $x^\top Q x$ regardless..

$$V_H^\star(x) = x^\top Q x, \ \forall x \in \mathbb{R}^d$$

Base case at H

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$$\min_{\pi_0,\ldots,\pi_{H-1}} \mathbb{E}\left[x_H^\top Q x_H + \sum_{h=0}^{H-1}(x_h^\top Q x_h + u_h^\top R u_h)\right]$$

So, at time step H, given x, the cost-to-go is $x^\top Q x$ regardless..

$$V_H^\star(x) = x^\top Q x,\ \forall x \in \mathbb{R}^d \rightarrow := x^\top P_H x + p_H$$

Denote $P_H := Q, p_H = 0,$

we write $V_H^\star(x) = x^\top P_H x + p_H$

(Goal: derive recursive formulation of P_h, & p_h)

Induction Step:

inductive hypothesis

$$\min_{\pi_0,\ldots,\pi_{H-1}} \quad E\left[x_H^\top Q x_H + \sum_{h=0}^{H-1}(x_h^\top Q x_h + u_h^\top R u_h)\right]$$

such that $\quad x_{h+1} = Ax_h + Bu_h + w_h, \; u_h = \pi_h(x_h) \quad x_0 \sim \mu_0, \; w_h \sim N(0,\sigma^2 I)$,

Assume $V^\star_{h+1}(x) = x^\top P_{h+1} x + p_{h+1}$, for all x, where $P_{h+1} \in \mathbb{R}^{d\times d}, p_{h+1} \in \mathbb{R}$

prove

$\hookrightarrow \quad V^\star_h(x) := x^\top P_h x + p_h$

Induction Step:

$$\min_{\pi_0,\ldots,\pi_{H-1}} \quad E\left[x_H^\top Q x_H + \sum_{h=0}^{H-1} (x_h^\top Q x_h + u_h^\top R u_h) \right]$$

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$$Q^\star_h(x, u) = c(x, u) + \mathbb{E}_{x'\sim P(x,u)} V^\star_{h+1}(x')$$

Induction Step:

$x \sim N(0, \sigma^2)$

$c + x \sim N(c, \sigma^2)$

$c \cdot x \sim N(0, c^2\sigma^2)$

$$\min_{\pi_0,\ldots,\pi_{H-1}} E\left[x_H^\top Q x_H + \sum_{h=0}^{H-1} (x_h^\top Q x_h + u_h^\top R u_h)\right]$$

such that $x_{h+1} = Ax_h + Bu_h + w_h,\ u_h = \pi_h(x_h) \quad x_0 \sim \mu_0,\ w_h \sim N(0,\sigma^2 I)$,

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$$Q^\star_h(x,u) = c(x,u) + \mathbb{E}_{x'\sim P(x,u)} V^\star_{h+1}(x')$$

$$= \underbrace{x^\top Q x + u^\top R u}_{= c(x,u)} + \mathbb{E}_{x'\sim P(x,u)} V^\star_{h+1}(x')$$

$x' = Ax + Bu + w, \quad w \sim N(0, \sigma^2 I)$

(1) $P(x,u) = N(Ax+Bu, \sigma^2 I)$

(2) $V^\star_{h+1}(x') = (x')^\top P_{h+1} x' + p_{h+1}$ $\leftarrow$ inductive hypothesis

Induction Step:

$$\min_{\pi_0,\ldots,\pi_{H-1}} E\left[x_H^\top Q x_H + \sum_{h=0}^{H-1}(x_h^\top Q x_h + u_h^\top R u_h)\right]$$

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$$= x^\top Q x + u^\top R u + \mathbb{E}_{x'\sim P(x,u)} V^\star_{h+1}(x')$$

$Ax+Bu+w,\ w\sim N(0,\sigma^2 I)$

$$= x^\top Q x + u^\top R u + \mathbb{E}_{w\sim\mathcal{N}(0,\sigma^2 I)}\left[V^\star_{h+1}(Ax+Bu+w)\right]$$

Induction Step:

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$$= x^\top Q x + u^\top R u + \mathbb{E}_{w\sim\mathcal{N}(0,\sigma^2 I)} \left[V^\star_{h+1}(Ax + Bu + w) \right]$$

$$= x^\top Q x + u^\top R u + \mathbb{E}_{w\sim\mathcal{N}(0,\sigma^2 I)} \left[\underbrace{(Ax+Bu+w)^\top P_{h+1}(Ax+Bu+w)}_{\Delta} + \underbrace{p_{h+1}}_{\Delta} \right]$$

$\mathbb{E}_{x\sim N(0,\sigma^2 I)} x^\top v = 0$

$(AB)^\top = B^\top A^\top$

$(ABC)^\top = C^\top B^\top A^\top = (BC)^\top A^\top = C^\top B^\top A^\top$

$V^\star_{h+1}(x) = x^\top P_{h+1} x + p_{h+1}$

inductive hypoth...

$(Ax + Bu + w)^\top P_{h+1} (Ax + Bu + w)$

$= x^\top A^\top P_{h+1} A x + u^\top B^\top P_{h+1} B u + w^\top P_{h+1} w + 2\underline{x^\top A^\top P_{h+1} B u} + 2 w^\top P_{h+1} A x + 2 w^\top P_{h+1} B u$

Induction Step:

$$\min_{\pi_0,\ldots,\pi_{H-1}} \quad E\left[x_H^\top Q x_H + \sum_{h=0}^{H-1}(x_h^\top Q x_h + u_h^\top R u_h)\right]$$

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$$Q_h^\star(x,u) = c(x,u) + \mathbb{E}_{x'\sim P(x,u)} V^\star_{h+1}(x')$$

$$= x^\top Q x + u^\top R u + \mathbb{E}_{x'\sim P(x,u)} V^\star_{h+1}(x')$$

$$= x^\top Q x + u^\top R u + \mathbb{E}_{w\sim\mathcal{N}(0,\sigma^2 I)}\left[V^\star_{h+1}(Ax + Bu + w)\right]$$

$$= x^\top Q x + u^\top R u + \mathbb{E}_{w\sim\mathcal{N}(0,\sigma^2 I)}\left[(Ax + Bu + w)^\top P_{h+1}(Ax + Bu + w) + p_{h+1}\right]$$

$$= x^\top\left(Q + A^\top P_{h+1} A\right)x + u^\top\left(R + B^\top P_{h+1} B\right)u + 2x^\top A^\top P_{h+1} B u + \mathbb{E}_{w\sim\mathcal{N}(0,\sigma^2 I)}\left[w^\top P_{h+1} w\right] + p_{h+1}$$

complete square:

$= x^\top A^\top P_{h+1} A x$

$+ u^\top B^\top P_{h+1} B u$

$+ 2x^\top A^\top P_{h+1} B u$

$+ 2w^\top P_{h+1}(Ax+Bu)$

$+ w^\top P_{h+1} w$

$= v^\top \left[\mathbb{E}_{w\sim N(0,\sigma^2 I)} w\right]$

$\mathbb{E}_{w\sim N(0,\sigma^2 I)}\left[w^\top v\right] = 0$

$\hookrightarrow \mathrm{Tr}(\sigma^2 I P_{h+1})$

Induction Step:

$$\min_{\pi_0,\ldots,\pi_{H-1}} E\left[x_H^\top Q x_H + \sum_{h=0}^{H-1}(x_h^\top Q x_h + u_h^\top R u_h)\right]$$

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$$= x^\top Q x + u^\top R u + \mathbb{E}_{w\sim\mathcal{N}(0,\sigma^2 I)}\left[V^\star_{h+1}(Ax+Bu+w)\right]$$

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$$= x^\top\left(Q + A^\top P_{h+1}A\right)x + u^\top\left(R + B^\top P_{h+1}B\right)u + 2x^\top A^\top P_{h+1}Bu + \mathbb{E}_{w\sim\mathcal{N}(0,\sigma^2 I)}\left[w^\top P_{h+1} w\right] + p_{h+1}$$

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$$\mathop{E}_{w\sim N(0,\sigma^2 I)} w^\top P_{h+1} w$$
$$= \mathop{E}_{w\sim N(0,\sigma^2 I)} \mathrm{Tr}\left(w^\top P_{h+1} w\right)$$
$$= \mathop{E}_{w\sim N(0,\sigma^2 I)} \mathrm{Tr}\left(ww^\top P_{h+1}\right)$$
$$= \mathrm{Tr}\left[\mathop{E}_{w\sim N(0,\sigma^2 I)} ww^\top P_{h+1}\right]$$
$$= \mathrm{Tr}\left(\sigma^2 I \cdot P_{h+1}\right)$$
$$= \mathrm{Tr}\left(\sigma^2 P_{h+1}\right)$$

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

Induction Step (continue)

$$Q_h^\star(x,u) = c(x,u) + \mathbb{E}_{x' \sim P(x,u)}\left[V_{h+1}^\star(x')\right]$$

$$= x^\top \left(Q + A^\top P_{h+1} A\right) x + u^\top \left(R + B^\top P_{h+1} B\right) u + 2x^\top A^\top P_{h+1} B u + \mathrm{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

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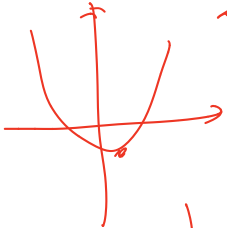
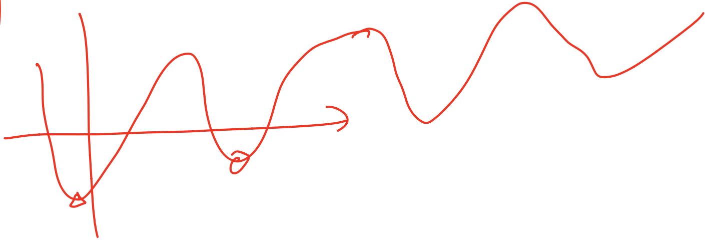
$$\pi_h^\star(x) = \arg\min_u Q_h^\star(x,u)$$



$\rightsquigarrow$ $\min ax^2 + bx + c$

where $a > 0$



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$$\pi_h^\star(x) = \arg\min_u Q_h^\star(x,u)$$

Set $\nabla_u Q_h^\star(x,u) = 0$, and solve for u:

Induction Step (continue)

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$$= \underbrace{x^\top\left(Q + A^\top P_{h+1} A\right)x}\, + \underbrace{u^\top\left(R + B^\top P_{h+1} B\right)u}\, + \underbrace{2x^\top A^\top P_{h+1} B u}\, + \mathrm{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

$2(R+B^\top P_{h+1}B)\cdot u + 2B^\top P_{h+1}Ax = 0$

$\Rightarrow (R+B^\top P_{h+1}B)u = -B^\top P_{h+1}Ax$

$$\pi_h^\star(x) = \arg\min_u Q_h^\star(x,u)$$

Set $\nabla_u Q_h^\star(x,u) = 0$, and solve for u:

$$\pi_h^\star(x) = -\underbrace{(R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A}_{:=K_h^\star}\, x$$

Induction Step (continue)

$$Q_h^\star(x,u) = c(x,u) + \mathbb{E}_{x'\sim P(x,u)}\left[V_{h+1}^\star(x')\right]$$

$$= x^\top \big(\underbrace{Q + A^\top P_{h+1} A}_{PD}\big)\, x + u^\top \big(\underbrace{R + B^\top P_{h+1} B}_{PD}\big)\, u + 2x^\top A^\top P_{h+1} B u + \mathrm{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

$$\pi_h^\star(x) = \arg\min_u Q_h^\star(x,u)$$

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$$\pi_h^\star(x) = -\underbrace{(R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A}_{:=K_h^\star}\, x$$

$$:= -K_h^\star x$$

$K_h^\star \in \mathbb{R}^{k\times d}$

Concluding the Induction step:

$$Q_h^\star(x,u) = x^\top \left(Q + A^\top P_{h+1} A\right) x + u^\top \left(R + B^\top P_{h+1} B\right) u + 2x^\top A^\top P_{h+1} B u + \mathrm{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

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$$\pi_h^\star(x) = -\underbrace{(R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A}_{:=K_h^\star}\, x$$

$$V_h^\star(x) = Q_h^\star(x, \pi_h^\star(x))$$

Concluding the Induction step:

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$-K_h^\star x$ $-K_h^\star \cdot x$ $-K_h^\star \cdot x$

$$\pi_h^\star(x) = -\underbrace{(R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A}_{:=K_h^\star}\, x$$

$$V_h^\star(x) = Q_h^\star(x, \pi_h^\star(x))$$

We can express $V_h^\star(x)$ as $V_h^\star(x) = x^\top P_h x + p_h,$ where:

Concluding the Induction step:

$$Q_h^\star(x,u) = x^\top\left(Q + A^\top P_{h+1} A\right)x + u^\top\left(R + B^\top P_{h+1} B\right)u + 2x^\top A^\top P_{h+1} B u + \operatorname{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

$$\pi_h^\star(x) = -\underbrace{(R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A}_{:=K_h^\star}\, x$$

$$-\,2x^\top A^\top P_{h+1} B K_h^\star x$$

$$= (-2)x^\top A^\top P_{h+1} B (R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A x$$

$$V_h^\star(x) = Q_h^\star(x, \pi_h^\star(x))$$

We can express $V_h^\star(x)$ as $V_h^\star(x) = x^\top P_h x + p_h$, where:

Recursive form for computing P_h

$$P_h = Q + A^\top P_{h+1} A - A^\top P_{h+1} B (R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A,$$

$$p_h = \operatorname{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

Concluding the Induction step:

$$Q_h^\star(x,u) = x^\top\left(Q + A^\top P_{h+1} A\right)x + u^\top\left(R + B^\top P_{h+1} B\right)u + 2x^\top A^\top P_{h+1} B u + \mathrm{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

$$\pi_h^\star(x) = -\underbrace{(R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A}_{:=K_h^\star}\, x$$

$$V_h^\star(x) = Q_h^\star(x, \pi_h^\star(x))$$

We can express $V_h^\star(x)$ as $V_h^\star(x) = x^\top P_h x + p_h$, where:

✓ $P_h = Q + A^\top P_{h+1} A - A^\top P_{h+1} B (R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A,$ **Ricatti Equation**

✓ $p_h = \mathrm{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$

Repeat: $-K_0^\star, -K_1^\star \ldots -K_{H-1}^\star$

Summary:

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Base

$$V_H^\star(x) = x^\top Q x, \text{ define } P_H = Q, p_H = 0,$$

Summary:

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We have shown that $V_h^\star(x) = x^\top P_h x + p_h$, where:

Riccati-Eqn.

$$P_h = Q + A^\top P_{h+1} A - A^\top P_{h+1} B (R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A,$$

$$p_h = \mathrm{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

Summary:

$P_H = Q \Leftarrow PD$

$$V^\star_H(x) = x^\top Q x, \text{ define } P_H = Q, p_H = 0,$$

We have shown that $V^\star_h(x) = x^\top P_h x + p_h$, where:

Assume P_{h+1} is PD

$$P_h = Q + A^\top P_{h+1} A - A^\top P_{h+1} B \underline{(R + B^\top P_{h+1} B)}^{-1} B^\top P_{h+1} A,$$

$$p_h = \operatorname{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

Along the way, we also have shown that $\pi^\star_h(x) = -K^\star_h x$ where:

$$\pi^\star_h(x) = -\underbrace{(R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A}_{:=K^\star_h}\, x$$

Start from $P_H = Q$, $p_H = 0$

$\forall h = H-1, \ldots 0$, $P_h, p_h \Leftarrow P_{h+1}, p_{h+1}$

Summary:

$w \sim N(0, \sigma^2 I)$

$$V_H^\star(x) = x^\top Q x, \text{ define } P_H = Q, p_H = 0,$$

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$$\pi_h^\star(x) = -\underbrace{(R + B^\top P_{h+1} B)^{-1} B^\top P_{h+1} A}_{:= K_h^\star} x$$

Optimal control has nothing to do with initial distribution, and the noise!

Some Basic Extension:

Time dependent costs and transitions:

$$\min_{\pi_0,\ldots,\pi_{H-1}} \quad E\left[x_H^\top Q_H x_H + \sum_{h=0}^{H-1} (x_h^\top Q_h x_h + u_h^\top R_h u_h)\right]$$

such that $\quad x_{h+1} = A_h x_h + B_h u_h + w_h, \;\; u_h = \pi_h(x_h) \quad x_0 \sim \mu_0, \; w_h \sim N(0,\sigma^2 I)\,,$

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Same derivation, we will have the following Ricatti Equation:

Ricatti-Eq n

$$P_h = Q_h + A_h^\top P_{h+1} A_h - A_h^\top P_{h+1} B_h (R_h + B_h^\top P_{h+1} B_h)^{-1} B_h^\top P_{h+1} A_h,$$

$$p_h = \mathrm{tr}\left(\sigma^2 P_{h+1}\right) + p_{h+1}$$

Some Basic Extension:

More generally…

$q_h \in \mathbb{R}^d$, $c_h \in \mathbb{R}$

$r_h \in \mathbb{R}^k$,

$$\min_{\pi_0,\ldots,\pi_{H-1}} E\left[x_H^\top Q_H x_H + x_H^\top q_H + c_H + \sum_{h=0}^{H-1} \left(x_h^\top Q_h x_h + u_h^\top R_h u_h + u_h^\top M_h x_h + x_h^\top q_h + u_h^\top r_h + c_h\right)\right]$$

such that $x_{h+1} = A_h x_h + B_h u_h + v_h + w_h,\ u_h = \pi_h(x_h) \quad x_0 \sim \mu_0,\ w_h \sim N(0,\sigma^2 I)$,

$v_h \in \mathbb{R}$

Some Basic Extension:

More generally…

$$\min_{\pi_0,\ldots,\pi_{H-1}} \quad E\left[x_H^\top Q_H x_H + x_H^\top q_H + c_H + \sum_{h=0}^{H-1}(x_h^\top Q_h x_h + u_h^\top R_h u_h + u_h^\top M_h x_h + x_h^\top q_h + u_h^\top r_h + c_h)\right]$$

such that $\quad x_{h+1} = A_h x_h + B_h u_h + v_h + w_h,\ \ u_h = \pi_h(x_h) \quad x_0 \sim \mu_0,\ w_h \sim N(0,\sigma^2 I)\,,$

$\pi_h^\star(x) = -K_h^\star x + \underline{k_h^\star}$

Same DP idea and similar derivation
(HW1 question)

Some Basic Extension:

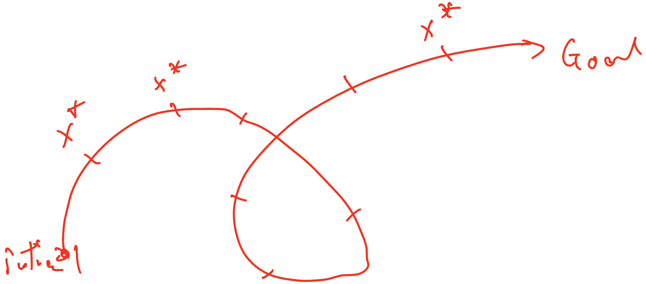
Tracking a pre-defined trajectory: $\leftarrow \left(x^\star_h, u^\star_h\right)_{h=0}^{H-1}$

$$\min_{\pi_0,\ldots,\pi_{H-1}} \; E\left[(x_H - x_H^\star)^\top Q_H (x_H - x_H^\star) + \sum_{h=0}^{H-1} (x_h - x_h^\star)^\top Q_h (x_h - x_h^\star) + (u_h - u_h^\star)^\top R_h (u_h - u_h^\star) \right]$$

such that $\quad x_{h+1} = A_h x_h + B_h u_h + w_h, \; u_h = \pi_h(x_h) \quad x_0 \sim \mu_0, \; w_h \sim N(0, \sigma^2 I)$,

$$\left(x_h - x_h^\star\right)^\top Q_h \left(x_h - x_h^\star\right)$$

$$= x_h^\top Q_h x_h - 2 x_h^\top Q_h x_h^\star + x_h^{\star\top} Q_h x_h^\star$$



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Tracking a pre-defined trajectory:

$$\min_{\pi_0,\ldots,\pi_{H-1}} \quad E\left[(x_H - x_H^\star)^\top Q_H (x_H - x_H^\star) + \sum_{h=0}^{H-1} (x_h - x_h^\star)^\top Q_h (x_h - x_h^\star) + (u_h - u_h^\star)^\top R_h (u_h - u_h^\star)\right]$$

such that $\quad x_{h+1} = A_h x_h + B_h u_h + w_h,\ \ u_h = \pi_h(x_h) \quad x_0 \sim \mu_0,\ w_h \sim N(0,\sigma^2 I)$,

We can simply complete the square
and we reduce back to the setting in the previous slide!

Known transition versus black-box access

So far, we studied Policy Evaluation, Policy Iteration, Value Iteration, and DP-based approach,

we have assumed that transition is **unknown**,
i.e., $P(s' \mid s, a), \forall s, a, s'$ is known, or $A, B, \mathcal{N}(0, \sigma^2 I)$ are known

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Starting from this Thursday, we start considering **unknown** transition:

We start w/ black-box access to P, or $f(x, u, w)$:

We can **reset the system to any** (s, a), and observe $s' \sim P(\cdot \mid s, a)$,

Or we can reset to any (x, u), and observe $x' = f(x, u, w)$
(w being some unknown noisy disturbance)

Summary today:

1. We use DP to derive the optimal control for LQR (Ricatti equation)!

2. Never try to remember the exact form!
Only need to understand the way we derive it (again DP!)

$$\mathbb{E}_{x,y}[f(x)] = \mathbb{E}_{x}[f(x)]$$

$$x \sim \sum_y p(x, y)$$

Next Lecture:

Control for Nonlinear system w/ black-box access to $f(x, u)$
(In general, very hard, we will study approximate algorithm and only aim for locally optimal solutions)

$\pi_h^* \ \forall s, h$

$P(x,y)$ is valid distr...

s_0

then $p(x) := \sum_y p(x,y)$ is valid distri...

$P_h^{\pi}(s,a; s_0) \geq 0$

$\sum_{s_t} P_h^{\pi}(s,a; s_0) = 1$

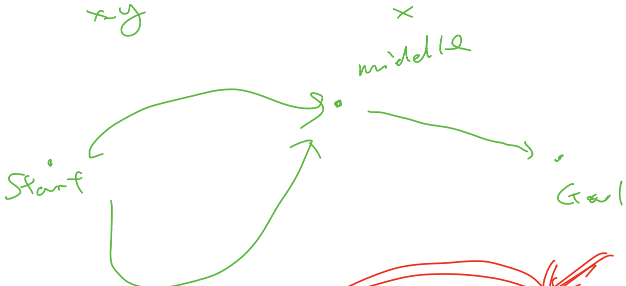
$(1-\gamma)$

s_0

$$\mathbb{E}_{a_0,s_1,a_1,s_2,a_2}\left[r(s_2,a_2)\gamma^2\right]$$

$$= \left[\sum_{a_0,s_1,a_1,s_2,a_2} \Pr(a_0,s_1,a_1,s_2,a_2) \cdot r(s_2,a_2)\right] \cdot \gamma^2$$

$$\mathbb{E}_{x,y}[f(x)] = \mathbb{E}_{x}[f(x)]$$



Marginization → $P_h^{\pi}(s,a; s_0)$ ← Def has nothing to do with γ.

$$d^{\pi}(s,a) = (1-\gamma)\sum_h \gamma^h P_h^{\pi}(s,a; s_0)$$