Basics of Markov Decision Process

Hello!!

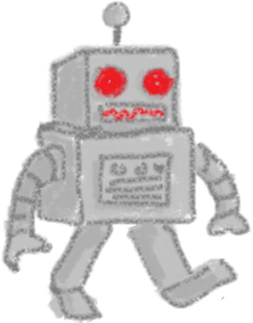
Announcement:

1. For wait list, we will need to prioritize CS students and seniors (cs-course-enroll@cornell.edu)

2. Clarification on the attendance bonus (5%)

Recap:

Learning Agent



$$\pi(s) \rightarrow a$$

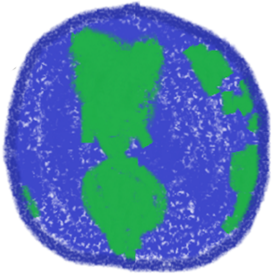
Policy: determine **action** based on **state**

Multiple Steps

Send **reward** and **next state** from a Markovian transition dynamics

$$r(s, a), s' \sim P(\cdot \mid s, a)$$

Environment



Recap:

$$\mathscr{M} = \{S, A, P, r, \gamma\}$$

$$P : S \times A \mapsto \Delta(S), \quad r : S \times A \to [0,1], \quad \gamma \in [0,1)$$

Policy $\pi : S \mapsto (A)$

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Policy $\pi : S \mapsto (A)$

Value function $V^\pi(s) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \middle| s_0 = s, a_h = \pi(s_h), s_{h+1} \sim P(\cdot \mid s_h, a_h)\right]$

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Q function $Q^\pi(s, a) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \,\middle|\, (s_0, a_0) = (s, a), a_h = \pi(s_h), s_{h+1} \sim P(\cdot \mid s_h, a_h)\right]$

Bellman Equation for V/Q-function:

$$V^\pi(s) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \,\middle|\, s_0 = s, a_h = \pi(s_h), s_{h+1} \sim P(\cdot \mid s_h, a_h)\right]$$

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$$Q^\pi(s, a) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \middle| (s_0, a_0) = (s, a), a_h = \pi(s_h), s_{h+1} \sim P(\cdot \mid s_h, a_h)\right]$$

What is the BE for Q function??

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What is the BE for Q function??

$$Q^\pi(s, a) = r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot \mid s, a)} V^\pi(s')$$

Bellman Equation for Q-function:

$$\forall s, a: \quad Q^\pi(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^\pi(s')$$

$$Q^\pi(s,a)$$

$$= \mathbb{E}\left[r(s_0,a_0) + \gamma\, r(s_1,a_1) + \gamma^2\, r(s_2,a_2) + \cdots \;\middle|\; s_0 = s,\ a_0 = a,\ a_h = \pi(s_h),\ s_{h+1} \sim P(\cdot|s_h,a_h) \right]$$

$$= r(s_0,a) + \gamma \mathbb{E}_{s_1 \sim P(\cdot|s,a)} \underbrace{\left[r(s_1,a_1) + \gamma\, r(s_2,a_2) + \cdots \;\middle|\; \cdots \right]}_{V^\pi(s_1)}$$

$$V^\pi(s) = Q^\pi(s, \pi(s))$$

Today:

1. We have A^S many policies, which one is the optimal policy $\pi^\star$?

2. Key property of the optimal policy $\pi^\star$

DP

3. State-action distributions

Definition of Optimal Policy $\pi^\star$

For infinite horizon discounted MDP, there always exists a deterministic policy

$$\pi^\star : S \mapsto A, \text{ s.t., } V^{\pi^\star}(s) \geq V^\pi(s), \forall s, \pi$$

[Puterman 94 chapter 6, also see theorem ~~1.4~~ in the RL monograph—no need to understand the proof]

1.7

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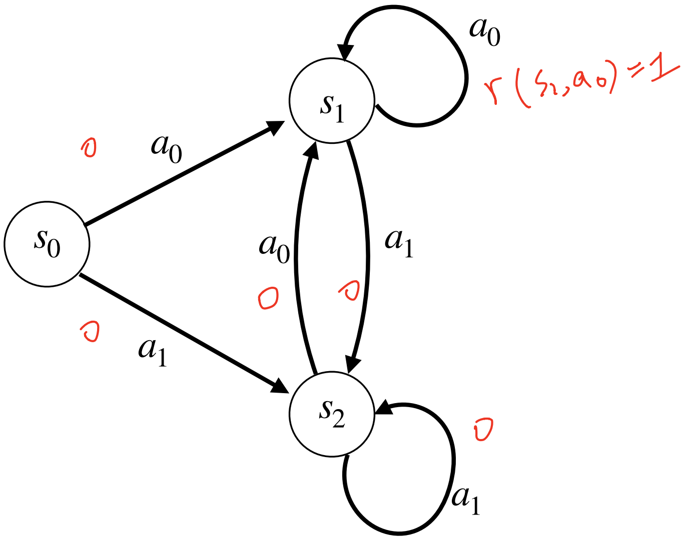
1.7 AJKS-Book,

i.e., $\pi^\star$ dominates any other policy π, everywhere!

We often denote $V^\star$, $Q^\star$ in short for $V^{\pi^\star}$, $Q^{\pi^\star}$

Example of Optimal Policy $\pi^\star$

Consider the following **deterministic** MDP w/ 3 states & 2 actions



Reward: $r(s_1, a_0) = 1$, 0 everywhere else

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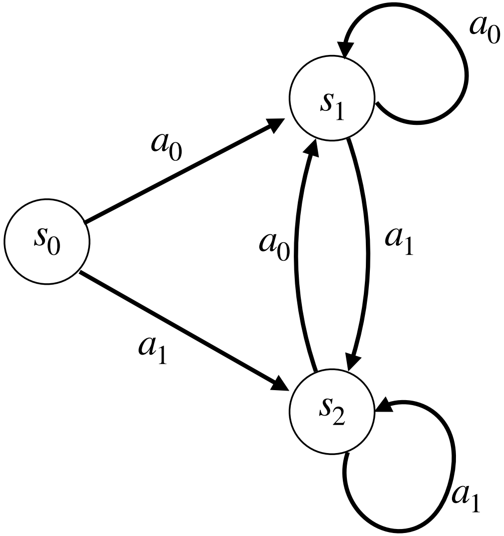

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What's the optimal policy?

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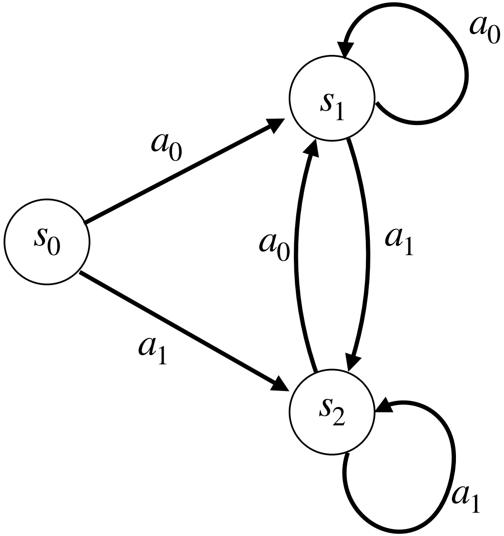

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$\pi^\star(s) = a_0, \forall s$

$\underline{\underline{V^\star}}$??

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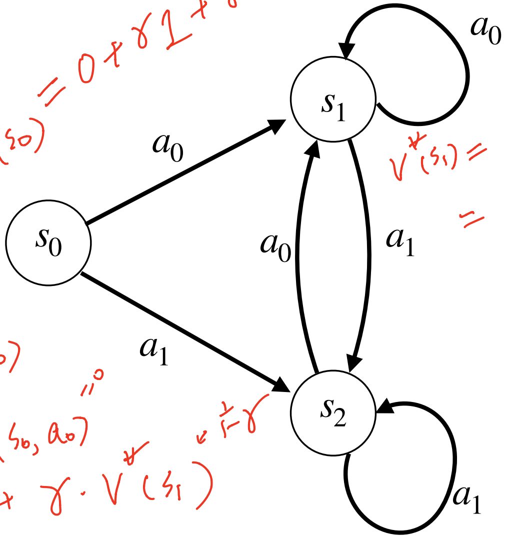
Consider the following **deterministic** MDP w/ 3 states & 2 actions

$V^\star(s_0) = 0 + \gamma 1 + \gamma^2 1 + \gamma^3 1 + \cdots = \frac{\gamma}{1-\gamma}$



$V^\star(s_1) = 1 + \gamma \cdot 1 + \gamma^2 \cdot 1 + \cdots = \frac{1}{1-\gamma}$

$V^\star(s_0) = \overset{=0}{r(s_0, a_0)} + \gamma \cdot \overset{\frac{1}{1-\gamma}}{V^\star(s_1)}$

Let's say $\gamma \in (0,1)$

What's the optimal policy?

$$\pi^\star(s) = a_0, \forall s$$

$$V^\star(s_0) = \frac{\gamma}{1-\gamma}, V^\star(s_1) = \frac{1}{1-\gamma}, V^\star(s_2) = \frac{\gamma}{1-\gamma}$$

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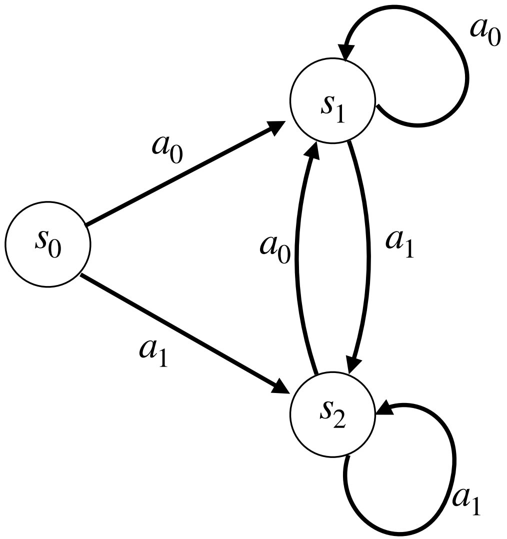

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$$\pi^\star(s) = a_0, \forall s$$

$$V^\star(s_0) = \frac{\gamma}{1-\gamma}, V^\star(s_1) = \frac{1}{1-\gamma}, V^\star(s_2) = \frac{\gamma}{1-\gamma}$$

What about policy $\pi(s) = a_1, \forall s$

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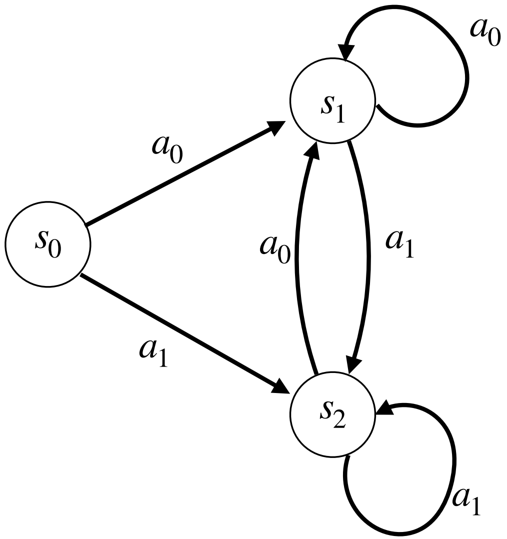

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Let's say $\gamma \in (0,1)$
What's the optimal policy?

$$\pi^\star(s) = a_0, \forall s$$

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What about policy $\pi(s) = a_1, \forall s$

$$V^\pi(s_0) = 0, V^\pi(s_1) = 0, V^\pi(s_2) = 0$$

π^* dominates π;

Summary so far:

Every discounted MDP has a deterministic optimal policy, that **dominates other policies everywhere** (proof is out of the scope)

$$V^\star(s) \geq V^\pi(s), \forall \pi, \forall s$$

$$\mathcal{M} = (S, A, P, \gamma, r)$$

$$V^\pi(s) = E\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h)\right] \quad r \in [0, 1)$$

Outline

1. We have A^S many policies, which one is the optimal policy $\pi^\star$?

2. Key property of the optimal policy $\pi^\star$

Bellman optimality

DP

3. State-action distributions

Bellman Optimality

$V^* := V^{\pi^*}$

Theorem 1: Bellman Optimality

$$V^\star(s) = \max_a \left[r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^\star(s') \right], \forall s$$

A

Understanding Bellman Optimality

$V^\star(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^\star(s') \right], \forall s$ **via *DP:***

Understanding Bellman Optimality

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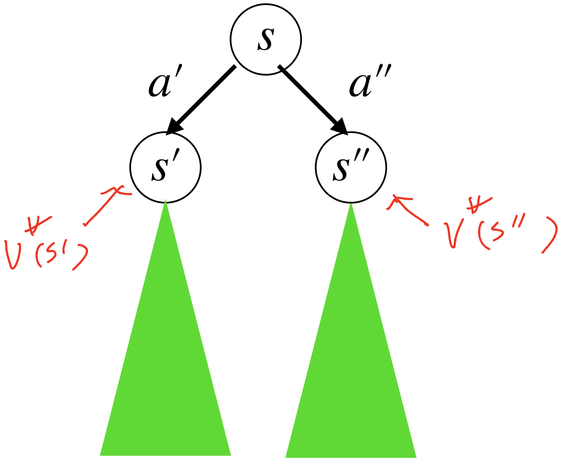
Q: If we know the optimal value at s', s'', i.e., $V^\star(s'), V^\star(s'')$, what we do at s?



Understanding Bellman Optimality

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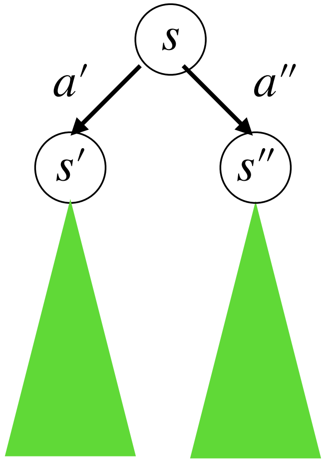
$$Q^\star(s,a') := r(s,a') + \gamma V^\star(s')$$



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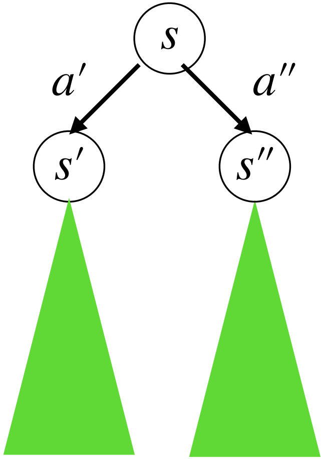

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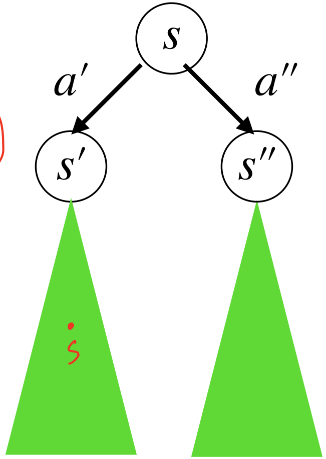

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$Q^\star(s,a'') := r(s,a'') + \gamma V^\star(s'')$

$$V^\star(s) = \max_{a',a''} \left\{ Q^\star(s,a'), Q^\star(s,a'') \right\}$$

Proof of Bellman Optimality

Theorem 1: Bellman Optimality

$$V^\star(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^\star(s') \right], \ \forall s \in S$$

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$Q^\star(s,a)$

Denote $\widehat{\pi}(s) := \arg\max_a Q^\star(s,a)$, we will prove $V^{\widehat{\pi}}(s) = V^\star(s), \forall s$

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BE for $\pi^\star$

$$V^\star(s) = r(s, \pi^\star(s)) + \gamma \mathbb{E}_{s' \sim P(s, \pi^\star(s))} V^\star(s')$$

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$$\leq \max_a \left[\underbrace{r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V^\star(s')}_{Q^\star(s,a)} \right] = r(s, \widehat{\pi}(s)) + \gamma \mathbb{E}_{s' \sim P(s, \widehat{\pi}(s))} V^\star(s')$$

Δ

Repeat

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BE

$$= r(s, \widehat{\pi}(s)) + \gamma \mathbb{E}_{s' \sim P(s, \widehat{\pi}(s))} \left[r(s', \pi^\star(s')) + \gamma \mathbb{E}_{s'' \sim P(s', \pi^\star(s'))} V^\star(s'') \right]$$

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$V^{\widehat{\pi}}(s) \le V^\star(s) \le V^{\widehat{\pi}}(s), \forall s.$

By Def of $\pi^\star$

$$V^\star(s) = r(s, \pi^\star(s)) + \gamma \mathbb{E}_{s' \sim P(s, \pi^\star(s))} V^\star(s')$$

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$$\le \mathbb{E} \left[r(s, \widehat{\pi}(s)) + \gamma r(s', \widehat{\pi}(s')) + \ldots \right] = V^{\widehat{\pi}}(s)$$

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This implies that $\underbrace{\arg\max_a Q^\star(s,a)}_{= \widehat{\pi}(s)}$ is an optimal policy

Proof of Bellman Optimality

Theorem 1: Bellman Optimality

$$V^\star(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^\star(s') \right]$$

Denote $\widehat{\pi}(s) := \arg\max_a Q^\star(s,a)$, we just proved $V^{\widehat{\pi}}(s) = V^\star(s), \forall s$

This implies that $\arg\max_a Q^\star(s,a)$ is an optimal policy

Q: why?

Summary so far:

Bellman Optimality and DP

Theorem 1: Bellman Optimality

$$V^\star(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^\star(s') \right], \forall s$$

Summary so far:

Bellman Optimality and DP

Theorem 1: Bellman Optimality

$$V^\star(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^\star(s') \right], \forall s$$

Next:

Any function $V(s)$ that satisfies Bellman Optimality, MUST be equal to $V^\star$

Bellman Optimality

Theorem 2:

For any $V : S \to \mathbb{R}$, if $V(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \right]$ for all s,

then $V(s) = V^\star(s), \forall s$

Bellman Optimality

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then $V(s) = V^\star(s), \forall s$

Bellman Opt allows us to focus on just one step,

i.e., to check if $V = V^\star$,

we only need to check if $\left| V(s) - \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \right] \right| = 0, \forall s$,

Bellman Optimality

Theorem 2:

For any $V : S \to \mathbb{R}$, if $V(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \right]$ for all s,

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then $V(s) = V^\star(s), \forall s$

$\forall s.$

$$|V(s) - V^\star(s)| = \Big| \underbrace{\max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s'))}_{\text{Condition}} - \underbrace{\max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s'))}_{\text{Bellman opt for } V^\star} \Big|$$

Bellman Optimality

Theorem 2:

For any $V : S \to \mathbb{R}$, if $V(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \right]$ for all s,

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$$|V(s) - V^\star(s)| = \left| \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right|$$

$$\leq \max_a \left| (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right|$$

$$\left| \max_a f(a) - \max_a g(a) \right| \leq \max_a \left| f(a) - g(a) \right| \checkmark$$

Bellman Optimality

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$$|V(s) - V^\star(s)| = \left| \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right|$$

$$\leq \max_a \left| (r\cancel{(s,a)} + \underbrace{\gamma \mathbb{E}_{s \sim P(s,a)} V(s')}_{A}) - (r\cancel{(s,a)} + \underbrace{\gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')}_{B}) \right|$$

$$\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left| V(s') - V^\star(s') \right|$$

$$\left| \mathbb{E}_{x \sim p} f(x) - \mathbb{E}_{x \sim p} g(x) \right| \leq \mathbb{E}_{x \sim p} \left| f(x) - g(x) \right|$$

$$|a+b| \leq |a| + |b|$$

Bellman Optimality

Theorem 2:

For any $V : S \to \mathbb{R}$, if $V(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \right]$ for all s,

then $V(s) = V^\star(s), \forall s$

$$
\begin{aligned}
|V(s) - V^\star(s)| &= \left| \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right| \\
&\le \max_a \left| (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right| \\
&\le \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left| V(s') - V^\star(s') \right| \\
&\le \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left(\max_{a'} \gamma \mathbb{E}_{s'' \sim P(s',a')} \left| V(s'') - V^\star(s'') \right| \right)
\end{aligned}
$$

Repeat

Repeat

Bellman Optimality

$V^\star(s) \in [0, \frac{1}{1-\gamma}]$

Theorem 2:

For any $V : S \to \mathbb{R}$, if $V(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \right]$ for all s,
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$$|V(s) - V^\star(s)| = \left| \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right|$$

$$\leq \max_a \left| (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right|$$

$$\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left| V(s') - V^\star(s') \right|$$

$\gamma \in [0,1)$

$$\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left(\max_{a'} \gamma \mathbb{E}_{s'' \sim P(s',a')} \left| V(s'') - V^\star(s'') \right| \right)$$

$\gamma^k \Rightarrow 0$

$$\leq \max_{a_1, a_2, \ldots a_{k-1}} \gamma^k \mathbb{E}_{s_k} | V(s_k) - V^\star(s_k) |$$ $\Leftarrow$ Repeat

$k \to \infty$, $|V(s) - V^\star(s)| \to 0$

Summary so far:

1. $V^\star$ satisfies Bellman Optimality:

$$V^\star(s) = \max_a \left[r(s,a) + \mathbb{E}_{s' \sim P(s,a)} V^\star(s') \right], \forall s$$

2. Any V that satisfies Bellman Optimality, i.e., $V(s) = \max_a \left[r(s,a) + \mathbb{E}_{s' \sim P(s,a)} V(s') \right], \forall s,$

MUST be that $V(s) = V^\star(s)$, for all s

key to compute V^*

Outline

1. We have A^S many policies, which one is the optimal policy $\pi^\star$?

2. Key property of the optimal policy $\pi^\star$: **Bellman Optimality**

3. State-action distributions

What's the probability of π visiting a particular state s?

Discounted State (action) Occupancy Measures

Assume we start at s_0, following π to step h, what's probability of seeing a trajectory:

$$(s_0, a_0, s_1, a_1, \ldots, s_h, a_h)?$$

Discounted State (action) Occupancy Measures

Assume we start at s_0, following π to step h, what's probability of seeing a trajectory:

$(s_0, a_0, s_1, a_1, \ldots, s_h, a_h)$?

Let's write π as a delta distribution, i.e., $\pi(a \mid s) = \begin{cases} 1, & a = \pi(s), \\ 0, & \text{else} \end{cases}$

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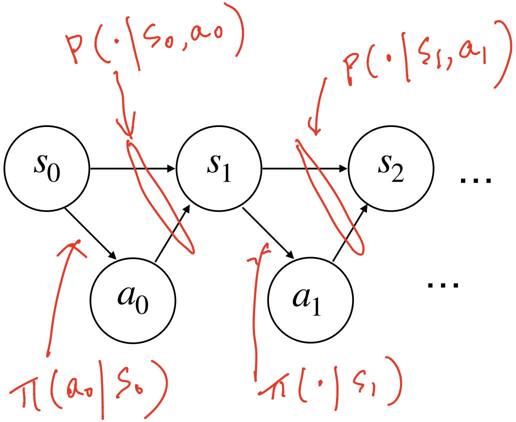

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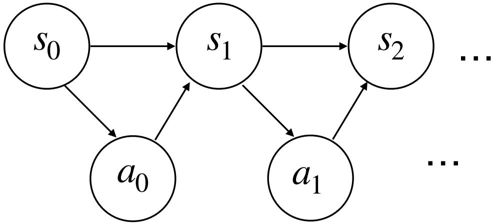

$\mathbb{P}^\pi(s_0, a_0, \ldots, s_h, a_h)$

Discounted State (action) Occupancy Measures

Assume we start at s_0, following π to step h, what's probability of seeing a trajectory:

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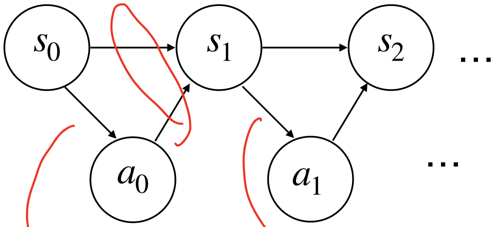
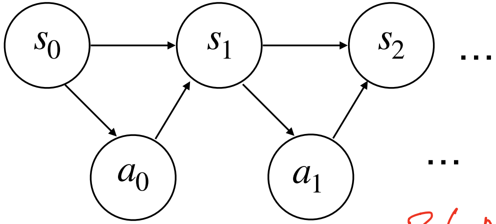


$$\mathbb{P}^\pi(s_0, a_0, \ldots, s_h, a_h)$$

$$= \pi(a_0 \mid s_0) P(s_1 \mid s_0, a_0) \pi(a_1 \mid s_1) P(s_2 \mid s_1, a_1) \ldots P(s_h \mid s_{h-1}, a_{h-1}) \pi(a_h \mid s_h)$$

State-action distribution at time step h



$$\mathbb{P}^{\pi}(s_0, a_0, \ldots, s_h, a_h)$$

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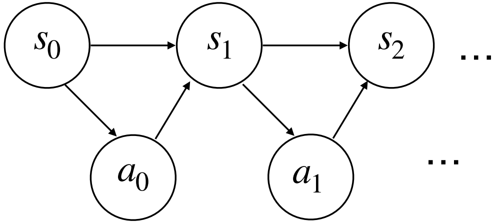
?

$P(A, B) \Rightarrow P(A=a)$ $\quad P(A=a) = \sum_{b} P(A=a, B=b)$

Q: what's the probability of π visiting state (s,a) at time step h?

State-action distribution at time step h



$$\mathbb{P}^{\pi}(s_0, a_0, \ldots, s_h, a_h)$$

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Q: what's the probability of π visiting state $(s$,a) at time step h?

$$\mathbb{P}_h^{\pi}(s, a; s_0) = \sum_{a_0, s_1, a_1, \ldots, s_{h-1}, a_{h-1}} \mathbb{P}^{\pi}(s_0, a_0, \ldots, s_{h-1}, a_{h-1}, s_h = s, a_h = a)$$

Discounted Average State-action distribution

Probability of π visiting (s, a) at h, starting from s_0

$$\mathbb{P}_h^{\pi}(s, a; s_0) = \sum_{a_0, s_1, a_1, \ldots, s_{h-1}, a_{h-1}} \mathbb{P}^{\pi}(s_0, a_0, \ldots, s_{h-1}, a_{h-1} s_h = s, a_h = a)$$

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$$d_{s_0}^\pi(s, a) = (1-\gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h^\pi(s, a; s_0)$$

$$(1-\gamma)\left(P_0^\pi(s, a) + \gamma P_1^\pi(s, a) + \gamma^2 P_2^\pi(s, a) + \cdots \right)$$

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Probability of π visiting (s, a) at h, starting from s_0

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$$d_{s_0}^\pi(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h^\pi(s, a; s_0)$$

Can you show that this is a valid distribution?

$d^\pi_{s_0}(s,a) \geq 0$

$\sum_{s,a} d^\pi_{s_0}(s,a) = 1$

Discounted Average State-action distribution

Probability of π visiting (s, a) at h, starting from s_0

$$\mathbb{P}_h^\pi(s, a; s_0) = \sum_{a_0, s_1, a_1, \ldots, s_{h-1}, a_{h-1}} \mathbb{P}^\pi(s_0, a_0, \ldots, s_{h-1}, a_{h-1} s_h = s, a_h = a)$$

$$d_{s_0}^\pi(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h^\pi(s, a; s_0)$$

Can you show that this is a valid distribution?

$$V^\pi(s_0) = \frac{1}{1 - \gamma} \sum_{s,a} d_{s_0}^\pi(s, a) r(s, a) = \frac{1}{1-\gamma} \mathbb{E}_{s\,a \sim d_{s_0}^\pi} \left[r(s,a) \right]$$

Discounted Average State-action distribution

Probability of π visiting (s, a) at h, starting from s_0

$$\mathbb{P}_h^\pi(s, a; s_0) = \sum_{a_0, s_1, a_1, \ldots, s_{h-1}, a_{h-1}} \mathbb{P}^\pi(s_0, a_0, \ldots, s_{h-1}, a_{h-1} s_h = s, a_h = a)$$

$$d_{s_0}^\pi(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h^\pi(s, a; s_0)$$

Can you show that this is a valid distribution?

$$V^\pi(s_0) = \frac{1}{1 - \gamma} \sum_{s,a} d_{s_0}^\pi(s, a) r(s, a)$$

Can you show the above is true?

Discounted Average State-action distribution

Probability of π visiting (s, a) at h, starting from s_0

$$\mathbb{P}_h^\pi(s, a; s_0) = \sum_{a_0, s_1, a_1, \ldots, s_{h-1}, a_{h-1}} \mathbb{P}^\pi(s_0, a_0, \ldots, s_{h-1}, a_{h-1} s_h = s, a_h = a)$$

$$d_{s_0}^\pi(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h^\pi(s, a; s_0)$$

HW0: Bellman-Eqn for $d^\pi_{s_0}(s,a)$

Can you show that this is a valid distribution?

$$V^\pi(s_0) = \frac{1}{1-\gamma} \sum_{s,a} d_{s_0}^\pi(s, a) r(s, a) = \mathbb{E}_{s,a \sim d^\pi_{s_0}} \left[r(s,a) \right]$$

HW0 questions!

Can you show the above is true?

Summary for today:

1. $\pi^\star$ **dominates** other policies, i.e., $V^\star(s) \geq V^\pi(s), \forall s, \pi$

2. Key property of the optimal policy $\pi^\star$: **Bellman Optimality**

(BE and B-Opt allow us to focus on one step)

3. State-action distribution: $d^\pi_{s_0}(s, a)$

Summary for today:

1. $\pi^\star$ **dominates** other policies, i.e., $V^\star(s) \geq V^\pi(s), \forall s, \pi$

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(BE and B-Opt allow us to focus on one step)

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RL is notation heavy! But we will see these over and over again during the semester.