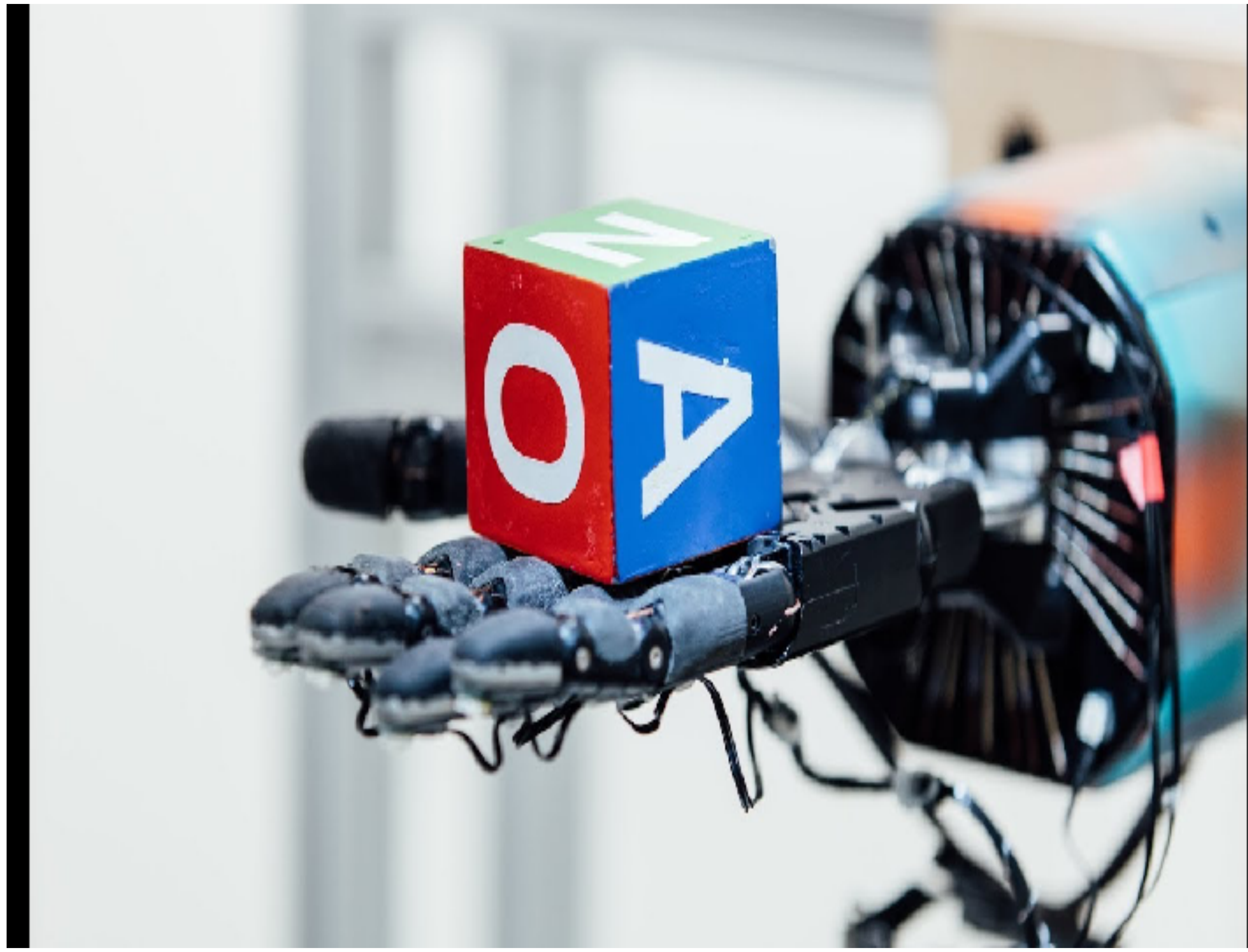
Policy Gradient

Policy Optimization

[AlphaZero, Silver et.al, 17]

[OpenAI Five, 18]



[OpenAI,19]

Recap of CPI

Recall CPI:

1. Greedy Policy Selector:
$$\pi' \in \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^{\pi^t}} \left[A^{\pi^t}(s, \pi(s)) \right]$$

2. If $\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^{\pi^t}}[A^{\pi^t}(s, \pi(s))] \leq \varepsilon$

   Return π^t

3. Incremental Update:
$$\pi^{t+1}(\cdot \mid s) = (1 - \alpha)\pi^t(\cdot \mid s) + \alpha \pi'(\cdot \mid s), \forall s$$

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1. Incremental update (Lemma 12.1 in AJKS)

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(By setting step size α properly…)

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Recap: two definitions of MDPs

$$\mathcal{M} = \{P, r, \gamma, \mu, S, A\}$$

where $s_0 \sim \mu$

Objective: $J(\pi) := \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \,\middle|\, s_0 \sim \mu, s_{h+1} \sim P_{s_h, a_h}, a_h \sim \pi(\cdot \mid s_h)\right]$

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Main question for today's lecture:
how to compute the gradient?

Outline for today

1. Recap on Gradient descent and stochastic gradient descent

2. Warm up: computing gradient using importance weighting

3. Policy Gradient formulations

Stochastic Gradient Descent

Given an objective function $J(\theta) : \mathbb{R}^d \mapsto \mathbb{R}$, (e.g., $J(\theta) = \mathbb{E}_{x,y}(f_\theta(x) - y)^2$)

SGD minimizes the above objective function as follows:

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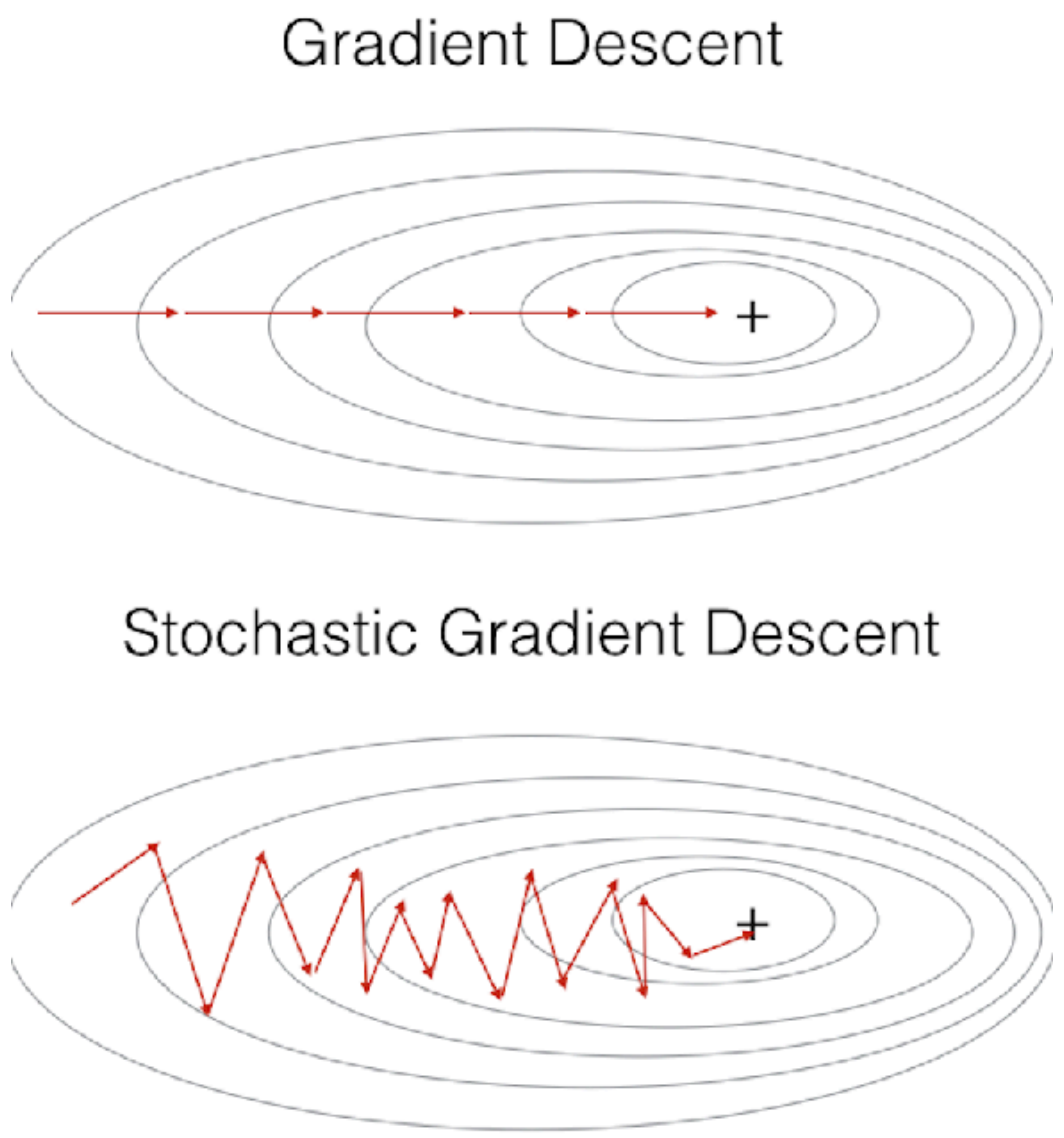

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$$\|\nabla_\theta J(\theta) - \nabla_\theta J(\theta_0)\|_2 \leq \beta \|\theta - \theta_0\|_2$$

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[**Theorem**] If $J(\theta)$ is β-smooth and $\sup_\theta |J(\theta)| \leq M$, and we run SGD: $\theta_{t+1} = \theta_t - \eta \widetilde{\nabla}_\theta J(\theta_t)$

where $\mathbb{E}\left[\widetilde{\nabla}_\theta J(\theta_t)\right] = \nabla_\theta J(\theta_t), \quad \mathbb{E}\left[\|\widetilde{\nabla}_\theta J(\theta_t)\|_2^2\right] \leq \sigma^2,$

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Proof of Convergence to Stationary Point (optional)

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Set $\eta = \sqrt{M/(\beta\sigma^2 T)}$

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Summary so far:

SGD is a simple algorithm that can find a locally optimal solution ($\|\nabla_\theta J(\hat{\theta})\|_2$ small in expectation — proof optional)

For convex function, this guarantees global optimality

Outline for today

✓ 1. Recap on Gradient descent and stochastic gradient descent

2. Warm up: computing gradient using importance weighting

3. Policy Gradient formulations

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$$\nabla_\theta J(\theta) = \nabla_\theta \mathbb{E}_{x \sim P_\theta} f(x) = \nabla_\theta \mathbb{E}_{x \sim \rho} \frac{P_\theta(x)}{\rho(x)} f(x)$$

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Warm Up

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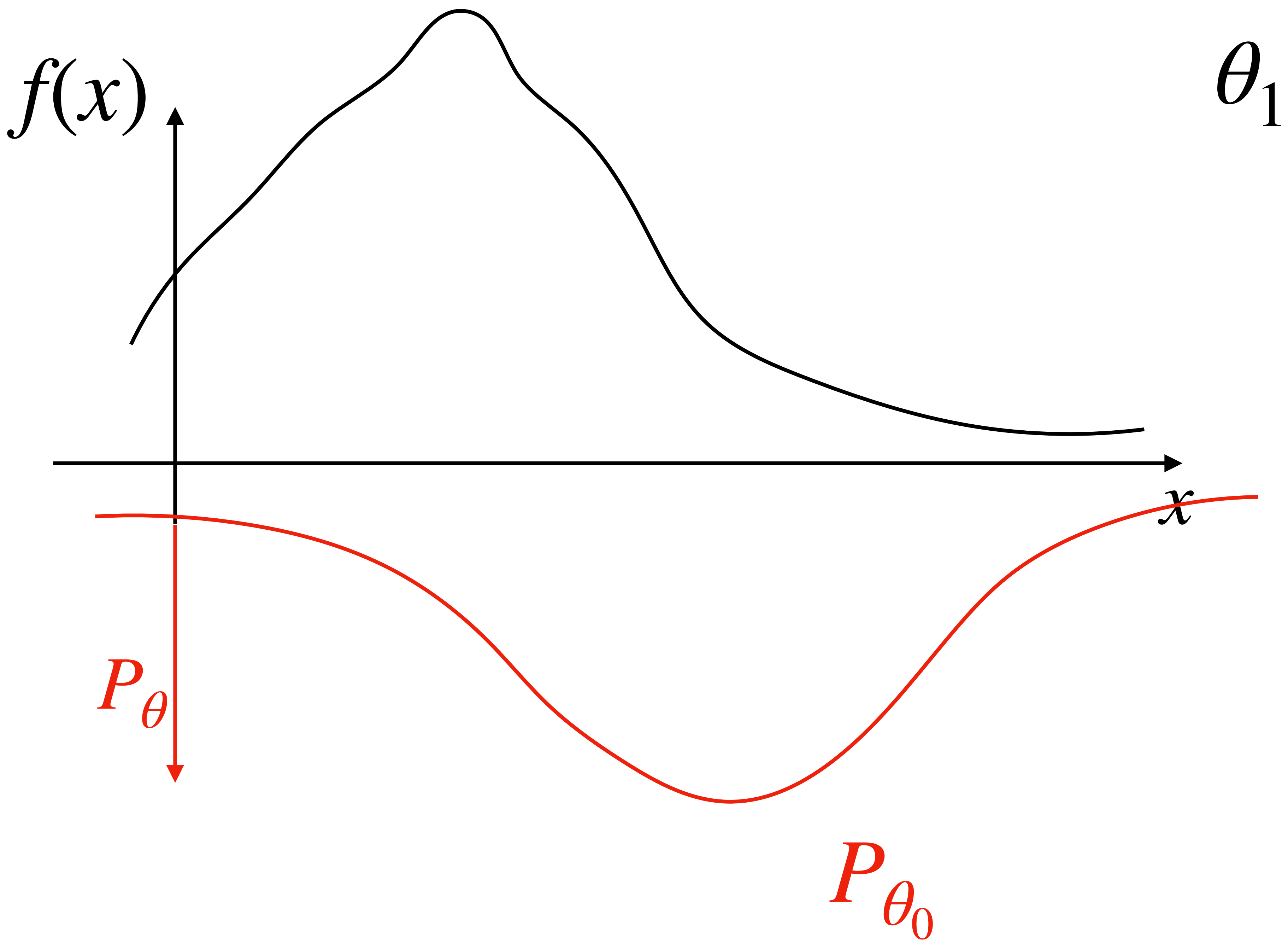

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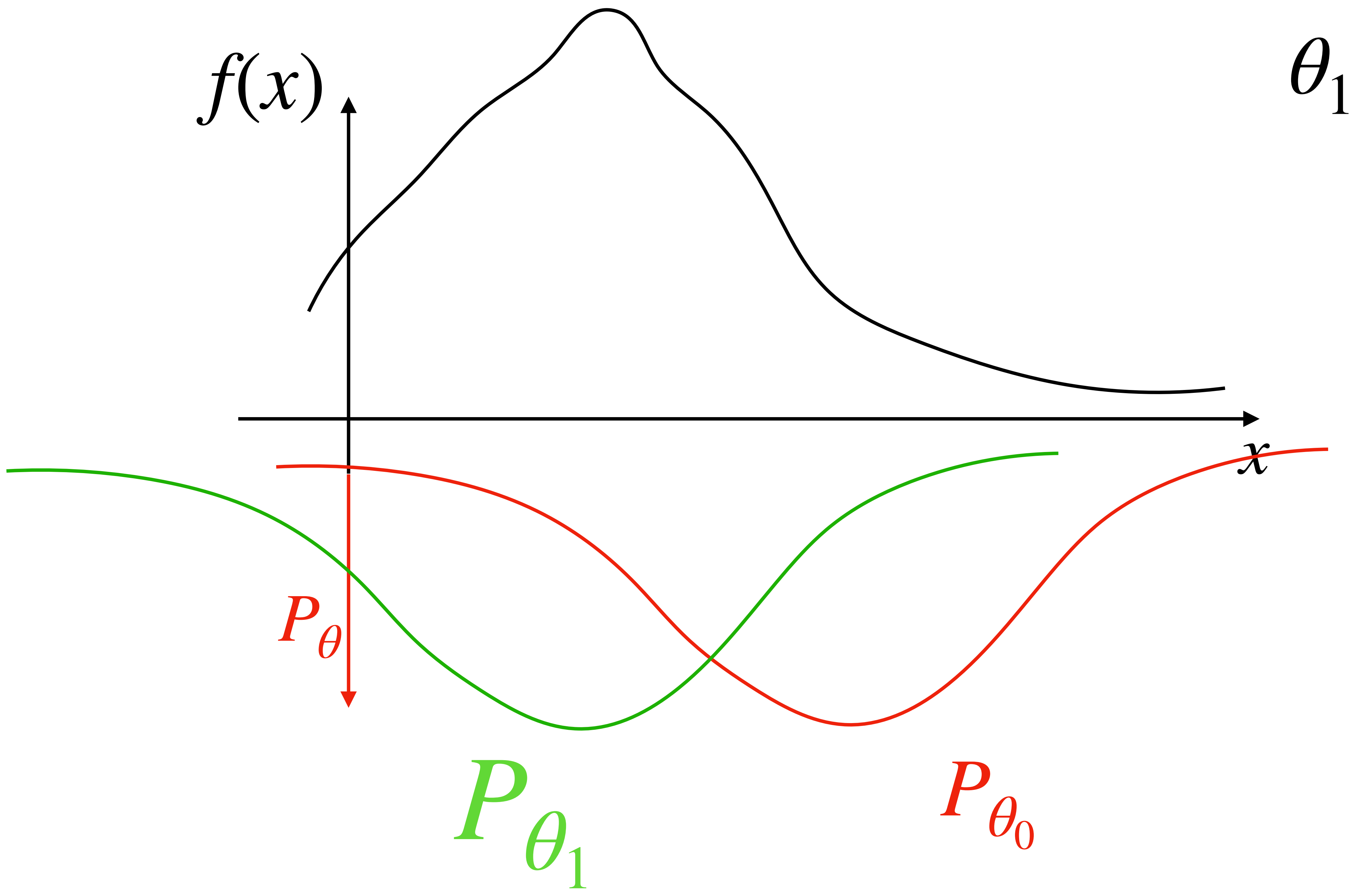

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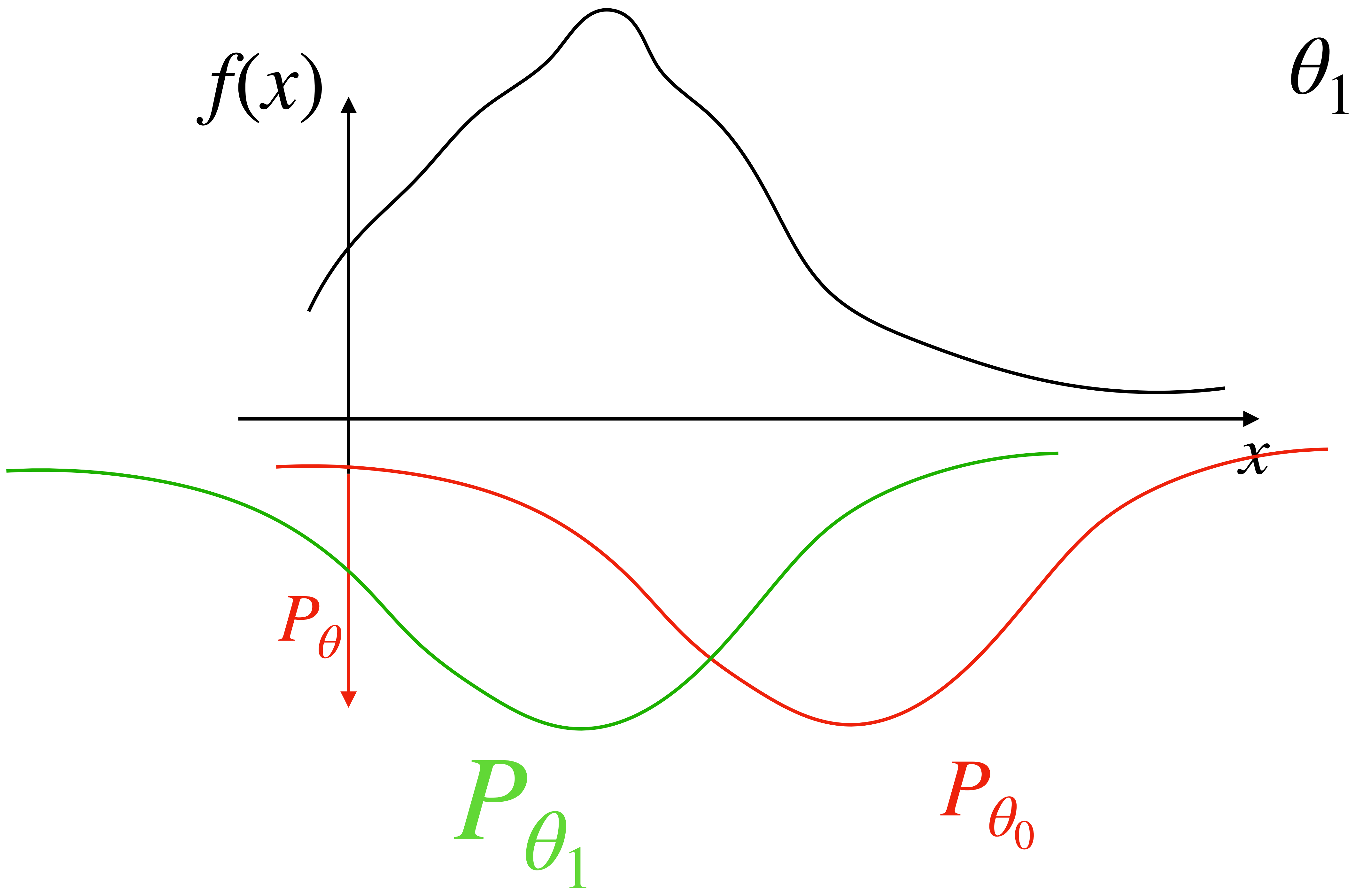

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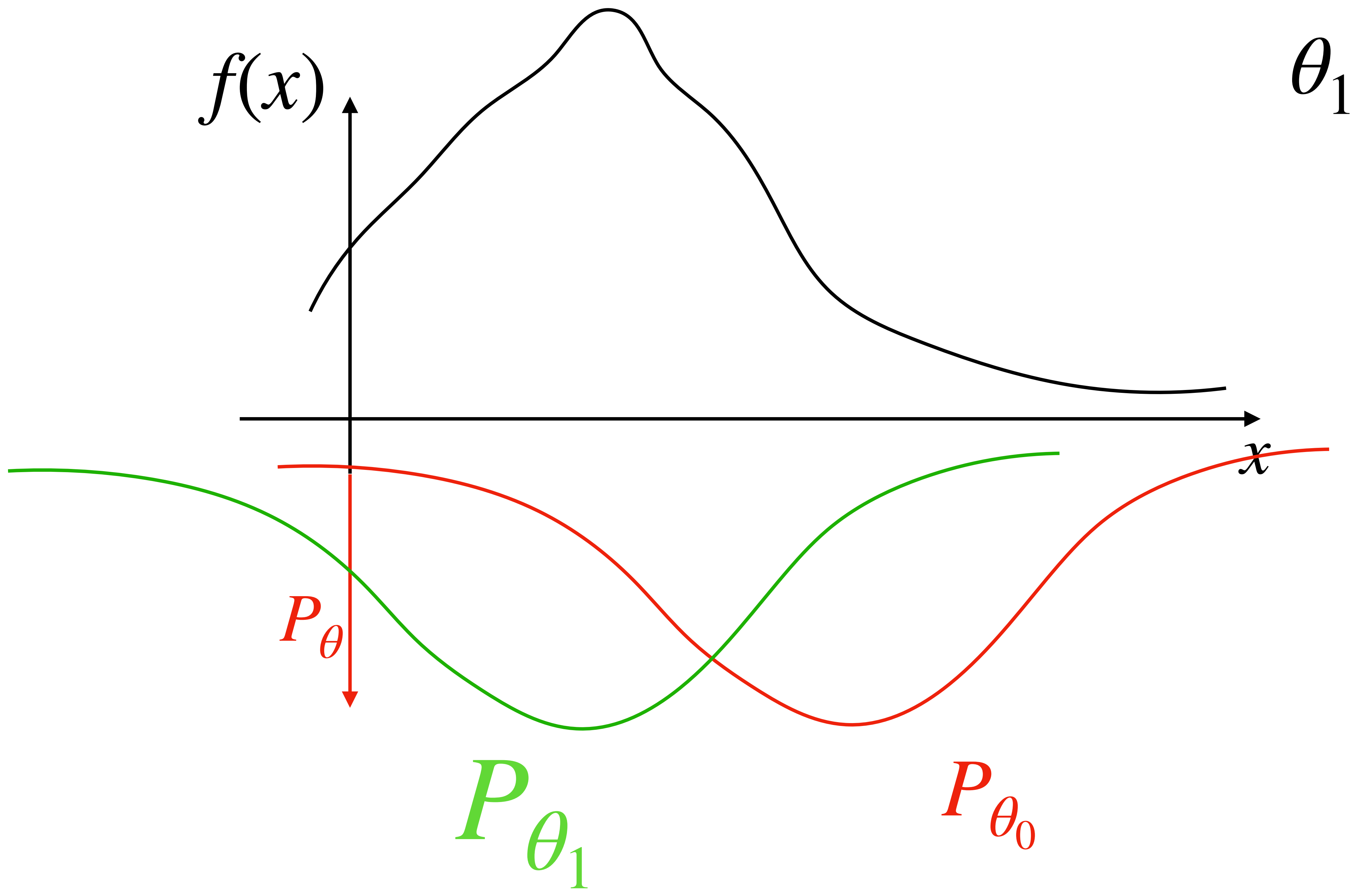

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Using same idea, now let's move on to RL…

Outline for today

✓ 1. Recap on Gradient descent and stochastic gradient descent

✓ 2. Warm up: computing gradient using importance weighting

3. Policy Gradient formulations

Policy Gradient: Examples of Policy Parameterization (discrete actions)

Recall that we consider parameterized policy $\pi_\theta(\cdot \mid s) \in \Delta(A), \forall s$

1. Softmax Policy for discrete MDPs:

$\theta_{s,a} \in \mathbb{R}, \forall s, a \in S \times A$

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In high level, think about π_θ as a classifier which has its parameters to be optimized

Derivation of Policy Gradient: REINFORCE

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Adjust policy such that larger reward traj has higher likelihood

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For finite horizon MDP (very common setting for PG):

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Exercise:

Show this simplified version is equivalent to REINFORCE

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1. Importance Weighting Trick

2. Policy Gradient:

REINFORCE (a direct application of our warm up example):

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2. Use unbiased estimate of $\nabla_\theta J(\theta)$, SG ascent converges to local optimal policy