Trust Region
Policy Optimization

Announcements

Thanks for providing midterm feedback!

1. HW2 will be out this Friday

2. I will have an additional office hour every Monday morning (11am - noon)

Recap Policy Gradient

$$J(\pi_\theta) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \,|\, s_0 \sim \mu, a \sim \pi_\theta\right]$$

Recap Policy Gradient

$$J(\pi_\theta) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \,|\, s_0 \sim \mu, a \sim \pi_\theta\right]$$

The most commonly used formulation:

$$\nabla_\theta J(\pi_{\theta_t}) = \mathbb{E}_{s,a \sim d_\mu^{\pi_{\theta_t}}}\left[\nabla_\theta \ln \pi_{\theta_t}(a \,|\, s) A^{\pi_{\theta_t}}(s, a)\right]$$

Recap Policy Gradient

$$J(\pi_\theta) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \,|\, s_0 \sim \mu, a \sim \pi_\theta\right]$$

The most commonly used formulation:

$$\nabla_\theta J(\pi_{\theta_t}) = \mathbb{E}_{s,a \sim d_\mu^{\pi_{\theta_t}}}\left[\nabla_\theta \ln \pi_{\theta_t}(a \,|\, s) A^{\pi_{\theta_t}}(s, a)\right]$$

Algorithm: Stochastic Gradient Ascent

Recap on Conservative Policy Iteration

For t = 0 …

1. Greedy Policy Selector:

$$\pi' \in \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^{\pi^t}} \left[A^{\pi^t}(s, \pi(s)) \right]$$

2. Incremental Update:

$$\pi^{t+1}(\cdot \mid s) = (1-\alpha)\pi^t(\cdot \mid s) + \alpha \pi'(\cdot \mid s), \forall s$$

Recap on Conservative Policy Iteration

For t = 0 …

1. Greedy Policy Selector:

$$\pi' \in \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^{\pi^t}} \left[A^{\pi^t}(s, \pi(s)) \right]$$

2. Incremental Update:

$$\pi^{t+1}(\cdot \mid s) = (1-\alpha)\pi^t(\cdot \mid s) + \alpha\pi'(\cdot \mid s), \forall s$$

Q: Why this is incremental? In what sense?

Q: Can we get monotonic policy improvement?

Recap of CPI:

Incremental update (Lemma 12.1 in AJKS)

$$\|d_\mu^{\pi^{t+1}} - d_\mu^{\pi^t}\|_1 \leq \frac{2\gamma\alpha}{1-\gamma}$$

Pros and Cons of CPI:

Pros:

This is fundamental!

The idea of incremental update and the theorem behind it are still being used today…

Cons:

Practical Issue (e.g., memory issue)

e.g., what if my policies are all extremely large neural networks…

Today's Question

Can we develop some practical version of CPI?

Outlines

1. Quick intro on KL-divergence

2. A Trust-Region Formulation for Policy Optimization

3. Algorithm: Natural Policy Gradient

Interesting videos from the today's algorithm

Train a robot to "run" forward as fast as possible:

State: joint angles, center of mass, velocity, etc

Action: torques on joints

Reward: distance of moving forward between two steps

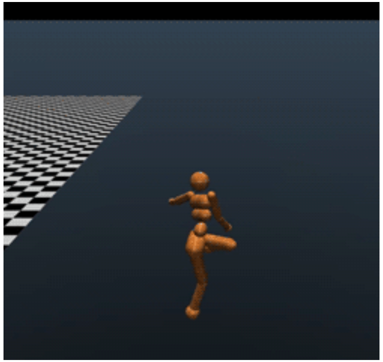
Interesting videos from the today’s algorithm

Train a robot to “run” forward as fast as possible:

State: joint angles, center of mass, velocity, etc

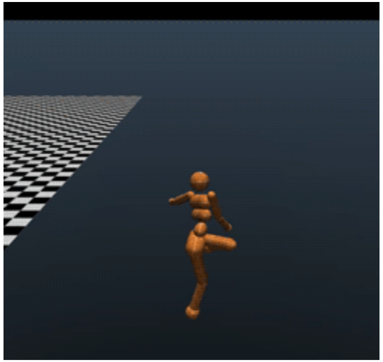
Action: torques on joints

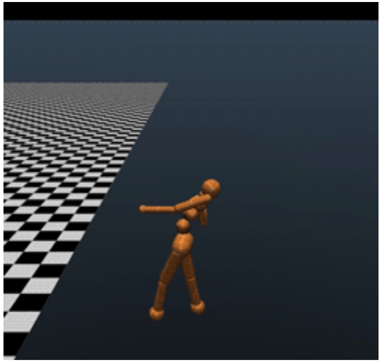
Reward: distance of moving forward between two steps



Interesting videos from the today's algorithm

Train a robot to "run" forward as fast as possible:
State: joint angles, center of mass, velocity, etc
Action: torques on joints
Reward: distance of moving forward between two steps





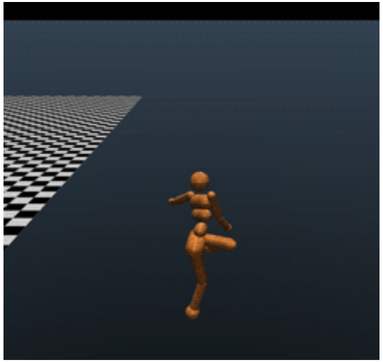
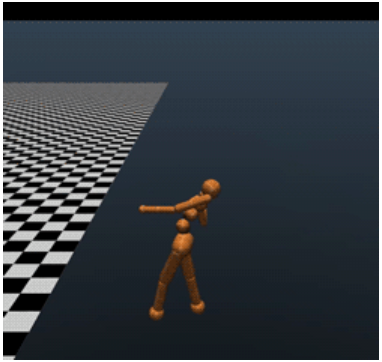
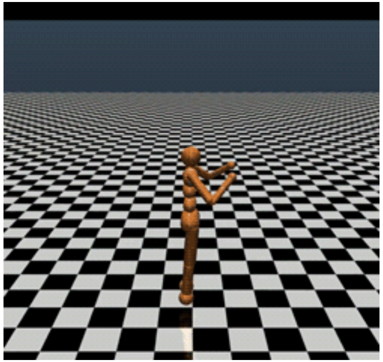
Interesting videos from the today's algorithm

Train a robot to "run" forward as fast as possible:

State: joint angles, center of mass, velocity, etc

Action: torques on joints

Reward: distance of moving forward between two steps

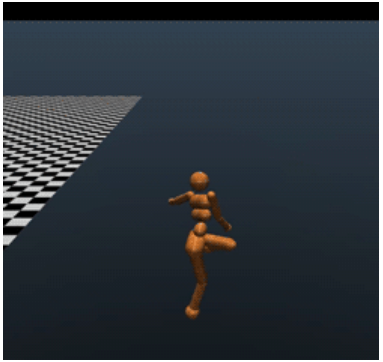
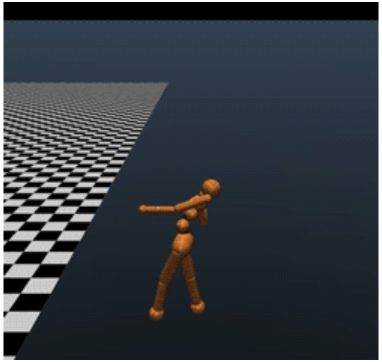
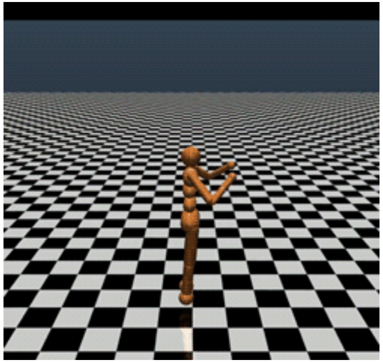
Interesting videos from the today's algorithm

Train a robot to "run" forward as fast as possible:

State: joint angles, center of mass, velocity, etc

Action: torques on joints

Reward: distance of moving forward between two steps

(BTW, This reveals an issue on reward design—we will study it in Learning from Demonstrations)

KL-divergence: measures the distance between two distributions

Given two distributions P & Q, where $P \in \Delta(X), Q \in \Delta(X)$,
KL Divergence is defined as:

$$KL(P \,|\, Q) = \mathbb{E}_{x \sim P}\left[\ln \frac{P(x)}{Q(x)}\right]$$

KL-divergence: measures the distance between two distributions

Given two distributions P & Q, where $P \in \Delta(X), Q \in \Delta(X)$,
KL Divergence is defined as:

$$KL(P | Q) = \mathbb{E}_{x \sim P}\left[\ln \frac{P(x)}{Q(x)}\right]$$

$P(x) = Q(x)$ $\forall x$

Examples:

If $Q = P$, then $KL(P | Q) = KL(Q | P) = 0$

KL-divergence: measures the distance between two distributions

Given two distributions P & Q, where $P \in \Delta(X), Q \in \Delta(X)$,
KL Divergence is defined as:

$$KL(P \,|\, Q) = \mathbb{E}_{x \sim P}\left[\ln \frac{P(x)}{Q(x)}\right]$$

$KL(P|Q)$

$\neq KL(Q|P)$

Examples:

If $Q = P$, then $KL(P \,|\, Q) = KL(Q \,|\, P) = 0$

If $P = \mathcal{N}(\mu_1, \sigma^2 I), Q = \mathcal{N}(\mu_2, \sigma^2 I)$, then $KL(P \,|\, Q) = \|\mu_1 - \mu_2\|_2^2/\sigma^2$ ✓

KL-divergence: measures the distance between two distributions

Given two distributions P & Q, where $P \in \Delta(X), Q \in \Delta(X)$,
KL Divergence is defined as:

$P(x)$.

Entropy: $\mathbb{E}_{x\sim P} -\log_2 P(x)$

$$KL(P\,|\,Q) = \mathbb{E}_{x\sim P}\left[\ln \frac{P(x)}{Q(x)}\right]$$

$\int_x P(x) = 1$

$\int_x Q(x) = 1$

$\mathbb{E}_{x\sim P}\left[f\left(\frac{P(x)}{Q(x)}\right)\right]$

$= \mathbb{E}_{x\sim P}[\ln P(x)] - \mathbb{E}_{x\sim P} \ln Q(x)$

f:

Examples:

If $Q = P$, then $KL(P\,|\,Q) = KL(Q\,|\,P) = 0$

If $P = \mathcal{N}(\mu_1, \sigma^2 I), Q = \mathcal{N}(\mu_2, \sigma^2 I)$, then $KL(P\,|\,Q) = \|\mu_1 - \mu_2\|_2^2/\sigma^2$

$\frac{1}{2}\|P - Q\|_1 \le \sqrt{KL(P\|Q)}$

Fact:

$KL(P\,|\,Q) \geq 0$, and being 0 if and only if $P = Q$

Outlines

1. Quick intro on KL-divergence

2. A Trust-Region Formulation for Policy Optimization

3. Algorithm: Natural Policy Gradient

Policy Parameterization

Recall that we consider parameterized policy $\pi_\theta(\cdot \mid s) \in \Delta(A), \forall s$

1. Softmax linear Policy (We will try this in HW2)

Feature vector $\phi(s, a) \in \mathbb{R}^d$, and parameter $\theta \in \mathbb{R}^d$

$$\pi_\theta(a \mid s) = \frac{\exp(\theta^\top \phi(s, a))}{\sum_{a'} \exp(\theta^\top \phi(s, a'))}$$

2. Neural Policy:

Neural network $f_\theta : S \times A \mapsto \mathbb{R}$

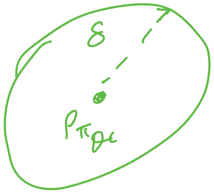
$$\pi_\theta(a \mid s) = \frac{\exp(f_\theta(s, a))}{\sum_{a'} \exp(f_\theta(s, a'))}$$

A trust region formulation for policy update:

At iteration t, with π_{θ_t} at hand, we compute θ_{t+1} as follows:

A trust region formulation for policy update:

At iteration t, with π_{θ_t} at hand, we compute θ_{t+1} as follows:



$$\max_{\pi_\theta} \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\underline{\mathbb{E}_{a\sim\pi_\theta(s)} A^{\pi_{\theta_t}}(s,a)}\right]$$

$$\text{s.t., } KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \le \delta$$

← hyper-parameter

Trust Region

Traj-Distribution

$\rho_\pi(\tau) = \mu(s_0)\,\pi(a_0|s_0)\,P(s_1|s_0,a_0)\,---$

A trust region formulation for policy update:

At iteration t, with π_{θ_t} at hand, we compute θ_{t+1} as follows:

$$\max_{\pi_\theta} \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}} \left[\mathbb{E}_{a\sim\pi_\theta(s)} A^{\pi_{\theta_t}}(s,a) \right]$$

$$\text{s.t., } KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \leq \delta$$

We want to maximize local advantage against π_{θ_t}, but we want the new policy to be close to π_{θ_t} (in the KL sense)

A trust region formulation for policy update:

At iteration t, with π_{θ_t} at hand, we compute θ_{t+1} as follows:

$$\max_{\pi_\theta} \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}} \left[\mathbb{E}_{a\sim\pi_\theta(s)} A^{\pi_{\theta_t}}(s,a) \right]$$

$$\text{s.t., } KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \leq \delta$$

=> simplify

We want to maximize local advantage against π_{θ_t}, but we want the new policy to be close to π_{θ_t} (in the KL sense)

How we can actually do the optimization here?
After all, we don't even know the analytical form of trajectory likelihood…

A trust region formulation for policy update:

At iteration t, with π_{θ_t} at hand, we compute θ_{t+1} as follows:

$$\max_{\pi_\theta} \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}} \left[\mathbb{E}_{a\sim\pi_\theta(s)} A^{\pi_{\theta_t}}(s,a) \right]$$ ← Linearize obj at θ_t

$$\text{s.t., } KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \leq \delta$$ ← second-order Taylor-Exp at θ_t

High-level strategy:

1. First-order Taylor expansion on the objective at θ_t
2. second-order Taylor expansion of the constraint at θ_t

Simplify Objective Function

$$\max_{\theta} \mathbb{E}_{s\sim d_{\mu}^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim \underline{\pi_\theta(s)}} A^{\pi_{\theta_t}}(s,a)\right]$$

$$\nabla_\theta \mathbb{E}_{s\sim d_{\mu}^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim \pi_\theta(s)} A^{\pi_{\theta_t}}(s,a)\right]$$

$$\Rightarrow \mathbb{E}_{s\sim d_{\mu}^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim \pi_{\theta_t}(s)} \nabla \ln \pi_{\theta_t}(a|s)\, A^{\pi_{\theta_t}}(s,a)\right]$$

Simplify Objective Function

$$\max_{\theta} \mathbb{E}_{s \sim d_{\mu}^{\pi_{\theta_t}}} \left[\mathbb{E}_{a \sim \pi_\theta(s)} A^{\pi_{\theta_t}}(s, a) \right]$$

Since the objective is also non-linear,
let’s do first order-talyor expansion on it:

Simplify Objective Function

$$\max_{\theta} \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_\theta(s)} A^{\pi_{\theta_t}}(s,a)\right]$$

Since the objective is also non-linear,
let's do first order-talyor expansion on it:

Inner product

$$\mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_\theta(s)} A^{\pi_{\theta_t}}(s,a)\right] \approx \underbrace{\mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\boxed{\mathbb{E}_{a\sim\pi_{\theta_t}(s)} A^{\pi_{\theta_t}}(s,a)}\right]}_{=0} + \underbrace{\mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_{\theta_t}(s)} \nabla_\theta \ln \pi_{\theta_t}(a\,|\,s) A^{\pi_{\theta_t}}(s,a)\right]}_{\nabla_\theta J(\pi_{\theta_t})} \cdot (\theta - \theta_t)$$

Simplify Objective Function

$$\max_{\theta} \mathbb{E}_{s\sim d_{\mu}^{\pi_{\theta_t}}} \left[\mathbb{E}_{a\sim\pi_{\theta}(s)} A^{\pi_{\theta_t}}(s,a) \right] \approx \underbrace{\nabla_{\theta} J(\theta_t)}_{a}^{\top} (\theta - \theta_t)$$

Since the objective is also non-linear,
let's do first order-talyor expansion on it:

$$\mathbb{E}_{s\sim d_{\mu}^{\pi_{\theta_t}}} \left[\mathbb{E}_{a\sim\pi_{\theta}(s)} A^{\pi_{\theta_t}}(s,a) \right] \approx \mathbb{E}_{s\sim d_{\mu}^{\pi_{\theta_t}}} \left[\mathbb{E}_{a\sim\pi_{\theta_t}(s)} A^{\pi_{\theta_t}}(s,a) \right] + \underbrace{\mathbb{E}_{s\sim d_{\mu}^{\pi_{\theta_t}}} \left[\mathbb{E}_{a\sim\pi_{\theta_t}(s)} \nabla_{\theta} \ln \pi_{\theta_t}(a \mid s) A^{\pi_{\theta_t}}(s,a) \right]}_{\nabla_{\theta} J(\pi_{\theta_t})} \cdot (\theta - \theta_t)$$

$$= \nabla_{\theta} J(\pi_{\theta_t})^{\top} (\theta - \theta_t)$$

Simplify Constraint via second-order Taylor Expansion:

Simplify Constraint via second-order Taylor Expansion:

$$KL(\rho_{\theta_t} | \rho_{\theta}) := \ell(\theta)$$

Δ

Simplify Constraint via second-order Taylor Expansion:

$$KL(\rho_{\theta_t} | \rho_\theta) := \ell(\theta)$$

$$\ell(\theta) \approx \underset{0}{\ell(\theta_t)} + \underbrace{\nabla \ell(\theta_t)}_{\text{First}}^\top (\theta - \theta_t) + \frac{1}{2}(\theta - \theta_t)^\top \underbrace{\nabla^2_\theta \ell(\theta_t)}_{\text{Hessian}}(\theta - \theta_t) \quad \cdots$$

Simplify Constraint via second-order Taylor Expansion:

$$KL(\rho_{\theta_t} | \rho_\theta) := \ell(\theta)$$

$$\ell(\theta) \approx \overset{=0}{\ell(\theta_t)} + \nabla \ell(\theta_t)^\top (\theta - \theta_t) + \frac{1}{2}(\theta - \theta_t)^\top \nabla_\theta^2 \ell(\theta_t)(\theta - \theta_t)$$

$$\ell(\theta_t) = KL(\underline{\rho_{\theta_t}} | \underline{\rho_{\theta_t}}) = 0$$

Simplify Constraint via second-order Taylor Expansion:

$$KL(\rho_{\theta_t} | \rho_\theta) := \ell(\theta)$$

$$\ell(\theta) \approx \underbrace{\ell(\theta_t)}_{=0} + \underbrace{\nabla \ell(\theta_t)^\top}_{=0}(\theta - \theta_t) + \frac{1}{2}(\theta - \theta_t)^\top \underline{\nabla_\theta^2 \ell(\theta_t)}(\theta - \theta_t)$$

$$\ell(\theta_t) = KL(\rho_{\theta_t} | \rho_{\theta_t}) = 0$$

We will show that $\nabla_\theta \ell(\theta_t) = 0$, and $\nabla^2 \ell(\theta_t)$ has a nice form!

$$KL(\rho_{\theta_t} | \rho_\theta) \approx \frac{1}{2}(\theta - \theta_t)^T \nabla_\theta^2 \ell(\theta_t)(\theta - \theta_t)$$

The gradient of the KL-divergence is zero at θ_t

Change from trajectory distribution to state-action distribution:

The gradient of the KL-divergence is zero at θ_t

Change from trajectory distribution to state-action distribution:

KL-Div

$$KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) = \mathbb{E}_{\tau \sim \rho_{\pi_{\theta_t}}} \ln \frac{\rho_{\pi_{\theta_t}}(\tau)}{\rho_{\pi_\theta}(\tau)} = \mathbb{E}_{\tau \sim \rho_{\pi_{\theta_t}}} \boxed{\sum_{h=0}^{\cancel{H-1}\ \infty} \ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_\theta(a_h | s_h)}}$$

$$\frac{\rho_{\pi_{\theta_t}}(\tau)}{\rho_{\pi_\theta}(\tau)} = \frac{\cancel{\mu_0(s_0)}\, \pi_{\theta_t}(a_0 | s_0)\, \cancel{P_1(s_1 | s_0, a_0)} \cdots}{\cancel{\mu_0(s_0)}\, \pi_{\theta}(a_0 | s_0)\, \cancel{P_1(s_1 | s_0, a_0)} \cdots}$$

$$\ln\left[\frac{\rho_{\pi_{\theta_t}}(\tau)}{\rho_{\pi_\theta}(\tau)}\right] = \sum_{h=0}^{\infty} \ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_{\theta_t}(a_h | s_h)}$$

The gradient of the KL-divergence is zero at θ_t

Change from trajectory distribution to state-action distribution: ← Finite Horizon setting

$$KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) = \mathbb{E}_{\tau \sim \rho_{\pi_{\theta_t}}} \ln \frac{\rho_{\pi_{\theta_t}}(\tau)}{\rho_{\pi_\theta}(\tau)} = \mathbb{E}_{\tau \sim \rho_{\pi_{\theta_t}}} \sum_{h=0}^{H-1} \ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_\theta(a_h | s_h)}$$

$$= \overset{H\cdot}{\cancel{\frac{1}{1-\gamma}}} \mathbb{E}_{s_h, a_h \sim d_\mu^{\pi_{\theta_t}}} \left[\ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_\theta(a_h | s_h)} \right] := \ell(\theta)$$

The gradient of the KL-divergence is zero at θ_t

Change from trajectory distribution to state-action distribution:

$$KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) = \mathbb{E}_{\tau\sim\rho_{\pi_{\theta_t}}} \ln \frac{\rho_{\pi_{\theta_t}}(\tau)}{\rho_{\pi_\theta}(\tau)} = \mathbb{E}_{\tau\sim\rho_{\pi_{\theta_t}}} \sum_{h=0}^{H-1} \ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_\theta(a_h | s_h)}$$

$$= \overset{H}{\cancel{\frac{1}{1-\gamma}}} \mathbb{E}_{s_h, a_h \sim d_\mu^{\pi_{\theta_t}}} \left[\ln \frac{\pi_{\theta_t}(a_h | s_h)}{\underset{\Delta}{\pi_\theta(a_h | s_h)}} \right] := \ell(\theta)$$

$$\ln\left(\frac{\pi_{\theta_t}}{\pi_\theta}\right) = \underline{\ln \pi_{\theta_t}} - \ln \pi_\theta$$

$$\nabla_\theta \ell(\theta) |_{\theta=\theta_t} = \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}} \sum_a \pi_{\theta_t}(a | s) \left(\underline{-\nabla_\theta \ln \pi_\theta(a_h | s_h) |_{\theta=\theta_t}} \right) = 0$$

The gradient of the KL-divergence is zero at θ_t

Change from trajectory distribution to state-action distribution:

$$KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) = \mathbb{E}_{\tau \sim \rho_{\pi_{\theta_t}}} \ln \frac{\rho_{\pi_{\theta_t}}(\tau)}{\rho_{\pi_\theta}(\tau)} = \mathbb{E}_{\tau \sim \rho_{\pi_{\theta_t}}} \sum_{h=0}^{H-1} \ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_\theta(a_h | s_h)}$$

$$= \frac{1}{1-\gamma} \mathbb{E}_{s_h, a_h \sim d_\mu^{\pi_{\theta_t}}} \left[\ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_\theta(a_h | s_h)} \right] := \ell(\theta)$$

$$\nabla_\theta \ell(\theta) |_{\theta=\theta_t} = \mathbb{E}_{s \sim d_\mu^{\pi_{\theta_t}}} \sum_a \pi_{\theta_t}(a | s) \left(- \underline{\nabla_\theta \ln \pi_\theta(a_h | s_h)} |_{\theta = \theta_t} \right)$$

$$= - \mathbb{E}_{s \sim d_\mu^{\pi_{\theta_t}}} \sum_a \cancel{\pi_{\theta_t}(a | s)} \frac{\nabla_\theta \pi_{\theta_t}(a | s)}{\cancel{\pi_{\theta_t}(a | s)}}$$

$$= - \mathbb{E}_s \nabla_\theta \sum_a \pi_{\theta_t}(a|s)$$

$$= - \mathbb{E}_s \nabla_\theta \cdot 1 = 0$$

The gradient of the KL-divergence is zero at θ_t

Change from trajectory distribution to state-action distribution:

$$KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) = \mathbb{E}_{\tau \sim \rho_{\pi_{\theta_t}}} \ln \frac{\rho_{\pi_{\theta_t}}(\tau)}{\rho_{\pi_\theta}(\tau)} = \mathbb{E}_{\tau \sim \rho_{\pi_{\theta_t}}} \sum_{h=0}^{H-1} \ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_\theta(a_h | s_h)}$$

$$= \frac{1}{1-\gamma} \mathbb{E}_{s_h, a_h \sim d_\mu^{\pi_{\theta_t}}} \left[\ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_\theta(a_h | s_h)} \right] := \ell(\theta)$$

$$\nabla_\theta \ell(\theta) |_{\theta=\theta_t} = \mathbb{E}_{s \sim d_\mu^{\pi_{\theta_t}}} \sum_a \pi_{\theta_t}(a | s) \left(-\nabla_\theta \ln \pi_\theta(a_h | s_h) |_{\theta=\theta_t} \right)$$

$$= -\mathbb{E}_{s \sim d_\mu^{\pi_{\theta_t}}} \sum_a \pi_{\theta_t}(a | s) \frac{\nabla_\theta \pi_{\theta_t}(a | s)}{\pi_{\theta_t}(a | s)} = 0$$

Let's compute the Hessian of the KL-divergence at θ_t

$$\mathbb{E}_{s,a\sim d_\mu^{\pi_{\theta_t}}}\left[\ln \frac{\pi_{\theta_t}(a_h \mid s_h)}{\pi_\theta(a_h \mid s_h)}\right] := \ell(\theta)$$

Let's compute the Hessian of the KL-divergence at θ_t

$$\nabla^2 \quad \mathbb{E}_{s,a\sim d_\mu^{\pi_{\theta_t}}}\left[\ln \frac{\pi_{\theta_t}(a_h \mid s_h)}{\pi_\theta(a_h \mid s_h)}\right] := \ell(\theta)$$

$$\left(\frac{f}{g}\right)' = \frac{f'}{g} - \frac{f \cdot g'}{g^2}$$

$$\ln \pi_{\theta_t} - \ln \pi_\theta$$

$$\nabla_\theta^2 \ell(\theta)\,|_{\theta=\theta_t} = \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}} \sum_a \pi_{\theta_t}(a \mid s)\left(-\underline{\nabla_\theta^2 \ln \pi_\theta(a \mid s)\,|_{\theta=\theta_t}}\right)$$

$$\nabla_\theta \ln \pi_\theta(a|s) = \frac{\nabla_\theta \pi_\theta(a|s)}{\pi_\theta(a|s)}$$

$$\nabla_\theta\left[\nabla_\theta \ln \pi_\theta(a|s)\right] = \nabla_\theta\left[\frac{\nabla_\theta \pi_\theta(a|s)}{\pi_\theta(a|s)}\right]$$

$$= \frac{\nabla_\theta^2 \pi_\theta(a|s)}{\pi_\theta(a|s)} - \frac{\nabla_\theta \pi_\theta(a|s) \cdot \nabla_\theta \pi(a|s)^\top}{\pi_\theta^2(a|s)}$$

Let's compute the Hessian of the KL-divergence at θ_t

$$\mathbb{E}_{s,a\sim d_\mu^{\pi_{\theta_t}}}\left[\ln\frac{\pi_{\theta_t}(a_h\,|\,s_h)}{\pi_\theta(a_h\,|\,s_h)}\right] := \ell(\theta)$$

$$\nabla_\theta^2 \ell(\theta)\,|_{\theta=\theta_t} = \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}} \sum_a \pi_{\theta_t}(a\,|\,s)\left(-\nabla_\theta^2 \ln \pi_\theta(a\,|\,s)\,|_{\theta=\theta_t}\right)$$

$$= -\,\mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\sum_a \pi_{\theta_t}(a\,|\,s)\left(\frac{\nabla_\theta^2\pi_{\theta_t}(\cancel{a}\,|\,s)}{\pi_{\cancel{\theta_t}}(a\,|\,s)} - \frac{\nabla_\theta\pi_{\theta_t}(a\,|\,s)\,\nabla_\theta\pi_{\theta_t}(a\,|\,s)^\top}{\pi_{\theta_t}^2(a\,|\,s)}\right)$$

$=0$

$$\sum_a \pi_{\theta_t}(a|s)\,\frac{\nabla_\theta^2\,\pi_{\theta_t}(a|s)}{\pi_{\theta_t}(a|s)} = \nabla_\theta^2\left[\sum_a \pi_{\theta_t}(a|s)\right] = \nabla_\theta^2\, 1 = 0$$

Let's compute the Hessian of the KL-divergence at θ_t

$$\mathbb{E}_{s,a\sim d_\mu^{\pi_{\theta_t}}}\left[\ln \frac{\pi_{\theta_t}(a_h \mid s_h)}{\pi_\theta(a_h \mid s_h)}\right] := \ell(\theta)$$

$$\nabla_\theta^2 \ell(\theta)\,|_{\theta=\theta_t} = \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}} \sum_a \pi_{\theta_t}(a \mid s)\left(-\nabla_\theta^2 \ln \pi_\theta(a \mid s)\,|_{\theta=\theta_t}\right)$$

$$= -\mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}} \sum_a \pi_{\theta_t}(a \mid s)\left(\frac{\nabla_\theta^2 \pi_{\theta_t}(a \mid s)}{\pi_{\theta_t}(a \mid s)} - \frac{\nabla_\theta \pi_{\theta_t}(a \mid s)\,\nabla_\theta \pi_{\theta_t}(a \mid s)^\top}{\pi_{\theta_t}^2(a \mid s)}\right)$$

$$= \mathbb{E}_{s,a\sim d_\mu^{\pi_{\theta_t}}}\left[\underbrace{\nabla_\theta \ln \pi_{\theta_t}(a \mid s)}_{\in R^{dim\_\theta}}\left(\nabla_\theta \ln \pi_{\theta_t}(a \mid s)\right)^\top\right] \in \mathbb{R}^{dim_\theta \times dim_\theta}$$

Let's compute the Hessian of the KL-divergence at θ_t

$$\mathbb{E}_{s,a\sim d_\mu^{\pi_{\theta_t}}}\left[\ln \frac{\pi_{\theta_t}(a_h \mid s_h)}{\pi_\theta(a_h \mid s_h)}\right] := \ell(\theta)$$

$$\nabla^2_\theta \ell(\theta)\mid_{\theta=\theta_t} = \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}} \sum_a \pi_{\theta_t}(a \mid s)\left(-\nabla^2_\theta \ln \pi_\theta(a \mid s)\mid_{\theta=\theta_t}\right)$$

$$= -\mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}} \sum_a \pi_{\theta_t}(a \mid s)\left(\frac{\nabla^2_\theta \pi_{\theta_t}(a \mid s)}{\pi_{\theta_t}(a \mid s)} - \frac{\nabla_\theta \pi_{\theta_t}(a \mid s)\nabla_\theta \pi_{\theta_t}(a \mid s)^\top}{\pi^2_{\theta_t}(a \mid s)}\right)$$

$$\frac{\nabla_\theta \pi_{\theta_t}(a \mid s)}{\pi_{\theta_t}(a \mid s)}\left[\frac{\nabla_\theta \pi_{\theta_t}(a \mid s)}{\pi_{\theta_t}(a \mid s)}\right]^\top$$

$$= \mathbb{E}_{s,a\sim d_\mu^{\pi_{\theta_t}}}\left[\nabla_\theta \ln \pi_{\theta_t}(a \mid s)\left(\nabla_\theta \ln \pi_{\theta_t}(a \mid s)\right)^\top\right] \in \mathbb{R}^{dim_\theta \times dim_\theta}$$

It's called fisher Information Matrix!

Summary so far:

We did second-order Taylor expansion on the KL constraint, and we get:

$$\frac{1}{H} KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \approx \frac{1}{2}(\theta - \theta_t)^\top F_{\theta_t}(\theta - \theta_t)$$

$$F_{\theta_t} := \mathbb{E}_{s,a \sim d_\mu^{\pi_{\theta_t}}}\left[\nabla_\theta \ln \pi_{\theta_t}(a \mid s)\left(\nabla_\theta \ln \pi_{\theta_t}(a \mid s)\right)^\top\right] \in \mathbb{R}^{dim_\theta \times dim_\theta}$$

Summary so far:

We did second-order Taylor expansion on the KL constraint, and we get:

$$\frac{1}{H} KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \approx \frac{1}{2}(\theta - \theta_t)^\top F_{\theta_t}(\theta - \theta_t)$$

$$F_{\theta_t} := \mathbb{E}_{s,a \sim d_\mu^{\pi_{\theta_t}}}\left[\nabla_\theta \ln \pi_{\theta_t}(a | s)\left(\nabla_\theta \ln \pi_{\theta_t}(a | s)\right)^\top\right] \in \mathbb{R}^{dim_\theta \times dim_\theta}$$

This leads to the following much simplified constrained optimization:

Summary so far:

We did second-order Taylor expansion on the KL constraint, and we get:

$$\frac{1}{H} KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \approx \frac{1}{2}(\theta - \theta_t)^\top F_{\theta_t}(\theta - \theta_t)$$

$xx^\top$

PSD

$$F_{\theta_t} := \mathbb{E}_{s,a \sim d_\mu^{\pi_{\theta_t}}}\left[\underbrace{\nabla_\theta \ln \pi_{\theta_t}(a|s)\left(\nabla_\theta \ln \pi_{\theta_t}(a|s)\right)^\top}_{\text{PSD}}\right] \in \mathbb{R}^{dim_\theta \times dim_\theta}$$

This leads to the following much simplified constrained optimization:

PG

$$\max_\theta \nabla_\theta J(\pi_{\theta_t})^\top \left(\theta - \theta_t\right)$$

$$\text{s.t. } (\theta - \theta_t)^\top F_{\theta_t}(\theta - \theta_t) \le \delta$$

PSD

Outlines

1. Quick intro on KL-divergence

2. A Trust-Region Formulation for Policy Optimization

$$\mathbb{E}_{s,a}\left[\nabla_\theta \ln \pi(a|s) \nabla_\theta \ln \pi(a|s)^T\right]$$

is PSD

3. Algorithm: Natural Policy Gradient

Put everything together, we get:

At iteration t, we update to θ_{t+1} via:

$$\max_{\theta} \nabla_\theta J(\pi_{\theta_t})^\top (\theta - \theta_t)$$

$$\text{s.t. } (\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t) \leq \delta$$

Put everything together, we get:

At iteration t, we update to θ_{t+1} via:

$$\max_{\theta} \nabla_\theta J(\pi_{\theta_t})^\top (\theta - \theta_t)$$

$$\text{s.t. } (\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t) \leq \delta$$

Linear objective and quadratic convex constraint, we can solve it optimally!

Put everything together, we get:

At iteration t, we update to θ_{t+1} via:

$$\max_{\theta} \nabla_\theta J(\pi_{\theta_t})^\top (\theta - \theta_t)$$

invertible

$F_{\theta_t} + \lambda I$

1e-7

$$\text{s.t. } (\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t) \leq \delta$$

Linear objective and quadratic convex constraint, we can solve it optimally!

Indeed this gives us:

$$\theta_{t+1} = \theta_t + \eta F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t})$$

Put everything together, we get:

At iteration t, we update to θ_{t+1} via:

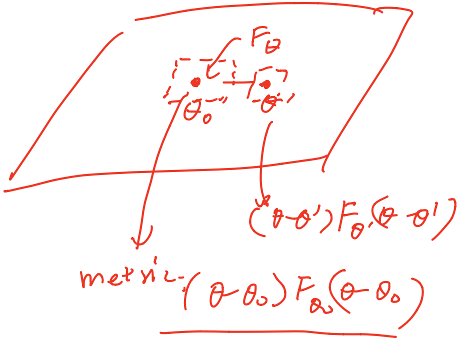
$\max_{\theta} \nabla J^{\top}(\theta - \theta_t)$

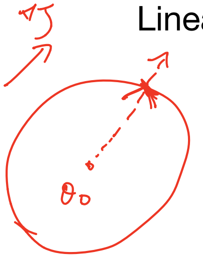
$$\max_{\theta} \nabla_\theta J(\pi_{\theta_t})^\top (\theta - \theta_t)$$

$(\theta - \theta_0) \,//\, \nabla J$

$$\text{s.t. } (\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t) \leq \delta$$

Δ





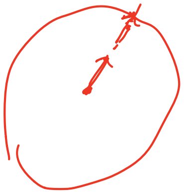
Linear objective and quadratic convex constraint, we can solve it optimally!

Indeed this gives us:

$$\theta_{t+1} = \theta_t + \eta F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t})$$

$$\text{Where } \eta = \sqrt{\frac{\delta}{\nabla_\theta J(\pi_{\theta_t})^\top F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t})}}$$

Δ



Algorithm: Natural Policy Gradient

Initialize θ_0

For t = 0, …

Algorithm: Natural Policy Gradient

Initialize θ_0

For t = 0, …

Estimate PG $\nabla_\theta J(\pi_{\theta_t})$

Algorithm: Natural Policy Gradient

Initialize θ_0

For t = 0, …

Estimate PG $\nabla_\theta J(\pi_{\theta_t})$

finite # of samples

Estimate Fisher info-matrix $F_{\theta_t} := \mathbb{E}_{s,a \sim d_\mu^{\pi_{\theta_t}}} \nabla_\theta \ln \pi_{\theta_t}(a \mid s) (\nabla_\theta \ln \pi_{\theta_t}(a \mid s))^\top$

Algorithm: Natural Policy Gradient

Initialize θ_0

For t = 0, …

Estimate PG $\nabla_\theta J(\pi_{\theta_t})$

Estimate Fisher info-matrix $F_{\theta_t} := \mathbb{E}_{s,a\sim d_\mu^{\pi_{\theta_t}}} \nabla_\theta \ln \pi_{\theta_t}(a \mid s)(\nabla_\theta \ln \pi_{\theta_t}(a \mid s))^\top$

Natural Gradient Ascent: $\theta_{t+1} = \theta_t + \eta \underbrace{F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t})}_{\text{Natural Gradient}}$

Algorithm: Natural Policy Gradient

Initialize θ_0

For t = 0, …

$\nabla_\theta J(\pi_\theta)\big|_{\theta=\theta_t}$

Estimate PG $\nabla_\theta J(\pi_{\theta_t})$

Estimate Fisher info-matrix $F_{\theta_t} := \mathbb{E}_{s,a\sim d_\mu^{\pi_{\theta_t}}} \nabla_\theta \ln \pi_{\theta_t}(a\,|\,s)(\nabla_\theta \ln \pi_{\theta_t}(a\,|\,s))^\top$

Natural Gradient Ascent: $\theta_{t+1} = \theta_t + \eta F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t})$

Tune 1e-2

Where $\eta = \sqrt{\dfrac{\delta}{\nabla_\theta J(\pi_{\theta_t})^\top F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t})}}$

Algorithm: Natural Policy Gradient

Initialize θ_0

For t = 0, …

Estimate PG $\nabla_\theta J(\pi_{\theta_t})$

Estimate Fisher info-matrix $F_{\theta_t} := \mathbb{E}_{s,a \sim d_\mu^{\pi_{\theta_t}}} \nabla_\theta \ln \pi_{\theta_t}(a \mid s) (\nabla_\theta \ln \pi_{\theta_t}(a \mid s))^\top$

Natural Gradient Ascent: $\theta_{t+1} = \theta_t + \eta F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t})$

Where $\eta = \sqrt{\dfrac{\delta}{\nabla_\theta J(\pi_{\theta_t})^\top F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t})}}$

(We will implement it in HW2 on Cartpole)

Summary for today:

Trust Region Policy Optimization and NPG

At iteration t:

$$\max_{\pi_\theta} \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_\theta(s)} A^{\pi_{\theta_t}}(s,a)\right]$$

$$\text{s.t., } KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \leq \delta$$

Intuition: maximize local adv subject
to being incremental (in KL);

Summary for today:

Trust Region Policy Optimization and NPG

At iteration t:

$$\max_{\pi_\theta} \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_\theta(s)} A^{\pi_{\theta_t}}(s,a)\right] \longrightarrow$$ First-order Taylor expansion at θ_t

$$\text{s.t., } KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \le \delta \longrightarrow$$ second-order Taylor expansion at θ_t

Intuition: maximize local adv subject
to being incremental (in KL);

Summary for today:

Trust Region Policy Optimization and NPG

At iteration t:

$$\max_{\pi_\theta} \mathbb{E}_{s \sim d_\mu^{\pi_{\theta_t}}} \left[\mathbb{E}_{a \sim \pi_\theta(s)} A^{\pi_{\theta_t}}(s,a) \right]$$ → First-order Taylor expansion at θ_t

s.t., $KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \leq \delta$ → second-order Taylor expansion at θ_t

Intuition: maximize local adv subject to being incremental (in KL);

$$\max_{\theta} \nabla_\theta J(\pi_{\theta_t})^\top (\theta - \theta_t)$$

$$\text{s.t. } \underbrace{(\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t)}_{\text{symmetric}} \leq \delta$$

Summary for today:

Trust Region Policy Optimization and NPG

At iteration t:

$$\max_{\pi_\theta} \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_\theta(s)} A^{\pi_{\theta_t}}(s,a)\right] \longrightarrow$$ First-order Taylor expansion at θ_t

$$\text{s.t., } KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \le \delta \longrightarrow$$ second-order Taylor expansion at θ_t

Intuition: maximize local adv subject to being incremental (in KL);

$$\max_{\theta} \nabla_\theta J(\pi_{\theta_t})^\top (\theta - \theta_t)$$

$$\text{s.t. } (\theta - \theta_t)^\top F_{\theta_t}(\theta - \theta_t) \le \delta$$

(Exercise: work out the $\arg\max_\theta$)

Summary for today:

Trust Region Policy Optimization and NPG

At iteration t:

$$\max_{\pi_\theta} \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_\theta(s)} A^{\pi_{\theta_t}}(s,a)\right] \longrightarrow$$ First-order Taylor expansion at θ_t

s.t., $KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \le \delta \longrightarrow$ second-order Taylor expansion at θ_t

Intuition: maximize local adv subject to being incremental (in KL);

$$\max_\theta \nabla_\theta J(\pi_{\theta_t})^\top(\theta - \theta_t)$$

s.t. $(\theta-\theta_t)^\top F_{\theta_t}(\theta-\theta_t) \le \delta$

$$\theta_{t+1} = \theta_t + \underbrace{\eta F_{\theta_t}^{-1}\nabla_\theta J(\pi_{\theta_t})}_{\text{NPG}} \longleftarrow$$

(Exercise: work out the $\arg\max_\theta$)

$$F^{-1} g \qquad F = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{bmatrix}$$

≈ 0

$$= \begin{bmatrix} \frac{1}{\sigma_1} g_1 \\ \vdots \\ \frac{1}{\sigma_d} g_d \end{bmatrix}$$

$\sigma_d = 0.1$

$$KL(P|Q) = \mathop{E}_{x\sim P} \ln \frac{P(x)}{Q(x)}$$

$$KL(U|Q) = \mathop{E}_{x\sim U} \ln \frac{U(x)}{Q(x)}$$

$$= \underline{\mathop{E}_{x\sim U} \ln U(x)} - \underline{\mathop{E}_{x\sim U} \ln Q(x)}$$

$$KL(Q|U) = \mathop{E}_{x\sim Q} \ln Q(x) - \boxed{\mathop{E}_{x\sim Q} \ln U(x)}$$

$$= \mathop{E}_{x\sim Q} \ln Q(x) - \ln \frac{1}{|X|}$$

$$KL(P|Q) = \mathop{E}_{x\sim P} \ln \frac{P(x)}{Q(x)}$$