Trust Region Policy Optimization & NPG
Recall on NPG:

At iteration $t$:

$$\max_{\pi_\theta} \mathbb{E}_{s \sim d_{\mu}} \mathbb{E}_{a \sim \pi_\theta(s)} A_{\pi_\theta}(s, a) .$$

s.t., $KL\left(\rho_{\pi_\theta} \mid \rho_{\pi_\theta}\right) \leq \delta$

Intuition: maximize local adv subject to being incremental (in KL);
Recap on NPG:

At iteration $t$:

$$\max_{\pi_{\theta}} \mathbb{E}_{s \sim d_{\mu \pi_{\theta}}} \left[ \mathbb{E}_{a \sim \pi_{\theta}(s)} A_{\pi_{\theta}}(s, a) \right]$$

s.t., $KL\left( \rho_{\pi_{\theta_{t}}} \mid \rho_{\pi_{\theta}} \right) \leq \delta$

Intuition: maximize local adv subject to being incremental (in KL);

First-order Taylor expansion at $\theta_t$

Second-order Taylor expansion at $\theta_t$
Recap on NPG:

At iteration $t$:

$$\max_{\pi_{\theta}} \mathbb{E}_{s \sim d_{\pi_{\theta}t}} \left[ \mathbb{E}_{a \sim \pi_{\theta}(s)} A_{\pi_{\theta}t}(s, a) \right]$$

subject to

$$KL \left( \rho_{\pi_{\theta}t} \| \rho_{\pi_{\theta}} \right) \leq \delta$$

Intuition: maximize local adv subject to being incremental (in KL);

- First-order Taylor expansion at $\theta_t$
- Second-order Taylor expansion at $\theta_t$

$$\max_{\theta} \nabla_{\theta} J(\pi_{\theta})^T (\theta - \theta_t)$$

subject to

$$(\theta - \theta_t)^T F_{\theta t} (\theta - \theta_t) \leq \delta$$
Recap on NPG:

At iteration $t$:

$$\max_{\pi_\theta} \mathbb{E}_{s \sim d_{\pi_\theta}} \left[ \mathbb{E}_{a \sim \pi_\theta(s)} A_{\pi_\theta}(s, a) \right]$$

First-order Taylor expansion at $\theta_t$

s.t., $KL\left( \rho_{\pi_{\theta_i}} | \rho_{\pi_\theta} \right) \leq \delta$

second-order Taylor expansion at $\theta_t$

Intuition: maximize local adv subject to being incremental (in KL):

$$\theta_{t+1} = \theta_t + \eta F_{\theta_t}^{-1} \nabla J(\pi_{\theta_t})$$

NPG

$$\max_{\theta} \nabla J(\pi_{\theta_t}) \top (\theta - \theta_t)$$

s.t. $$(\theta - \theta_t) \top F_{\theta_t}(\theta - \theta_t) \leq \delta$$
Recap on NPG:

At iteration $t$:

$$\max_{\pi_{\theta}} \mathbb{E}_{s \sim d_\mu} \left[ \mathbb{E}_{a \sim \pi_{\theta}(s)} A_{\pi_{\theta}}(s, a) \right]$$

s.t., $KL\left( \rho_{\pi_{\theta}} \| \rho_{\theta} \right) \leq \delta$

Intuition: maximize local adv subject to being incremental (in KL);

$\theta_{t+1} = \theta_t + \eta F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t})$

NPG

$\max_{\theta} \nabla_\theta J(\pi_{\theta_t})^T(\theta - \theta_t)$

s.t. $(\theta - \theta_t)^T F_{\theta_t}(\theta - \theta_t) \leq \delta$

$F_{\theta_t} := \mathbb{E}_{s,a \sim d_\mu} \left[ \nabla_\theta \ln \pi_{\theta_t}(a | s) \left( \nabla_\theta \ln \pi_{\theta_t}(a | s) \right)^T \right] \in \mathbb{R}^{\text{dim}_\theta \times \text{dim}_\theta}$

First-order Taylor expansion at $\theta_t$

second-order Taylor expansion at $\theta_t$
Outline for Today:

1. Derivation of the closed-form NPG update

2. Intuitive Explanation of Natural (Policy) Gradient

3. Review of Policy Optimization (API, CPI, PG, and NPG) & a new algorithm
At iteration \( t \), NPG solves a convex constrained optimization problem:

\[
\max_{\theta} \nabla_{\theta} J(\pi_{\theta_t})^T (\theta - \theta_t)
\]

s.t. \((\theta - \theta_t)^T F_{\theta_t}(\theta - \theta_t) \leq \delta\)
At iteration $t$, NPG solves a convex constrained optimization problem:

$$\begin{align*}
\max_{\theta} & \quad \nabla_{\theta} J(\pi_{\theta_t})^T (\theta - \theta_t) \\
\text{s.t.} & \quad (\theta - \theta_t)^T F_{\theta_t} (\theta - \theta_t) \leq \delta
\end{align*}$$

Notation simplification
At iteration $t$, NPG solves a convex constrained optimization problem:

$$\max_{\theta} \nabla_\theta J(\pi_\theta)\top (\theta - \theta_t)$$

s.t. $(\theta - \theta_t)\top F_{\theta_t}(\theta - \theta_t) \leq \delta$

Notation simplification

$$\max_{\Delta} \nabla^\top \Delta,$$

s.t. $\Delta^\top F \Delta \leq \delta$
At iteration $t$, NPG solves a convex constrained optimization problem:

$$\max_{\theta} \nabla_{\theta} J(\pi_{\theta_t})^\top (\theta - \theta_t) \quad \text{s.t.} \quad (\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t) \leq \delta$$

Notation simplification

$$\max_{\Delta} \nabla^T \Delta, \quad \text{s.t.} \quad \Delta^T F \Delta \leq \delta$$

$$\widetilde{\Delta} := F^{1/2} \Delta$$

$$F^t := \sqrt{F}$$

$$F = u \Xi u^\top$$

$$F^t = u \Xi u^\top$$

$$(F^t)^2 = F$$
At iteration $t$, NPG solves a convex constrained optimization problem:

\[
\begin{align*}
\max_{\theta} \nabla_{\theta} J(\pi_{\theta_t})^T (\theta - \theta_t) \\
\text{s.t.} \ (\theta - \theta_t)^T F_{\theta_t} (\theta - \theta_t) \leq \delta
\end{align*}
\]

Notation simplification

\[
\begin{align*}
\max_{\Delta} \nabla^T \Delta, \\
\text{s.t.} \ (\Delta^T F \Delta) \leq \delta \\
\Rightarrow \ (F_{\theta_t})^T \Delta \\
\Delta := F^{1/2} \Delta
\end{align*}
\]

\[
\begin{align*}
&= (F^{1/2} \Delta)^T (F^{1/2} \Delta) \\
&= \widehat{A}^T \widehat{\Delta} \\
&\text{s.t.} \ (\Delta^T \Delta) \leq \delta
\end{align*}
\]
At iteration $t$, NPG solves a convex constrained optimization problem:

$$\max_{\theta} \nabla_{\theta} J(\pi_{\theta_t})(\theta - \theta_t) \quad \text{s.t.} \quad (\theta - \theta_t)^T F_{\theta_t}(\theta - \theta_t) \leq \delta$$

Notation simplification:

$$\max_{\Delta} \nabla^T \Delta, \quad \text{s.t.} \quad \Delta^T F \Delta \leq \delta$$

$$\hat{\Delta} := F^{1/2}$$

$$\max_{\hat{\Delta}} (F^{-1/2} \nabla)^T \hat{\Delta}, \quad \text{s.t.} \quad \hat{\Delta}^T \hat{\Delta} \leq \delta$$
At iteration $t$, NPG solves a convex constrained optimization problem:

$$\max_{\theta} \nabla_{\theta} J(\pi_{\theta_t})^T(\theta - \theta_t)$$

$$\text{s.t. } (\theta - \theta_t)^T F_{\theta_t}(\theta - \theta_t) \leq \delta$$

Notation simplification:

$$\max_{\Delta} \nabla^T \Delta,$$

$$\text{s.t. } \Delta^T F \Delta \leq \delta$$

$$\tilde{\Delta} := F^{1/2} \Delta$$

$$\max_{\tilde{\Delta}} (F^{-1/2} \nabla)^T \tilde{\Delta},$$

$$\text{s.t. } \tilde{\Delta}^T \tilde{\Delta} \leq \delta$$

Radius = $\sqrt{\delta}$
At iteration \( t \), NPG solves a convex constrained optimization problem:

\[
\begin{align*}
\max_{\theta} & \quad \nabla_{\theta} J(\pi_{\theta_t})^T (\theta - \theta_t) \\
\text{s.t.} & \quad (\theta - \theta_t)^T F_{\theta_t} (\theta - \theta_t) \leq \delta
\end{align*}
\]

Notation simplification:

\[
\begin{align*}
\max_{\Delta} & \quad \nabla^T \Delta, \\
\text{s.t.} & \quad \Delta^T F \Delta \leq 2^{-1} \delta
\end{align*}
\]

\[
\tilde{\Delta} := F^{1/2} \Delta
\]

\[
\begin{align*}
\max_{\tilde{\Delta}} & \quad (F^{-1/2} \nabla)^T \tilde{\Delta}, \\
\text{s.t.} & \quad \tilde{\Delta}^T \tilde{\Delta} \leq 2^{-1} \delta
\end{align*}
\]
At iteration $t$, NPG solves a convex constrained optimization problem:

$$\begin{align*}
\max_{\theta} & \nabla_{\theta} J(\pi_{\theta_t})^T (\theta - \theta_t) \\
\text{s.t.} & \ (\theta - \theta_t)^T F_{\theta_t} (\theta - \theta_t) \leq \delta
\end{align*}$$

Notation simplification:

$$\begin{align*}
\max_{\Delta} & \ \nabla^T \Delta, \\
\text{s.t.} & \ \Delta^T F \Delta \leq \delta
\end{align*}$$

Diagram:

$$\tilde{\Delta}_{\text{max}} = \eta F^{-1/2} \nabla$$

$$\| \eta F^{-1/2} \nabla \|_2 = \sqrt{\delta}$$

$$\tilde{\Delta} := F^{1/2} \Delta$$

$$\begin{align*}
\max_{\tilde{\Delta}} & \ (F^{-1/2} \nabla)^T \tilde{\Delta}, \\
\text{s.t.} & \ \tilde{\Delta}^T \tilde{\Delta} \leq \delta
\end{align*}$$
At iteration $t$, NPG solves a convex constrained optimization problem:

$$\begin{align*}
\max_{\theta} & \quad \nabla_\theta J(\pi_{\theta_t})^T (\theta - \theta_t) \\
\text{s.t.} & \quad (\theta - \theta_t)^T F_{\theta_t} (\theta - \theta_t) \leq \delta
\end{align*}$$

Notation simplification:

$$\begin{align*}
\max_{\Delta} & \quad \nabla^T \Delta, \\
\text{s.t.} & \quad \Delta^T F \Delta \leq \delta
\end{align*}$$

$$\tilde{\Delta} := F^{1/2} \Delta$$

$$\begin{align*}
\max_{\tilde{\Delta}} & \quad (F^{-1/2} \nabla)^T \tilde{\Delta}, \\
\text{s.t.} & \quad \tilde{\Delta}^T \tilde{\Delta} \leq \delta
\end{align*}$$

Diagram:

- $\tilde{\Delta}_{\text{max}} = \eta F^{-1/2} \nabla$
- $\|\eta F^{-1/2} \nabla\|_2 = \sqrt{\delta}$
- $\Rightarrow \eta = \sqrt{\frac{\delta}{\nabla^T F^{-1} \nabla}}$
- radius = $\sqrt{\delta}$
At iteration $t$, NPG solves a convex constrained optimization problem:

$$\max_{\theta} \nabla_{\theta} J(\pi_{\theta_t})^T (\theta - \theta_t)$$

subject to $(\theta - \theta_t)^T F_{\theta_t} (\theta - \theta_t) \leq \delta$

Notation simplification:

$$\max_{\Delta} \nabla^T \Delta,$$

subject to $\Delta^T F \Delta \leq \delta$

$$\widetilde{\Delta} := F^{1/2} \Delta$$

$$\max_{\widetilde{\Delta}} \left( F^{-1/2} \nabla \right)^T \widetilde{\Delta},$$

subject to $\widetilde{\Delta}^T \widetilde{\Delta} \leq \delta$

Notation simplification:

$$\widetilde{\Delta} := \frac{1}{\sqrt{\delta}} \Delta$$

$$\Delta = F^{1/2} \widetilde{\Delta}$$

$\Delta_{\max}$ simplification:

$$\|\eta F^{-1/2} \nabla\|_2 = \sqrt{\delta}$$

$$\Rightarrow \eta = \sqrt{\frac{\delta}{\nabla^T F^{-1} \nabla}}$$

$\widetilde{\Delta}_{\max}$:

$$\widetilde{\Delta}_{\max} = \sqrt{\frac{\delta}{\nabla^T F^{-1} \nabla}} F^{-1/2} \nabla$$
At iteration $t$, NPG solves a convex constrained optimization problem:

$$\max_{\theta} \nabla_{\theta} J(\pi_{\theta_t})^T(\theta - \theta_t)$$

s.t. $(\theta - \theta_t)^T F_{\theta_t}(\theta - \theta_t) \leq \delta$

Notation simplification

$$\max_{\Delta} \nabla^T \Delta,$$

s.t. $\Delta^T F \Delta \leq \delta$

$$\widetilde{\Delta} := F^{1/2} \Delta$$

$$\max_{\widetilde{\Delta}} \left( F^{-1/2} \nabla \right)^T \widetilde{\Delta},$$

s.t. $\widetilde{\Delta}^T \widetilde{\Delta} \leq \delta$

$$\Delta_{max} := \sqrt{\frac{\delta}{\nabla^T F^{-1} \nabla}} F^{-1/2} \nabla$$
At iteration $t$, NPG solves a convex constrained optimization problem:

$$\max_{\theta} \nabla_{\theta} J(\pi_{\theta_t})^T (\theta - \theta_t)$$

s.t. $(\theta - \theta_t)^T F_{\theta_t}(\theta - \theta_t) \leq \delta$

A more standard and straightforward way is to use Lagrange multiplier $\lambda \leq 0$: 

At iteration $t$, NPG solves a convex constrained optimization problem:

$$\max_{\theta} \nabla_{\theta} J(\pi_{\theta_t})^T(\theta - \theta_t)$$

s.t. $(\theta - \theta_t)^T F_{\theta_t}(\theta - \theta_t) \leq \delta$

A more standard and straightway is to use Lagrange multiplier $\lambda \leq 0$:

$$\min_{\lambda \leq 0} \max_{\theta} \nabla_{\theta} J(\pi_{\theta_t})^T(\theta - \theta_t) + \lambda \left( (\theta - \theta_t)^T F_{\theta_t}(\theta - \theta_t) - \delta \right)$$
At iteration $t$, NPG solves a convex constrained optimization problem:

$$\max_{\theta} \nabla_{\theta} J(\pi_{\theta_t})^T(\theta - \theta_t)$$

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A more standard and straightway is to use Lagrange multiplier $\lambda \leq 0$:

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(This is optional: Lagrange formulation is out of scope)
At iteration $t$, NPG solves a convex constrained optimization problem:

$$
\max_\theta \nabla_\theta J(\pi_{\theta_t})^T(\theta - \theta_t)
\quad \text{s.t. } (\theta - \theta_t)^T F_{\theta_t} (\theta - \theta_t) \leq \delta
$$

A more standard and straightforward way is to use Lagrange multiplier $\lambda \leq 0$:

$$
\min_{\lambda \leq 0} \max_\theta \nabla_\theta J(\pi_{\theta_t})^T(\theta - \theta_t) + \lambda \left( (\theta - \theta_t)^T F_{\theta_t} (\theta - \theta_t) - \delta \right)
$$

(This is optional: Lagrange formulation is out of scope)

Summary: at this stage, we complete the NPG algorithm derivation
Outline for Today:

1. Derivation of the closed-form NPG update

2. Intuitive Explanation of Natural (Policy) Gradient

3. Review of Policy Optimization (API, CPI, PG, and NPG) & a new algorithm
NPG update: $\theta_1 = \theta_0 + \eta F_{\theta_0}^{-1} \nabla_{\theta_0} \theta_0$

Natural Gradient
NPG update: \( \theta_1 = \theta_0 + \eta F_{\theta_0}^{-1} \nabla \theta_0 \)

\[
KL \left( \rho_{\pi_{\theta_0}} | \rho_{\pi_{\theta}} \right) \leq \delta \Rightarrow (\theta - \theta_0)^\top F_{\theta_0} (\theta - \theta_0) \leq \delta
\]

metric, \( F_{\theta_0} \) is PD
NPG update: $\theta_1 = \theta_0 + \eta F_{\theta_0}^{-1} \nabla \theta_0$

$$KL \left( \rho_{\pi_{\theta_0}} \mid \rho_{\pi_{\theta}} \right) \leq \delta \Rightarrow (\theta - \theta_0)^\top F_{\theta_0} (\theta - \theta_0) \leq \delta$$

Our goal is to make sure two distributions do not change too much, but parameters $\theta$ could potentially change a lot!
NPG update: $\theta_1 = \theta_0 + \eta F_{\theta_0}^{-1} \nabla \theta_0$

$$KL \left( \frac{\rho_{\pi_0}}{\rho_{\pi_0}} \right) \leq \delta \Rightarrow (\theta - \theta_0)^\top F_{\theta_0} (\theta - \theta_0) \leq \delta$$

Our goal is to make sure two distributions do not change too much, but parameters $\theta$ could potentially change a lot!

Consider special case where $F_{\theta_0}$ is a diagonal matrix: $F_{\theta_0} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$
**NPG update:** \( \theta_1 = \theta_0 + \eta F_{\theta_0}^{-1} \nabla \theta_0 \)

\[
KL \left( \rho_{\pi_{\theta_0}} \middle| \rho_{\pi_{\theta}} \right) \leq \delta \Rightarrow (\theta - \theta_0)^{\top} F_{\theta_0} (\theta - \theta_0) \leq \delta
\]

Our goal is to make sure two distributions do not change too much, but parameters \( \theta \) could potentially change a lot!

Consider special case where \( F_{\theta_0} \) is a diagonal matrix: \( F_{\theta_0} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \)

\[ \forall i : \; \theta_1[i] = \theta_0[i] + (\eta \sigma_i^{-1}) \nabla_{\theta_0}[i] \]
NPG update: \( \theta_1 = \theta_0 + \eta F_{\theta_0}^{-1} \nabla \theta_0 \)

\[
KL \left( \rho_{\pi_{\theta_0}} | \rho_{\pi_\theta} \right) \leq \delta \Rightarrow (\theta - \theta_0)^\top F_{\theta_0} (\theta - \theta_0) \leq \delta
\]

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\[
\forall i : \theta_1[i] = \theta_0[i] + (\eta \sigma_i^{-1}) \nabla \theta_0[i]
\]

For tiny \( \sigma_i \), we indeed have a huge learning rate, i.e., \( \eta \sigma_i^{-1} \), at coordinate \( i \)!
NPG update: $\theta_1 = \theta_0 + \eta F_{\theta_0}^{-1} \nabla \theta_0$

$$KL \left( \rho_{\pi_0} \mid \rho_{\pi_0} \right) \leq \delta \Rightarrow (\theta - \theta_0)\left( F_{\theta_0}\right) (\theta - \theta_0) \leq \delta$$

Our goal is to make sure two distributions do not change too much, but parameters $\theta$ could potentially change a lot!

Consider special case where $F_{\theta_0}$ is a diagonal matrix: $F_{\theta_0} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$

$$\forall i : \theta_1[i] = \theta_0[i] + (\eta \sigma_i^{-1}) \nabla \theta_0[i]$$

For tiny $\sigma_i$, we indeed have a huge learning rate, i.e., $\eta \sigma_i^{-1}$, at coordinate $i$!

In other words, NPG allows a big jump on some coordinates which do not affect KL-div too much.
Example of Natural Gradient on 1-d problem:

\[ p_{\theta} = \left( \frac{\exp(\theta)}{1 + \exp(\theta)}, \frac{1}{1 + \exp(\theta)} \right) \]

\[ g(\theta) = 100 \cdot p_{\theta}[1] + 1 \cdot p_{\theta}[2] \]

\[ \Delta \theta^+ = \arg\max_\theta g(\theta) \]

\[ p_{\theta^+}[1] = 1 \]

\[ p_{\theta^+}[2] = 0 \]
Example of Natural Gradient on 1-d problem:

\[
p_\theta = \left( \frac{\exp(\theta)}{1 + \exp(\theta)} , \frac{1}{1 + \exp(\theta)} \right)
\]

\[
g(\theta) = 100 \cdot p_\theta[1] + 1 \cdot p_\theta[2]
\]

\[
\theta^* = \arg\max_\theta g(\theta)
\]

\[
\theta' = \theta + g'(\theta)
\]
Example of Natural Gradient on 1-d problem:

\[ p_\theta = \left( \frac{\exp(\theta)}{1 + \exp(\theta)}, \frac{1}{1 + \exp(\theta)} \right) \]

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Example of Natural Gradient on 1-d problem:

\[ p_\theta = \left( \frac{\exp(\theta)}{1 + \exp(\theta)}, \frac{1}{1 + \exp(\theta)} \right) \]

\[ g(\theta) = 100 \cdot p_\theta[1] + 1 \cdot p_\theta[2] \]

\[ (p_{\theta_0[1]}, p_{\theta_0[2]}) := \left( \frac{\exp(\theta_0)}{1 + \exp(\theta_0)}, \frac{1}{1 + \exp(\theta_0)} \right) = \left( \frac{1}{2}, \frac{1}{2} \right) \]
Example of Natural Gradient on 1-d problem:

\[ p_\theta = \left( \frac{\exp(\theta)}{1 + \exp(\theta)}, \frac{1}{1 + \exp(\theta)} \right) \]

\[ g(\theta) = 100 \cdot p_\theta[1] + 1 \cdot p_\theta[2] \]

\[ \kappa_l \left( p_{\theta_0} \parallel p_\theta \right) \bigg| \theta = \theta_0 \]

Fisher information scalar:
\[ f_{\theta_0} = \frac{\exp(\theta_0)}{(1 + \exp(\theta_0))^2} \]
Example of Natural Gradient on 1-d problem:

\[
p_\theta = \left( \frac{\exp(\theta)}{1 + \exp(\theta)}, \frac{1}{1 + \exp(\theta)} \right)
\]

\[
g(\theta) = 100 \cdot p_\theta[1] + 1 \cdot p_\theta[2]
\]

Fisher information scalar: \( f_\theta = \frac{\exp(\theta)}{(1 + \exp(\theta))^2} \)

Hence: \( f_\theta \to 0^+, \text{ as } \theta \to \infty \)

\[
\lim_{\theta \to \infty} \frac{\exp(\theta)}{(1 + \exp(\theta))^2} = 0^+
\]
Example of Natural Gradient on 1-d problem:

\[ p_{\theta} = \left( \frac{\exp(\theta)}{1 + \exp(\theta)}, \frac{1}{1 + \exp(\theta)} \right) \]

\[ g(\theta) = 100 \cdot p_{\theta}[1] + 1 \cdot p_{\theta}[2] \]

Fisher information scalar:
\[ f_{\theta_0} = \frac{\exp(\theta_0)}{(1 + \exp(\theta_0))^2} \]

Hence: \( f_{\theta_0} \to 0^+ \), as \( \theta_0 \to \infty \)

NPG:
\[ \theta_1 = \theta_0 + \eta \frac{g'(\theta_0)}{f_{\theta_0}} \]

\[ \theta_1 \to \infty^+, \quad \theta_0 \to \infty^+ \]
Example of Natural Gradient on 1-d problem:

\[
p_\theta = \left( \frac{\exp(\theta)}{1 + \exp(\theta)}, \frac{1}{1 + \exp(\theta)} \right)
\]

\[
g(\theta) = 100 \cdot p_\theta[1] + 1 \cdot p_\theta[2]
\]

Fisher information scalar:

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f_{\theta_0} = \frac{\exp(\theta_0)}{(1 + \exp(\theta_0))^2}
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Hence: \( f_{\theta_0} \to 0^+ \), as \( \theta_0 \to \infty \)

NPG: \( \theta_1 = \theta_0 + \eta \frac{g'(\theta_0)}{f_{\theta_0}} \)

GA: \( \theta_1 = \theta_0 + \eta g'(\theta_0) \)
Example of Natural Gradient on 1-d problem:

\[
p_\theta = \left( \frac{\exp(\theta)}{1 + \exp(\theta)}, \frac{1}{1 + \exp(\theta)} \right)
\]

\[g(\theta) = 100 \cdot p_\theta[1] + 1 \cdot p_\theta[2]\]

Fisher information scalar:

\[f_{\theta_0} = \frac{\exp(\theta_0)}{(1 + \exp(\theta_0))^2}\]

Hence: \(f_{\theta_0} \to 0^+\), as \(\theta_0 \to \infty\)

NPG:

\[
\theta_1 = \theta_0 + \eta \frac{g'(\theta_0)}{f_{\theta_0}}
\]

GA:

\[
\theta_1 = \theta_0 + \eta g'(\theta_0)
\]

i.e., Plain GA in \(\theta\) will move to \(\theta = \infty\) at a constant speed, while Natural GA can traverse faster and faster when \(\theta\) gets bigger (subject to the same learning rate)
Outline for Today:

1. Derivation of the closed-form NPG update

2. Intuitive Explanation of Natural (Policy) Gradient

(In HW2, try to compare PG and NPG, see how they perform differently in practice!)

3. Review of Policy Optimization (API, CPI, PG, and NPG) & a new algorithm
Review on Policy Optimization:

We have huge space, i.e., $|S|$ might be $255^{3 \times 512 \times 512}$.

We can only reset from initial state distribution $s_0 \sim \mu$. 
Review on Policy Optimization:

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Numeration over state (e.g., a for loop) is not possible!
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Goal: learn w/ function approximation
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**Goal:** learn w/ function approximation

A Policy is a classifier w/ $A$ many classes
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Goal: learn w/ function approximation

A Policy is a classifier w/ $A$ many classes.

What about continuous actions $a \in \mathbb{R}^d$?
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Goal: learn w/ function approximation

A Policy is a classifier w/ $A$ many classes

What about continuous actions $a \in \mathbb{R}^d$?

$$\pi_{\beta, \alpha}(\cdot \mid s) = \mathcal{N}(\mu_{\beta}(s), \exp(\alpha)I_{d \times d})$$

$\theta := [\beta, \alpha]$
Review on Policy Optimization:

We have huge space space, i.e., $|S|$ might be $255^{3 \times 512 \times 512}$

We can only reset from initial state distribution $s_0 \sim \mu$

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$$\pi_{\beta, \alpha}(\cdot | s) = \mathcal{N}(\mu_{\beta}(s), \exp(\alpha)I_{d \times d})$$

$$\theta := [\beta, \alpha]$$
Review on Policy Optimization: API

Given an current policy \( \pi^t \), we perform policy update to \( \pi^{t+1} \)

First attempt: **Approximate Policy Iteration**

\[
\pi^{t+1} = \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi_t} \left[ A_t(s, \pi(s)) \right]
\]
Review on Policy Optimization: API

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

First attempt: **Approximate Policy Iteration**

$$
\pi^{t+1} = \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\mu_t} \left[ A^{\pi^t}(s, \pi(s)) \right]
$$

i.e., find the greedy policy that maximizes the local advantage (e.g., via regression)
Review on Policy Optimization: API

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

First attempt: Approximate Policy Iteration

$$\pi^{t+1} = \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi_t} \left[ A^\pi_t(s, \pi(s)) \right]$$

i.e., find the greedy policy that maximizes the local advantage (e.g., via regression)

Unfortunately, $\pi^{t+1}$ might be very different from $\pi^t$, and API could fail to make any progress
Review on Policy Optimization: CPI

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

second attempt: **Conservative Policy Iteration**

$$\pi_{grd} = \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi_t} \left[ A^\pi_t(s, \pi(s)) \right]$$
Review on Policy Optimization: CPI

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

second attempt: **Conservative Policy Iteration**

$$
\pi_{grd} = \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}} \left[ A^{\pi^t}(s, \pi(s)) \right]
$$

$$
\forall s : \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha \pi_{grd}(\cdot | s)
$$
Review on Policy Optimization: CPI

Given an current policy \( \pi^t \), we perform policy update to \( \pi^{t+1} \)

second attempt: Conservative Policy Iteration

\[
\pi_{\text{grd}} = \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu} \left[ A^{\pi^t}(s, \pi(s)) \right]
\]

\[
\forall s : \pi^{t+1}(\cdot | s) = (1 - \alpha) \pi^t(\cdot | s) + \alpha \pi_{\text{grd}}(\cdot | s)
\]

i.e., CPI find the greedy policy, and move towards it a little bit!
Review on Policy Optimization: CPI

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

second attempt: **Conservative Policy Iteration**

$$
\pi_{grd} = \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu} \left[ A^{\pi^t}(s, \pi(s)) \right]
$$

$$
\forall s : \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha \pi_{grd}(\cdot | s)
$$

i.e., CPI find the greedy policy, and move towards it a little bit!

**Two nice properties:**
Review on Policy Optimization: CPI

Given an current policy \( \pi^t \), we perform policy update to \( \pi^{t+1} \)

second attempt: **Conservative Policy Iteration**

\[
\pi_{grd} = \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi_t} \left[ A^\pi_t(s, \pi(s)) \right]
\]

\[\forall s : \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha \pi_{grd}(\cdot | s)\]

i.e., CPI find the greedy policy, and move towards it a little bit!

**Two nice properties:**

\[
\| d^{\pi^t}(\cdot) - d^{\pi^{t+1}}(\cdot) \|_1 \leq O \left( \frac{\alpha}{1 - \gamma} \right), \quad V^{\pi^{t+1}} > V^{\pi^t} \text{ (if not terminate yet)}
\]
Review on Policy Optimization: PG

Given a current policy \( \pi^t \), we perform policy update to \( \pi^{t+1} \)

Third attempt: **PG on parameterized policy**

\[
\max_{\theta} \mathbb{E}_{s \sim \mu_{\pi_\theta}} \left[ \mathbb{E}_{a \sim \pi_\theta(\cdot|s)} A_{\pi_\theta}(s, a) \right]
\]
Review on Policy Optimization: PG

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

Third attempt: **PG on parameterized policy**

$$\max_{\theta_0} \mathbb{E}_{s \sim d_\mu^\pi_t} \left[ \mathbb{E}_{a \sim \pi_0(\cdot \mid s)} A^{\pi_\theta_t}(s, a) \right]$$

Locally Improve the local-adv a little bit via one-step gradient ascent:

$$\nabla_\theta \left[ \mathbb{E}_{s \sim d_\mu^\pi_t} \left[ \mathbb{E}_{a \sim \pi_0(\cdot \mid s)} \frac{\pi_\theta_t(a \mid s)}{T_{\theta}(a \mid s)} A^{T_{\theta}(a \mid s)}(s, a) \right] \right] = \nabla_\theta J(\theta_0)$$
Review on Policy Optimization: PG

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

Third attempt: PG on parameterized policy

$$
\max_{\theta} \mathbb{E}_{s \sim d_{\mu}^{\pi_{\theta}}} \left[ \mathbb{E}_{a \sim \pi_{\theta}(\cdot | s)} A^{\pi_{\theta}}(s, a) \right]
$$

Locally Improve the local-adv a little bit via one-step gradient ascent:

$$
\theta_{t+1} = \theta_t + \eta \cdot \mathbb{E}_{s \sim d_{\mu}^{\pi_{\theta}}} \left[ \mathbb{E}_{a \sim \pi_{\theta}(s)} \nabla \ln \pi_{\theta}(a | s) \cdot A^{\pi_{\theta}}(s, a) \right]
$$
Review on Policy Optimization: PG

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

Third attempt: PG on parameterized policy

$$\max_{\theta} \mathbb{E}_{s \sim d_\mu} \left[ \mathbb{E}_{a \sim \pi_\theta(\cdot \mid s)} A_{\pi_\theta}(s, a) \right]$$

Locally Improve the local-adv a little bit via one-step gradient ascent:

$$\theta_{t+1} = \theta_t + \eta \cdot \mathbb{E}_{s \sim d_\mu} \left[ \mathbb{E}_{a \sim \pi_\theta(s)} \nabla \ln \pi_\theta(a \mid s) \cdot A_{\pi_\theta}(s, a) \right]$$

When $\eta \to 0^+$, gradient ascent ensures we improve the objective function
Review on Policy Optimization: NPG

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

Fourth attempt: **Natural Policy Gradient**

$$\max_{\theta} \mathbb{E}_{s \sim d_{\pi^t}} \left[ \mathbb{E}_{a \sim \pi_{\theta}(\cdot|s)} A_{\pi^t}(s, a) \right]$$
Review on Policy Optimization: NPG

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

Fourth attempt: Natural Policy Gradient

$$\max_\theta \mathbb{E}_{s \sim d^{\pi^{t+1}}_\mu} \left[ \mathbb{E}_{a \sim \pi(\cdot | s)} A^{\pi^{t+1}}(s, a) \right]$$

s.t. $KL(\rho^{\pi^{t+1}} | \rho) \leq \delta$
Review on Policy Optimization: NPG

Given a current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

Fourth attempt: **Natural Policy Gradient**

$$\max_\theta \mathbb{E}_{s \sim d_\mu} \left[ \mathbb{E}_{a \sim \pi_\theta \cdot | s} A_{\pi_\theta}(s, a) \right]$$

$$s . t . , \text{KL}(\rho_{\theta_t} | \rho_\theta) \leq \delta$$

Define fisher info-matrix $F_{\theta_t} = \nabla_\theta^2 \text{KL}(\rho_{\theta_t} | \rho_\theta) |_{\theta=\theta_t}$, a convex approximation, e.g., linearize obj and quadratize constraint, gives us the following NPG update:
Review on Policy Optimization: NPG

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

Fourth attempt: **Natural Policy Gradient**

$$
\max_\theta \mathbb{E}_{s \sim d_\mu} \left[ \mathbb{E}_{a \sim \pi(\cdot|s)} A^\pi(s, a) \right]
$$

$$
s.t., \ KL(\rho^t \mid \rho) \leq \delta
$$

Define fisher info-matrix $F^t_\theta = \nabla^2_\theta KL(\rho^t \mid \rho) |_{\theta = \theta^t}$, a convex approximation, e.g., linearize obj and quadratize constraint, gives us the following NPG update:

$$
\max_\theta \nabla_\theta J(\pi^t) ^\top (\theta - \theta^t), \ s.t., (\theta - \theta^t) ^\top F^t_\theta (\theta - \theta^t) \leq \delta
$$
An extension of NPG (even faster in practice):

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

fifth attempt (new): **Proximal Policy Optimization (PPO)**

$$\max_{\theta} \mathbb{E}_{s \sim d_{\mu}^{\pi_{\theta}}} \left[ \mathbb{E}_{a \sim \pi_{\theta}(\cdot | s)} A_{\pi_{\theta}}(s, a) \right]$$
An extension of NPG (even faster in practice):

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

fifth attempt (new): **Proximal Policy Optimization (PPO)**

$$\max_\theta \mathbb{E}_{s \sim d^{\pi_0}_t} \left[ \mathbb{E}_{a \sim \pi_\theta(\cdot|s)} A^{\pi_0_t}(s, a) \right] - \lambda \mathbb{E}_{s \sim d^{\pi_t}_t} \Delta \left[ KL \left( \pi_{\theta_t}(a | s) \| \pi_\theta(a | s) \right) \right]$$ 

regularization
An extension of NPG (even faster in practice):

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

fifth attempt (new): **Proximal Policy Optimization (PPO)**

\[
\max_{\theta} \mathbb{E}_{s \sim d^{\pi_0}_t} \left[ \mathbb{E}_{a \sim \pi_\theta(s | s)} A^{\pi_0}(s, a) \right] - \lambda \mathbb{E}_{s \sim d^t_\mu} \left[ \text{KL} \left( \pi_{\theta_i}(a | s) \mid \pi_\theta(a | s) \right) \right]
\]

regularization

Use importance weighting & expand KL divergence:
An extension of NPG (even faster in practice):

Given an current policy \( \pi^t \), we perform policy update to \( \pi^{t+1} \)

fifth attempt (new): **Proximal Policy Optimization (PPO)**

\[
\max_{\theta} \mathbb{E}_{s \sim d_{\mu}^{\pi_\theta t}} \left[ \mathbb{E}_{a \sim \pi_\theta(\cdot | s)} A^{\pi_\theta t}(s, a) \right] - \lambda \mathbb{E}_{s \sim d_{\mu}^{\pi_t}} \left[ \text{KL} \left( \pi_{\theta_t}(a | s) \parallel \pi_{\theta}(a | s) \right) \right]
\]

regularization

Use importance weighting & expand KL divergence:

\[
\ell(\theta) := \mathbb{E}_{s \sim d_{\mu}^{\pi_\theta t}} \left[ \frac{\pi_{\theta}(a | s)}{\pi_{\theta_t}(a | s)} A^{\pi_\theta}(s, a) \right] - \lambda \mathbb{E}_{s \sim d_{\mu}^{\pi_t}} \mathbb{E}_{a \sim \pi_{\theta_t}(\cdot | s)} \left[ -\ln \pi_{\theta}(a | s) \right]
\]

expand KL
An extension of NPG (even faster in practice):

Given an current policy $\pi^t$, we perform policy update to $\pi^{t+1}$

fifth attempt (new): **Proximal Policy Optimization (PPO)**

\[
\begin{align*}
    \max_{\theta} & \mathbb{E}_{s \sim d^{\pi_\theta}_t} \left[ \mathbb{E}_{a \sim \pi_\theta(\cdot | s)} A^{\pi_\theta}(s, a) \right] - \lambda \mathbb{E}_{s \sim d^{\pi_\theta}_t} \left[ \text{KL} \left( \pi_{\theta_t}(a | s) \mid \pi_\theta(a | s) \right) \right] \\
    \text{regularization}
\end{align*}
\]

Use importance weighting & expand KL divergence:

\[
\ell(\theta) = \mathbb{E}_{s \sim d^{\pi_\theta}_t} \left[ \mathbb{E}_{a \sim \pi_\theta(\cdot | s)} \frac{\pi_\theta(a | s)}{\pi_{\theta_t}(a | s)} A^{\pi_\theta}(s, a) \right] - \lambda \mathbb{E}_{s \sim d^{\pi_\theta}_t} \mathbb{E}_{a \sim \pi_\theta(\cdot | s)} \left[ -\ln \pi_\theta(a | s) \right]
\]

**PPO:** Perform a few steps of mini-batch SGA on $\ell(\theta)$ to approximate $\arg \max_{\theta} \ell(\theta)$
Next a few lectures:

Imitation Learning
(Learning from Demonstrations)

Can we learn a good policy purely from expert demonstrations?
\[
F_{\theta_0} = E_{s \sim S} \left[ \ln \pi_{\theta_0}(s) \ln \pi_{\theta}(s) \right] \in \mathbb{R}^{d \times d}
\]

\[
Hessian \quad F_{\theta_0} = U \Sigma U^T
\]

\[
F_{\theta_0}^{-1} = U \Sigma^{-1} U^T \quad U^T = I
\]

\[
F_{\theta_0}^{-1} \cdot \nabla_\theta = U \Sigma^{-1} \left( U^T \nabla_{\theta_0} \right)
\]

\[
= \sum_{i=1}^{d} \frac{1}{\sigma_i} (U_i^T \nabla_{\theta_0})
\]

\[
KL (\rho_{\theta_0} \| \rho_{\theta}) \approx (\theta - \theta_0)^T \nabla_{\theta} \sum_{i=1}^{d} \frac{1}{\sigma_i} (U_i^T \nabla_{\theta_0})
\]