Conservative Policy Iteration
Recap

Recall Policy Iteration (PI) with known \((P, r)\)
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Assume MDP is known, we compute \(A^{\pi_{old}}(s, a)\) exactly for all \(s, a\), PI updates policy as:
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i.e., pick an action that has the largest advantage against \(\pi_{old}\) at every state \(s\),
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i.e., pick an action that has the largest advantage against \(\pi_{\text{old}}\) at every state \(s\),

Maximize advantage is great, as it gives monotonic improvement:

\[
Q^{\pi_{\text{new}}}(s, a) \geq Q^{\pi_{\text{old}}}(s, a), \forall s, a
\]
Recap

Recall Policy Iteration (PI):

$$\pi_{\text{new}}(s) = \arg \max_a A^{\pi_{\text{old}}}(s, a)$$
Recap

Recall Policy Iteration (PI):

\[ \pi_{new}(s) = \arg \max_a A^{\pi_{old}}(s, a) \]

Performance Difference Lemma (PDL): for all \( s_0 \in S \)
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Recall Policy Iteration (PI):

$$\pi_{\text{new}}(s) = \arg \max_a A_{\pi_{\text{old}}}(s, a)$$

Performance Difference Lemma (PDL): for all $s_0 \in S$

$$V^{\pi_{\text{new}}}(s_0) - V^{\pi_{\text{old}}}(s_0) = \frac{1}{1 - \gamma} E_{s, a \sim d_{\pi_{\text{new}}}} [A_{\pi_{\text{old}}}(s, a)]$$
Recap

Recall Policy Iteration (PI):

$$\pi_{new}(s) = \arg \max_a A_{\pi_{old}}(s, a)$$

Performance Difference Lemma (PDL): for all $s_0 \in S$

$$V_{\pi_{new}}(s_0) - V_{\pi_{old}}(s_0) = \frac{1}{1 - \gamma} \mathbb{E}_{s, a \sim d_{\pi_{new}}} \left[ A_{\pi_{old}}(s, a) \right]$$

The advantage against $\pi_{old}$ averaged over $\pi_{new}$’s own distribution
Today:
Conservative Policy Iteration

Q: How to enforce incremental policy update and ensure monotonic improvement
Outline

1. Greedy Policy Selection (via reduction to regression) and recap of API

2. Conservative Policy Iteration

3. Monotonic Improvement of CPI
Setting and Notation

Discounted infinite horizon MDP:

\[ \mathcal{M} = \{S, A, \gamma, r, P, \mu\} \]
Setting and Notation

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State visitation:  
\[ d^\pi_\mu(s) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h P^\pi_h(s; \mu) \]
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As we will consider large scale unknown MDP here, we start with a (restricted) function class \( \Pi \):

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(We can think about each policy as a classifier)

Note that the optimal policy \( \pi^* \) may not be in \( \Pi \)
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As we will consider large scale unknown MDP here, we start with a (restricted) function class \( \Pi \):

\[ \Pi = \{ \pi : S \mapsto A \} \]

From now on, think about deterministic policy as a special stochastic policy

(We can think about each policy as a classifier)

Note that the optimal policy \( \pi^* \) may not be in \( \Pi \)
Recall **Approximate** Policy Iteration (API)
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Given the current policy $\pi^t$, let’s find a new policy that has large local adv over $\pi^t$ under $d_{\mu}^{\pi'}$.
Recall Approximate Policy Iteration (API)

Given the current policy $\pi^t$, let’s find a new policy that has large local adv over $\pi^t$ under $d_\mu^{\pi'}$ i.e., let’s aim to (approximately) solve the following program:

$$\arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^{\pi}} \left[ A^{\pi'}(s, \pi(s)) \right]$$
Recall Approximate Policy Iteration (API)

Given the current policy $\pi^t$, let’s find a new policy that has large local adv over $\pi^t$ under $d^*_{d_{\mu}}$

i.e., let’s aim to (approximately) solve the following program:

$$\arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d^t_\pi} \left[ A^{\pi'}(s, \pi(s)) \right]$$

Greedy Policy Selector
Recall Approximate Policy Iteration (API)

Given the current policy $\pi^t$, let’s find a new policy that has large local adv over $\pi^t$ under $d_\mu^{\pi'}$

i.e., let’s aim to (approximately) solve the following program:

$$\arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^t} \left[ A^{\pi'}(s, \pi(s)) \right]$$

How to implement such greedy policy selector?
We talked about a regression process..
Implementing Approximate Greedy Policy Selector via Regression

We can do a **reduction to Regression** via Advantage function approximation

\[ \mathcal{F} = \{ f : S \times A \mapsto \mathbb{R} \} \quad (\approx A^\pi) \]
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\[ \{ s_i, a_i, y_i \}, s_i \sim d^\pi, a_i \sim U(A), \mathbb{E}[y_i] = A^\pi(s_i, a_i) \]
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\[ \{ s_i, a_i, y_i \}, s_i \sim d^{\pi_t}, a_i \sim U(A), \mathbb{E} [y_i] = A^{\pi_t}(s_i, a_i) \]

Regression oracle:

\[ \widehat{A}^t = \arg \min_{f \in \mathcal{F}} \sum_i (f(s_i, a_i) - y_i)^2 \]
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**Regression oracle:**

$$\hat{A}^t = \arg \min_{f \in \mathcal{F}} \sum_i \left( f(s_i, a_i) - y_i \right)^2$$

Act greedily wrt the estimator $\hat{A}^t$ (as we hope $\hat{A}^t \approx A^\pi$):
Implementing Approximate Greedy Policy Selector via Regression

We can do a \textbf{reduction to Regression} via Advantage function approximation

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\mathcal{F} = \{ f : S \times A \mapsto \mathbb{R} \} \quad (\approx A^{\pi})
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$$
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\{ s_i, a_i, y_i \}, s_i \sim d^{\pi}_\mu, a_i \sim U(A), \mathbb{E}[y_i] = A^{\pi}(s_i, a_i)
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\hat{\pi}(s) = \arg \max_a \hat{A}^t(s, a), \forall s
$$
Successful Regression ensures approximate greedy operator

\[ \{s_i, a_i, y_i\}, s_i \sim d^\pi_t, a_i \sim U(A), \mathbb{E}[y_i] = A^\pi_t(s_i, a_i) \]

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Regression oracle:

\[ \hat{A}^t = \arg \min_{f \in \mathcal{F}} \sum_i (f(s_i, a_i) - y_i)^2 \]

Assume this regression is successful, i.e.,

\[ \mathbb{E}_{s \sim d_\pi^t, a \sim U(A)} \left( \hat{A}^t(s, a) - A^\pi^t(s, a) \right)^2 \leq \delta \]
Successful Regression ensures approximate greedy operator

\{s_i, a_i, y_i\}, s_i \sim d^{\pi_t}_\mu, a_i \sim U(A), \mathbb{E} [y_i] = A^{\pi_t'}(s_i, a_i)

Regression oracle:

\hat{A}^t = \arg\min_{f \in \mathcal{F}} \sum_i (f(s_i, a_i) - y_i)^2

Assume this regression is successful, i.e.,

\mathbb{E}_{s \sim d^{\pi_t}_\mu, a \sim U(A)} \left( (\hat{A}^t(s, a) - A^{\pi_t}(s, a))^2 \right) \leq \delta

Then, \( \hat{\pi}(s) = \arg\max_a \hat{A}^t(s, a) \) is an approximate greedy policy:

\mathbb{E}_{s \sim d^{\pi_t}_\mu} A^{\pi_t}(s, \hat{\pi}(s)) \geq \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^{\pi_t}_\mu} A^{\pi_t}(s, \pi(s)) - O\left(\sqrt{\delta}\right)
Summary So Far:

By reduction to **Supervised Learning** (i.e., via Regression to approximate $A^{\pi'}$ under $d^{\pi'}_{\mu}$), we can expect to find an approximate greedy optimizer $\hat{\pi}$, s.t.,
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By reduction to Supervised Learning (i.e., via Regression to approximate $A^{\pi^i}$ under $d^{\pi^i}_\mu$), we can expect to find an approximate greedy optimizer $\hat{\pi}$, s.t.,

$$\mathbb{E}_{s \sim d^{\pi^i}_\mu} \left[ A^{\pi^i}(s, \hat{\pi}(s)) \right] \approx \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^{\pi^i}_\mu} \left[ A^{\pi^i}(s, \pi(s)) \right]$$
Summary So Far:

**By reduction to Supervised Learning** (i.e., via Regression to approximate $A^{\pi^t}$ under $d_\mu^{\pi^t}$), we can expect to find an approximate greedy optimizer $\hat{\pi}$, s.t.,

$$\mathbb{E}_{s \sim d_\mu^{\pi^t}} \left[ A^{\pi^t}(s, \hat{\pi}(s)) \right] \approx \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^{\pi^t}} \left[ A^{\pi^t}(s, \pi(s)) \right]$$

Throughout this lecture, we will simply assume we can achieve $\arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^{\pi^t}} \left[ A^{\pi^t}(s, \pi(s)) \right]$.
Outline

1. Greedy Policy Selection

2. Conservative Policy Iteration

3. Monotonic Improvement of CPI
The Failure case of API: Abrupt distribution change

API cannot guarantee to succeed (let’s think about advantage function approximation setting)
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\[ A^\pi(s, a) \Rightarrow A^{t+1} \Rightarrow \pi^{t+1} \]
The Failure case of API: Abrupt distribution change

API cannot guarantee to succeed (let’s think about advantage function approximation setting)

Oscillation between two updates:
No monotonic improvement
Key Idea of CPI: Incremental Update—No Abrupt Distribution Change

Let’s design policy update rule such that $d^{\pi^i}_{\mu}$ and $d^{\pi^{i+1}}_{\mu}$ are not that different!
Key Idea of CPI: Incremental Update—No Abrupt Distribution Change

Let’s design policy update rule such that $d_{μ}^{π_t+1}$ and $d_{μ}^{π_t}$ are not that different!

Recall Performance Difference Lemma:

$$V^{π_t+1} - V^{π_t} = \frac{1}{1 - γ} \mathbb{E}_{s \sim d_{μ}^{π_t+1}} \left[ A^{π_t}(s, π^{t+1}(s)) \right]$$
Key Idea of CPI: Incremental Update—No Abrupt Distribution Change

Let’s design policy update rule such that $d_{\mu}^{\pi^t+1}$ and $d_{\mu}^{\pi^t}$ are not that different!

Recall Performance Difference Lemma:

$$V^{\pi^t+1} - V^{\pi^t} = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim d_{\mu}^{\pi^t+1}} \left[ A^{\pi^t}(s, \pi^{t+1}(s)) \right]$$

$$d^{\pi^t} \approx d^{\pi^t+1}$$
Key Idea of CPI: Incremental Update—No Abrupt Distribution Change

Let’s design policy update rule such that \(d^{\pi^t}_\mu\) and \(d^{\pi^t}_{\mu+1}\) are not that different!

Recall Performance Difference Lemma:

\[
V^{\pi_{t+1}} - V^{\pi_t} = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim d^{\pi^t}_{\mu+1}} \left[ A^{\pi^t}(s, \pi^{t+1}(s)) \right]
\]

\[d^\pi_t \approx d^{\pi^t}_{\mu+1}\]

s.t.,

\[
\mathbb{E}_{s \sim d^\pi_t} \left[ A^{\pi^t}(s, \pi^{t+1}(s)) \right] \approx \mathbb{E}_{s \sim d^{\pi^t}_{\mu+1}} \left[ A^{\pi^t}(s, \pi^{t+1}(s)) \right]
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Key Idea of CPI: Incremental Update—No Abrupt Distribution Change

Let’s design policy update rule such that \( d_{\mu}^{\pi_{t+1}} \) and \( d_{\mu}^{\pi_{t}} \) are not that different!

Recall Performance Difference Lemma:

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V_{\pi_{t+1}}^{\pi_{t}} = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim d_{\pi_{t}}^{\pi_{t+1}}} \left[ A_{\pi_{t}}^{\pi_{t}}(s, \pi_{t+1}(s)) \right]
\]

\[
d_{\pi_{t}} \approx d_{\pi_{t+1}}
\]

s.t., \[
\mathbb{E}_{s \sim d_{\pi_{t}}} \left[ A_{\pi_{t}}^{\pi_{t}}(s, \pi_{t+1}(s)) \right] \approx \mathbb{E}_{s \sim d_{\pi_{t+1}}} \left[ A_{\pi_{t}}^{\pi_{t}}(s, \pi_{t+1}(s)) \right]
\]

This we know how to optimize: the Greedy Policy Selector
The CPI Algorithm

Initialize $\pi^0$

For $t = 0$ ...
The CPI Algorithm

Initialize $\pi^0$

For $t = 0$ …

1. Greedy Policy Selector:

$$\pi' \in \arg \max_{\pi' \in \Pi} \mathbb{E}_{s \sim d_{\pi'}^t} \left[ A_{\pi'}(s, \pi(s)) \right]$$
The CPI Algorithm

Initialize $\pi^0$

For $t = 0$ …

1. Greedy Policy Selector:
   \[
   \pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\pi^t}} \left[ A_{\pi'}(s, \pi(s)) \right]
   \]

2. If $\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\pi^t}} [A_{\pi'}(s, \pi(s))] \leq \varepsilon$
   
   Return $\pi^t$
The CPI Algorithm

Initialize $\pi^0$

For $t = 0$ ...

1. Greedy Policy Selector:
   \[
   \pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi_t} \left[ A^\pi_t(s, \pi(s)) \right]
   \]

2. If $\max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi_t}[A^\pi_t(s, \pi(s))] \leq \varepsilon$
   
   Return $\pi^t$

3. Incremental Update:
   \[
   \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha\pi'(\cdot | s), \forall s
   \]
The CPI Algorithm

Initialize $\pi^0$

For $t = 0$ ...

1. Greedy Policy Selector:
   \[
   \pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\pi^t}} \left[ A^{\pi'}(s, \pi(s)) \right]
   \]

   Q: Why this is incremental? In what sense?

2. If $\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\pi^t}}[A^{\pi'}(s, \pi(s))] \leq \varepsilon$
   
   Return $\pi^t$

   Q: Can we get monotonic policy improvement?

3. Incremental Update:
   \[
   \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha \pi'(\cdot | s), \forall s
   \]
Today: Policy Optimization

1. Greedy Policy Selection

2. Conservative Policy Iteration

3. Monotonic Improvement of CPI
Q1: Why this is incremental? In what sense?

\[
\pi^{t+1}( \cdot | s) = (1 - \alpha)\pi^t( \cdot | s) + \alpha \pi'( \cdot | s), \forall s
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Q1: Why this is incremental? In what sense?

\[ \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha\pi'(\cdot | s), \forall s \]

Key observation 1:

For any state \( s \), we have \( \|\pi^{t+1}(\cdot | s) - \pi^t(\cdot | s)\|_1 \leq 2\alpha \)
Q1: Why this is incremental? In what sense?

\[ \pi^{t+1}(\cdot \mid s) = (1 - \alpha)\pi^t(\cdot \mid s) + \alpha\pi'(\cdot \mid s), \forall s \]

Key observation 1:
For any state \( s \), we have \( \left\| \pi^{t+1}(\cdot \mid s) - \pi^t(\cdot \mid s) \right\|_1 \leq 2\alpha \)

Key observation 2 (Lemma 12.1 in AJKS)
For any two policies \( \pi \) and \( \pi' \), if \( \left\| \pi(\cdot \mid s) - \pi'(\cdot \mid s) \right\|_1 \leq \delta, \forall s \), then \( \left\| d_\mu^\pi(\cdot) - d_\mu^\pi'(\cdot) \right\|_1 \leq \frac{\gamma\delta}{1 - \gamma} \)
Q1: Why this is incremental? In what sense?

\[ \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha\pi'(\cdot | s), \forall s \]

Key observation 1:
For any state \( s \), we have \( \|\pi^{t+1}(\cdot | s) - \pi^t(\cdot | s)\|_1 \leq 2\alpha \)

Key observation 2 (Lemma 12.1 in AJKS)
For any two policies \( \pi \) and \( \pi' \), if \( \|\pi(\cdot | s) - \pi'(\cdot | s)\|_1 \leq \delta \), \( \forall s \), then \( \|d^\pi_\mu(\cdot) - d^{\pi'}_\mu(\cdot)\|_1 \leq \frac{\gamma \delta}{1 - \gamma} \)

CPI ensures incremental update, i.e., \( \|d^\pi^{t+1}_\mu(\cdot) - d^{\pi^t}_\mu(\cdot)\|_1 \leq \frac{2\gamma\alpha}{1 - \gamma} \)
Monotonic Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

\[
\mathbb{A} := \mathbb{E}_{d_\mu^t} \left[ A^{\pi^t}(s, \pi'(s)) \right] \geq \varepsilon
\]

Recall CPI:

1. Greedy Policy Selector:
   \[
   \pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^t} \left[ A^{\pi^t}(s, \pi(s)) \right]
   \]

2. If \( \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^t}[A^{\pi^t}(s, \pi(s))] \leq \varepsilon \)
   
   Return \( \pi^t \)

3. Incremental Update:
   \[
   \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha \pi'(\cdot | s), \forall s
   \]
Monotonic Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

\[ A := \mathbb{E}_{d^t_\mu} \left[ A^{\pi'}(s, \pi'(s)) \right] \geq \epsilon \]

Can we translate local advantage $A$ to $V^{\pi^{t+1}} - V^{\pi'}$? (Yes, by PDL)

Recall CPI:

1. Greedy Policy Selector:
   \[ \pi' \in \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d^t_\mu} \left[ A^{\pi'}(s, \pi(s)) \right] \]

2. If \( \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^t_\mu} \left[ A^{\pi'}(s, \pi(s)) \right] \leq \epsilon \)

   Return $\pi'$

3. Incremental Update:
   \[ \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha\pi'(\cdot | s), \forall s \]
Monotonic Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

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Can we translate local advantage \( \mathbb{A} \) to \( V^{\pi_t+1} - V^{\pi'} \)? (Yes, by PDL)

\[
(1 - \gamma) \left( V^{\pi_t+1}_\mu - V^{\pi_t}_\mu \right) = \mathbb{E}_{s \sim d_{\mu}^{t+1}} \left[ \mathbb{E}_{a \sim \pi^{t+1}(\cdot | s)} A^{\pi_t}(s, a) \right]
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Recall CPI:

1. Greedy Policy Selector:
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\[
(1 - \gamma) \left( V^{\pi^t+1}_\mu - V^{\pi^t}_\mu \right) = \mathbb{E}_{s \sim d_{\mu}^{t+1}} \left[ \mathbb{E}_{a \sim \pi^{t+1}(\cdot | s)} A^{\pi^t}(s, a) \right] \\
= \mathbb{E}_{s \sim d_{\mu}^{t+1}} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right]
\]

Recall CPI:

1. Greedy Policy Selector:
   \[ \pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^{t}} \left[ A^{\pi^t}(s, \pi(s)) \right] \]

2. If \( \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^{t}} [A^{\pi^t}(s, \pi(s))] \leq \epsilon \)
   
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\[
(1 - \gamma) \left( V^{\pi^{t+1}}_{\mu} - V^{\pi^t}_{\mu} \right) = \mathbb{E}_{s \sim d_{\pi}^{t+1}} \left[ \mathbb{E}_{a \sim \pi^{t+1}(\cdot \mid s)} A^{\pi^t}(s, a) \right] \\
= \mathbb{E}_{s \sim d_{\pi}^{t+1}} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right] \quad \text{Why?}
\]

Recall CPI:

1. Greedy Policy Selector:
   \[ \pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^{t}} \left[ A^{\pi'}(s, \pi(s)) \right] \]

2. If \( \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^{t}} \left[ A^{\pi'}(s, \pi(s)) \right] \leq \epsilon \)
   Return \( \pi^t \)

3. Incremental Update:
   \( \pi^{t+1}(\cdot \mid s) = (1 - \alpha)\pi^t(\cdot \mid s) + \alpha \pi'(\cdot \mid s), \forall s \)
Monotonic Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

\[ A := \mathbb{E}_{d^t_\mu} \left[ A^{\pi^t}(s, \pi'(s)) \right] \geq \epsilon \]

Can we translate local advantage \( A \) to \( V^{\pi^t+1} - V^{\pi^t} \)? (Yes, by PDL)

\[
(1 - \gamma) \left( V^{\pi^t+1}_\mu - V^{\pi^t}_\mu \right) = \mathbb{E}_{s \sim d^{t+1}_\mu} \left[ \mathbb{E}_{a \sim \pi^{t+1}(. | s)} A^{\pi^t}(s, a) \right]
\]

\[
= \mathbb{E}_{s \sim d^{t+1}_\mu} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right] \quad \text{Why?}
\]

\[
= \mathbb{E}_{s \sim d^{t}_\mu} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right] + \mathbb{E}_{s \sim d^{t+1}_\mu} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right] - \mathbb{E}_{s \sim d^{t}_\mu} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right]
\]

Recall CPI:

1. Greedy Policy Selector:
   \[ \pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^t_\mu} \left[ A^{\pi}(s, \pi(s)) \right] \]

2. If \( \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^t_\mu} [A^{\pi}(s, \pi(s))] \leq \epsilon \)
   Return \( \pi^t \)

3. Incremental Update:
   \[ \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha \pi'(\cdot | s), \forall s \]
Monotonic Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

\[ \mathbb{A} := \mathbb{E}_{d_\mu^t} \left[ A^{\pi^t}(s, \pi'(s)) \right] \geq \varepsilon \]

Can we translate local advantage \( \mathbb{A} \) to \( V^{\pi^t+1} - V^{\pi'} \)? (Yes, by PDL)

\[
(1 - \gamma) \left( V^{\pi^t+1}_\mu - V^{\pi'}_\mu \right) = \mathbb{E}_{s \sim d_\mu^{t+1}} \left[ \mathbb{E}_{a \sim \pi^t+1(s)} A^{\pi^t}(s, a) \right] \\
= \mathbb{E}_{s \sim d_\mu^{t+1}} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right] \quad \text{Why?}
\]

\[
= \mathbb{E}_{s \sim d_\mu^t} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right] + \mathbb{E}_{s \sim d_\mu^{t+1}} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right] - \mathbb{E}_{s \sim d_\mu^t} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right] \\
\geq \mathbb{E}_{s \sim d_\mu^t} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right] - \frac{\alpha}{1 - \gamma} \| d_\mu^{t+1} - d_\mu^{t'} \|_1
\]
Monotonic Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

\[ \mathbb{A} := \mathbb{E}_{d_\mu^t} \left[ A^{\pi'(s, \pi'(s))} \right] \geq \epsilon \]

Can we translate local advantage \( \mathbb{A} \) to \( V^{\pi^{t+1}} - V^{\pi'} \)? (Yes, by PDL)

\[
(1 - \gamma) \left( V^{\pi^{t+1}}_\mu - V^{\pi'}_\mu \right) = \mathbb{E}_{s \sim d_\mu^{t+1}} \left[ \mathbb{E}_{a \sim \pi^{t+1}(.|s)} A^{\pi'(s, a)} \right] \\
= \mathbb{E}_{s \sim d_\mu^{t+1}} \left[ \alpha A^{\pi'(s, \pi'(s))} \right] \text{ Why?}
\]

\[
= \mathbb{E}_{s \sim d_\mu^{t}} \left[ \alpha A^{\pi'(s, \pi'(s))} \right] + \mathbb{E}_{s \sim d_\mu^{t+1}} \left[ \alpha A^{\pi'(s, \pi'(s))} \right] - \mathbb{E}_{s \sim d_\mu^{t}} \left[ \alpha A^{\pi'(s, \pi'(s))} \right] \\
\geq \mathbb{E}_{s \sim d_\mu^{t}} \left[ \alpha A^{\pi'(s, \pi'(s))} \right] - \frac{\alpha}{1 - \gamma} \|d^{\pi^{t+1}}_\mu - d^{\pi'}_\mu\|_1 \\
\geq \alpha \mathbb{A} - \frac{2\gamma \alpha^2}{(1 - \gamma)^2}
\]
Monotonic Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

\[ \mathbb{A} := \mathbb{E}_{d_{\mu}^{t}} \left[ A^{\pi^{t}}(s, \pi'(s)) \right] \geq \epsilon \]

Can we translate local advantage \( \mathbb{A} \) to \( V^{\pi^{t+1}} - V^{\pi'} \)? (Yes, by PDL)

\[
(1 - \gamma)(V^{\pi^{t+1}}_\mu - V^{\pi'}_\mu) = \mathbb{E}_{s \sim d^{t+1}_{\mu}} \left[ \mathbb{E}_{a \sim \pi^{t+1}(\cdot | s)} A^{\pi'}(s, a) \right] \\
= \mathbb{E}_{s \sim d^{t+1}_{\mu}} \left[ \alpha A^{\pi}(s, \pi'(s)) \right] \quad \text{Why?}
\]

\[
= \mathbb{E}_{s \sim d_{\mu}^{t}} \left[ \alpha A^{\pi}(s, \pi'(s)) \right] + \mathbb{E}_{s \sim d_{\mu}^{t+1}} \left[ \alpha A^{\pi'}(s, \pi'(s)) \right] - \mathbb{E}_{s \sim d_{\mu}^{t}} \left[ \alpha A^{\pi'}(s, \pi'(s)) \right] \\
\geq \mathbb{E}_{s \sim d_{\mu}^{t}} \left[ \alpha A^{\pi}(s, \pi'(s)) \right] - \alpha \frac{\| d^{t+1}_{\mu} \|_{1} - d^{\pi'}_{\mu} \|_{1}}{1 - \gamma} \\
\geq \alpha \mathbb{A} - \frac{2\gamma \alpha^2}{(1 - \gamma)^2} \quad \text{(Set } \alpha = \frac{(1 - \gamma)^2 \mathbb{A}}{4\gamma} \text{)}
\]

Recall CPI:

1. Greedy Policy Selector:
   \[ \pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^{t}} A^{\pi}(s, \pi(s)) \]

2. If \( \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^{t}} A^{\pi}(s, \pi(s)) \leq \epsilon \)
   
   Return \( \pi^{t} \)

3. Incremental Update:
   \[ \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^{t}(\cdot | s) + \alpha \pi'(\cdot | s), \forall s \]
Monotonic Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

\[ \mathbb{A} := \mathbb{E}_{d_\mu^{t+1}} \left[ A^{\pi_t}(s, \pi'(s)) \right] \geq \epsilon \]

Can we translate local advantage \( \mathbb{A} \) to \( V_{\pi_{t+1}}^{\pi_t} - V_{\pi'} \)? (Yes, by PDL)

\[
(1 - \gamma)(V_{\pi_{t+1}}^{\pi_t} - V_{\pi'}^{\pi_t}) = \mathbb{E}_{s \sim d_\mu^{t+1}} \left[ \mathbb{E}_{a \sim \pi_{t+1}(s)} A^{\pi_t}(s, a) \right]
\]

\[
= \mathbb{E}_{s \sim d_\mu^{t+1}} \left[ A^{\pi_t}(s, \pi'(s)) \right] \quad \text{Why?}
\]

\[
= \mathbb{E}_{s \sim d_\mu^{t+1}} \left[ \alpha A^{\pi_t}(s, \pi'(s)) \right] + \mathbb{E}_{s \sim d_\mu^{t+1}} \left[ \alpha A^{\pi_t}(s, \pi'(s)) \right] - \mathbb{E}_{s \sim d_\mu^{t+1}} \left[ \alpha A^{\pi_t}(s, \pi'(s)) \right]
\]

\[
\geq \mathbb{E}_{s \sim d_\mu^{t+1}} \left[ \alpha A^{\pi_t}(s, \pi'(s)) \right] - \frac{\alpha}{1 - \gamma} \| d_\mu^{t+1} - d_\mu^{t'} \|_1
\]

\[
\geq \alpha \mathbb{A} - \frac{2\gamma \alpha^2}{(1 - \gamma)^2} = \frac{\mathbb{A}^2 (1 - \gamma)}{8\gamma} \quad \text{(Set } \alpha = \frac{(1 - \gamma)^2 \mathbb{A}}{4\gamma})
\]
Summary of CPI so far:

1. Incremental update (Lemma 12.1 in AJKS)

\[
\|d_{\mu}^{\pi^{t+1}} - d_{\mu}^{\pi^t}\|_1 \leq \frac{2\gamma\alpha}{1 - \gamma}
\]

Recall CPI:

1. Greedy Policy Selector:

\[
\pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\pi}^t}[A^{\pi'}(s, \pi(s))]
\]

2. If \(\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\pi}^t}[A^{\pi'}(s, \pi(s))] \leq \epsilon\)

Return \(\pi'\)

3. Incremental Update:

\[
\pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha\pi'(\cdot | s), \forall s
\]

Local adv versus distribution change:
Summary of CPI so far:

1. Incremental update (Lemma 12.1 in AJKS)
   \[ \| d_{t+1}^\pi - d_\mu^\pi \|_1 \leq \frac{2\gamma \alpha}{1 - \gamma} \]

2. Before terminate, monotonic improvement (Thm 12.2 in AJKS):

Recall CPI:

1. Greedy Policy Selector:
   \[ \pi' \in \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^\pi} \left[ A^\pi'(s, \pi(s)) \right] \]

2. If \(\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^\pi} [A^\pi'(s, \pi(s))] \leq \epsilon\)
   Return \(\pi'\)

3. Incremental Update:
   \[ \pi_{t+1}(\cdot | s) = (1 - \alpha)\pi_t(\cdot | s) + \alpha\pi'(\cdot | s), \forall s \]

Local adv versus distribution change:
Summary of CPI so far:

1. Incremental update (Lemma 12.1 in AJKS)

\[ \| d_{\mu}^{t+1} - d_{\mu}^t \|_1 \leq \frac{2\gamma\alpha}{1 - \gamma} \]

2. Before terminate, monotonic improvement (Thm 12.2 in AJKS):

(By setting step size \( \alpha \) properly...)

Recall CPI:

1. Greedy Policy Selector:

\[ \pi' \in \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\pi^t}} \left[ A_{\pi'}(s, \pi(s)) \right] \]

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Summary of CPI so far:

1. Incremental update (Lemma 12.1 in AJKS)

\[ \|d_{\mu}^{\pi, t+1} - d_{\mu}^{\pi, t}\|_1 \leq \frac{2\gamma\alpha}{1 - \gamma} \]

2. Before terminate, monotonic improvement (Thm 12.2 in AJKS):

\[ V_{\mu}^{\pi, t+1} - V_{\mu}^{\pi, t} \geq \frac{\epsilon^2}{8\gamma} \]

(By setting step size \(\alpha\) properly...)

Recall CPI:

1. Greedy Policy Selector:

\[ \pi' \in \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^t}[A_{\pi'}(s, \pi(s))] \]

2. If \( \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^t}[A_{\pi'}(s, \pi(s))] \leq \epsilon \)

\[ \text{Return } \pi' \]

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\[ \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha\pi'(\cdot | s), \forall s \]

Local adv versus distribution change:
Summary of CPI so far:

1. Incremental update (Lemma 12.1 in AJKS)

\[ \| d_\mu^{\pi_{t+1}} - d_\mu^{\pi_t} \|_1 \leq \frac{2\gamma \alpha}{1 - \gamma} \]

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\[ V_\mu^{\pi_{t+1}} - V_\mu^{\pi_t} \geq \frac{\epsilon^2}{8\gamma} \]

(By setting step size $\alpha$ properly...)

Recall CPI:

1. Greedy Policy Selector:

\[ \pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim \hat{d}_\pi^t} \left[ A^{\pi'}(s, \pi(s)) \right] \]

2. If \[ \max_{\pi \in \Pi} \mathbb{E}_{s \sim \hat{d}_\pi^t}[A^{\pi'}(s, \pi(s))] \leq \epsilon \]

Return $\pi'$

3. Incremental Update:

\[ \pi^{t+1}(\cdot | s) = (1 - \alpha) \pi^t(\cdot | s) + \alpha \pi'(\cdot | s), \forall s \]

Local adv versus distribution change:
Summary of CPI so far:

1. Incremental update (Lemma 12.1 in AJKS)

\[
\|d^t_{\pi} - d^{t+1}_{\pi}\|_1 \leq \frac{2\gamma \alpha}{1 - \gamma}
\]

2. Before terminate, monotonic improvement (Thm 12.2 in AJKS):

\[
V^{\pi}_{\mu} - \mu^{\pi}_{\mu} \geq \frac{\epsilon^2}{8\gamma}
\]

(By setting step size $\alpha$ properly...)

Recall CPI:

1. Greedy Policy Selector:

\[
\pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^t_{\pi}} \left[ A^{\pi'}(s, \pi(s)) \right]
\]

2. If \( \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^t_{\pi}} [A^{\pi'}(s, \pi(s))] \leq \epsilon \)

Return $\pi'$

3. Incremental Update:

\[
\pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha \pi'(\cdot | s), \forall s
\]

Local adv versus distribution change:
Summary of CPI so far:

1. Incremental update (Lemma 12.1 in AJKS)

\[ \| d_{\mu}^{\pi^{t+1}} - d_{\mu}^{\pi^t} \|_1 \leq \frac{2\gamma\alpha}{1 - \gamma} \]

2. Before terminate, monotonic improvement (Thm 12.2 in AJKS):

\[ V_{\mu}^{\pi^{t+1}} - V_{\mu}^{\pi^t} \geq \frac{\epsilon^2}{8\gamma} \]

(By setting step size \( \alpha \) properly...)

Recall CPI:

1. Greedy Policy Selector:

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3. Incremental Update:

\[ \pi^{t+1}(\cdot \mid s) = (1 - \alpha)\pi^t(\cdot \mid s) + \alpha\pi'(\cdot \mid s), \forall s \]

Local adv versus distribution change:
Summary for today:

1. Algorithm: Conservative Policy Iteration:
   Find the local greedy policy, and move towards it a little bit

2. Small change in policies results small change in state distributions

2. Unlike API, incremental policy update ensures monotonic improvement