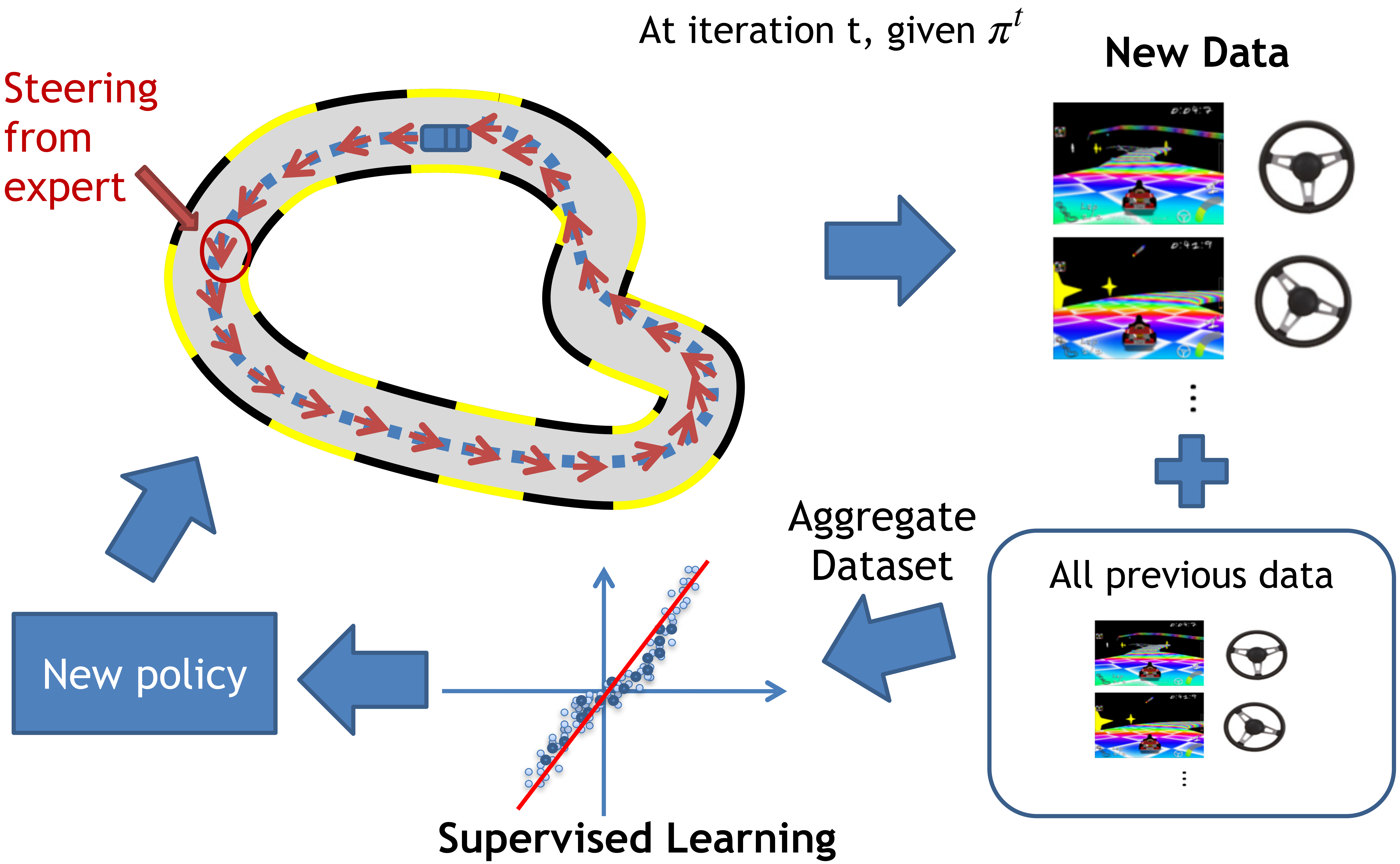
Maximum Entropy IRL

DAgger Recap



Data Aggregation = Follow-the-Regularized-Leader Online Learner

DAgger Performance Recap:

DAgger finds a policy $\widehat{\pi}$ such that it **matches to $\pi^\star$ under its own $d_\mu^{\widehat{\pi}}$**

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This is a significant improvement over BC in both theory and practice

Plan for Today:

1. The principle of Maximum Entropy

2. Constrained Optimization

2. The Algorithm: Maximum Entropy Inverse RL

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Finite horizon MDP $\mathscr{M} = \{S, A, H, c, P, \mu, \pi^\star\}$

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$c(s, a) = \langle \theta^\star, \phi(s, a) \rangle$, **linear w.r.t feature** $\phi(s, a)$

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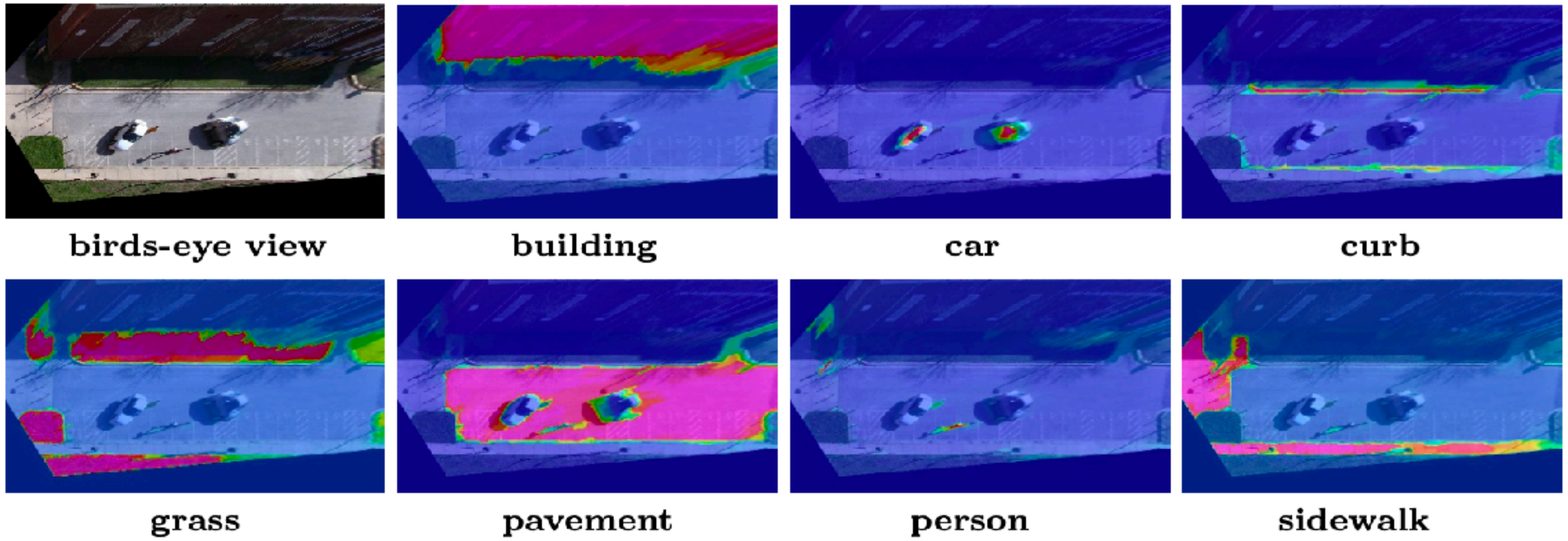

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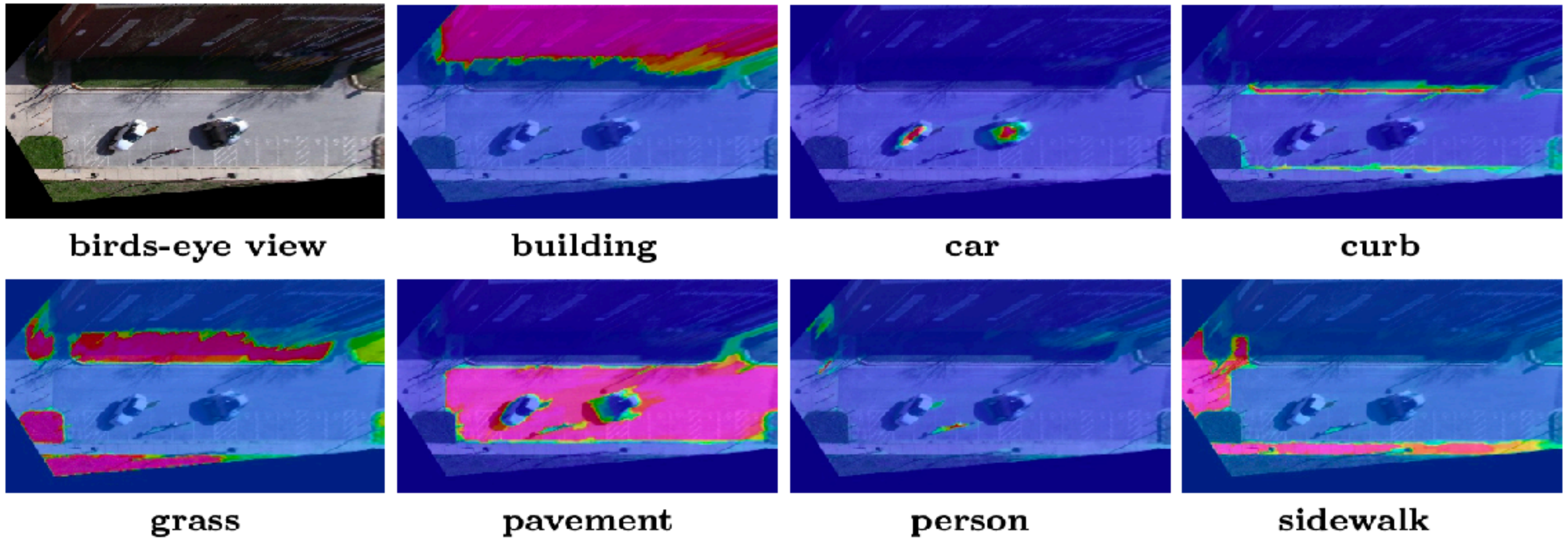

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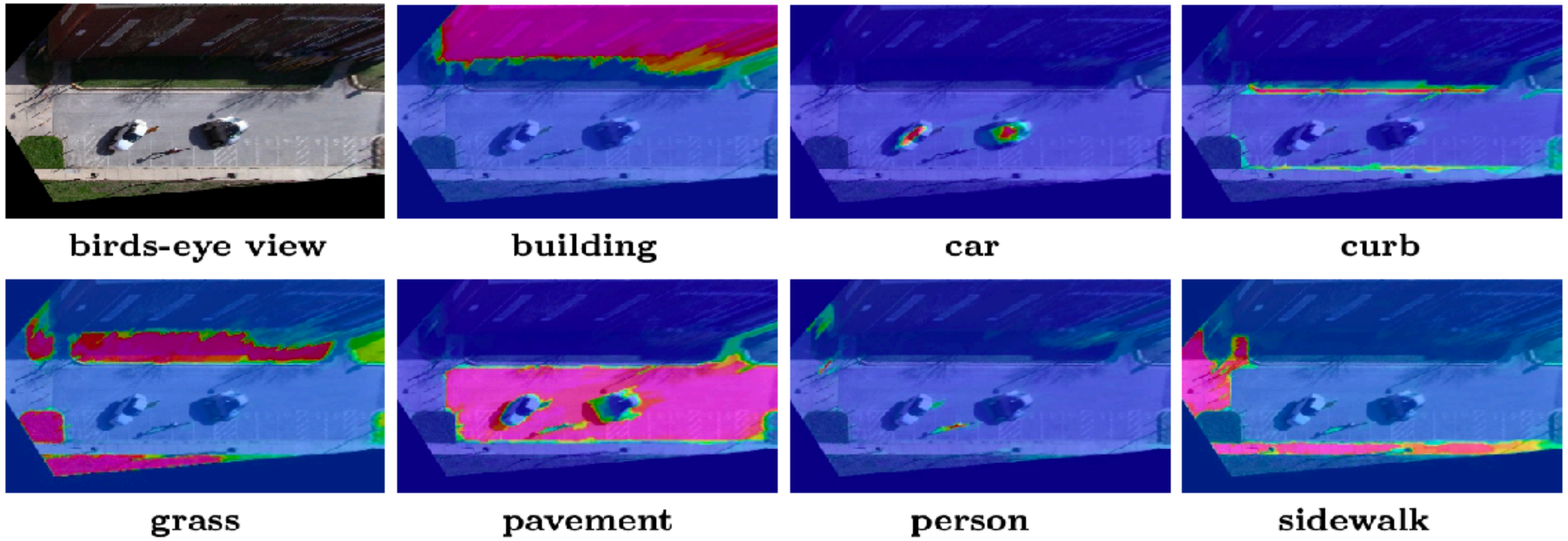

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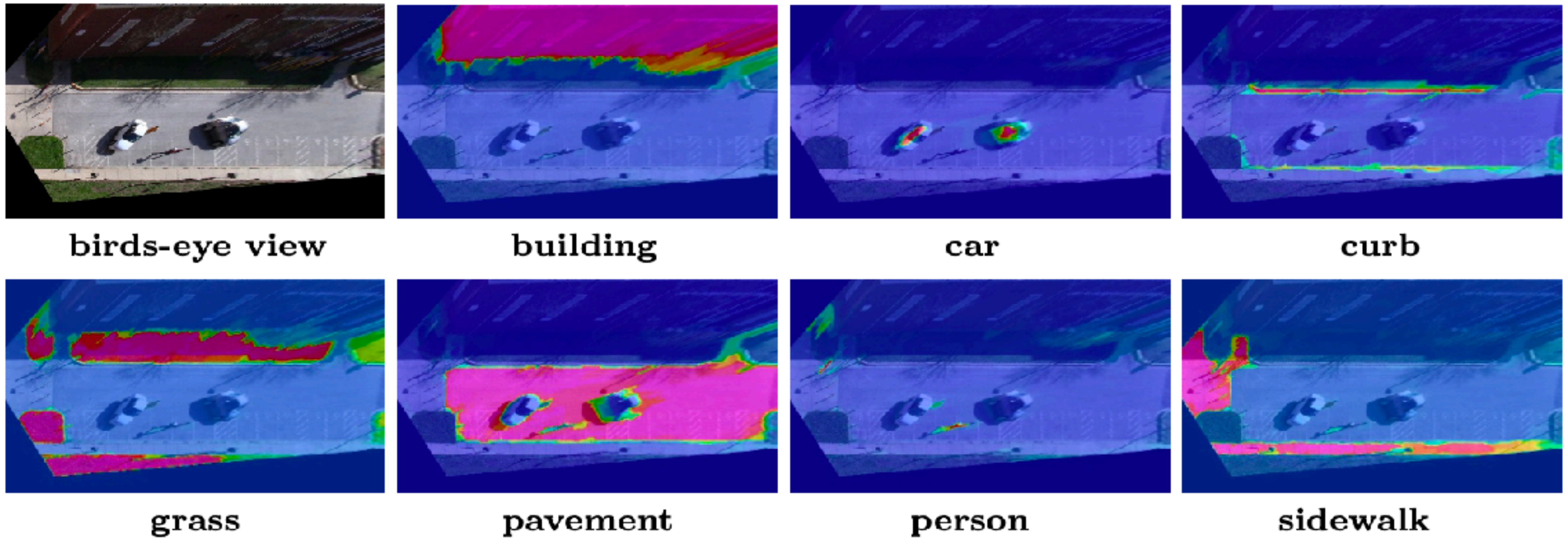

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Maybe colliding with cars or buildings has **high** cost, but walking on sideway or grass has **low** cost

Notation on Distributions

$\mathbb{P}_h^\pi(s, a; \mu)$: probability of visiting (s, a) at time step h following π

$d_\mu^\pi(s, a) = \sum_{h=0}^{H-1} \mathbb{P}_h^\pi(s, a; \mu)/H$: average state-action distribution

$\rho^\pi(\tau) := \mu_0(s_0)\pi(a_0 \mid s_0)P(s_1 \mid s_0, a_0)\pi(a_1 \mid s_1)\ldots\pi(a_{H-1} \mid s_{H-1})P(s_H \mid s_{H-1}, a_{H-1})$:
Likelihood of the trajectory τ under π, i.e., the prob of π generating τ

Detour: Principle of Maximum Entropy

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Deterministic distribution has zero entropy:
i.e., $\text{Entropy}(\delta(x_0)) = -1 \cdot \ln 1 - \sum_{x \neq x_0} 0 \ln 0 = 0$

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$$\max_{P \in \Delta(X)} \text{entropy}(P), \quad \text{s.t.,} \quad \mathbb{E}_{x \sim P}[x] = \mu, \quad \mathbb{E}_{x \sim P}[xx^\top] = \Sigma + \mu\mu^\top$$

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Solution: $P^\star = \mathcal{N}(\mu, \Sigma)$
(proof: out of scope)

Detour: Principle of Maximum Entropy

In summary:

Maximum Entropy Principle says that:

Among the distributions that satisfy pre-defined constraints (mean & variance),
let’s pick the one that is the most uncertain
(uncertainty measured in entropy)

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$$\min_{x} f(x)$$

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How to solve such constrained optimization problem?

Constrained Optimization:

Define two Lagrange multiplier $w_1, w_2 \in \mathbb{R}$, we consider the following Lagrange formulation:

$$\min_{x} \left[\max_{w_1, w_2} f(x) + w_1 g_1(x) + w_2 g_2(x) \right]$$

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In other words,

$$\max_{w_1, w_2} f(x) + w_1 g_1(x) + w_2 g_2(x) = \begin{cases} +\infty & g_1(x) \neq 0 \text{ or } g_2(x) \neq 0 \text{ i.e, infeasible} \\ f(x) & g_1(x) = g_2(x) = 0 \text{ i.e., feasible} \end{cases}$$

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Thus, solving the Lagrange formulation is equivalent to the original formulation:

$$\arg\min_x \left[\max_{w_1, w_2} f(x) + w_1 g_1(x) + w_2 g_2(x) \right] = x^\star$$

Constrained Optimization:

In summary, we have that

$$\arg\min_{x} \left[\max_{w_1, w_2} f(x) + w_1 g_1(x) + w_2 g_2(x) \right] = x^\star$$

Where $x^\star$ is the optimal solution of the original constrained program:

$$\min_{x} f(x)$$

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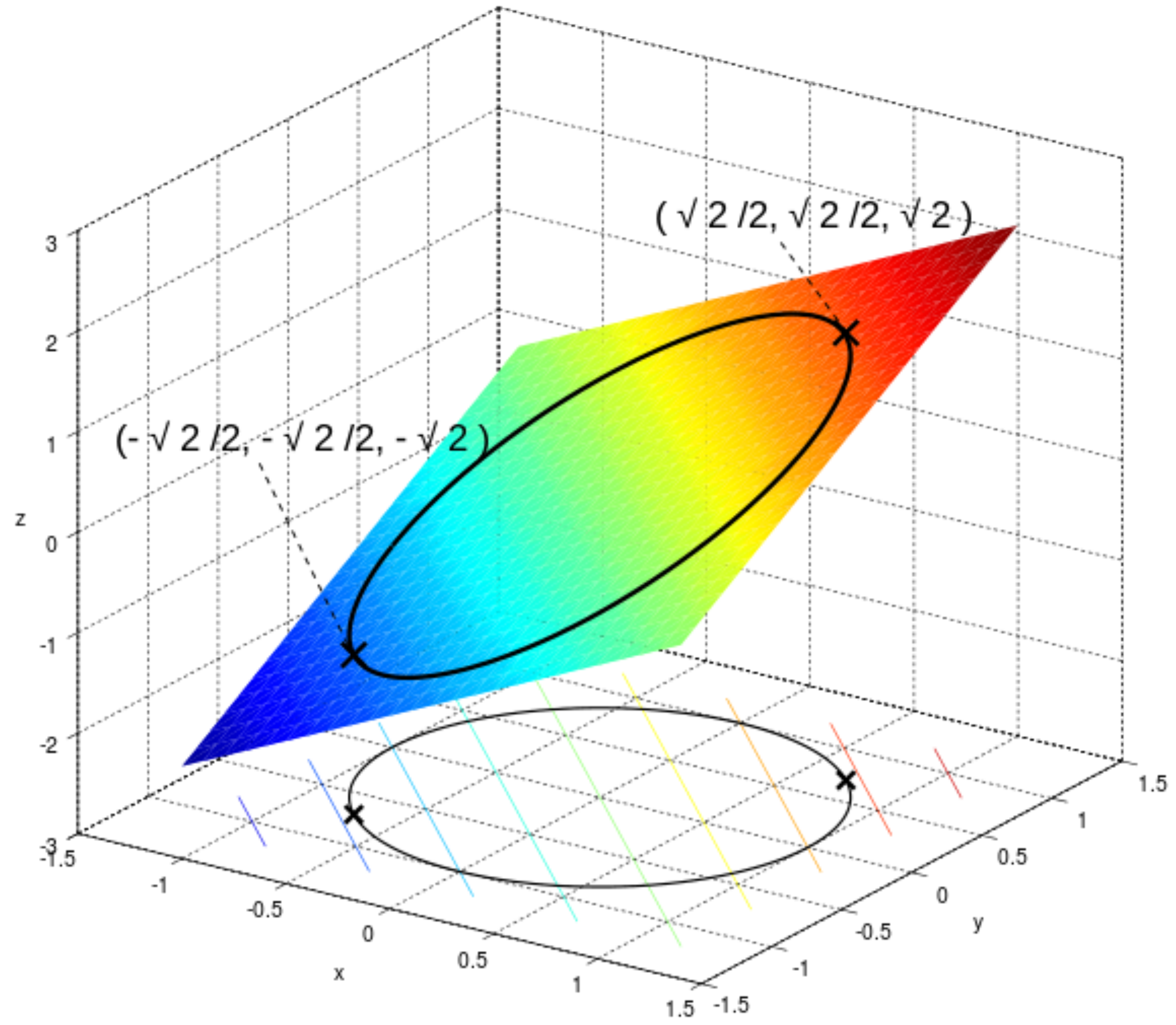

Example:

$$\min_{x,y} x + y, \text{ s.t.}, x^2 + y^2 = 1$$

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(#incremental update: $w^{t+1} = w^t + \eta \nabla_w \ell(x^t, w)$)

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Return: $\bar{x} = \sum_{t=0}^{T-1} x_t / T$ Informal theorem: when f, g are convex, $\bar{x} \to x^\star$, as $T \to \infty$

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This can be estimated using expert data: $\sum_{i=1}^{N} \phi(s_i^\star, a_i^\star)/N$

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$$\arg\max_{\pi} \mathbb{E}_{s \sim d_\mu^\pi} \left[\text{entropy}(\pi(\cdot \mid s)) \right] = \arg\min_{\pi} \mathbb{E}_{s,a \sim d_\mu^\pi} \ln \pi(a \mid s)$$

Maximum Entropy Inverse RL:

We arrive at the following constraint optimization problem:

$$\arg\min_{\pi} \mathbb{E}_{s,a\sim d_\mu^\pi} \ln \pi(a\,|\,s)$$

$$s.t, \mathbb{E}_{s,a\sim d_\mu^\pi}\phi(s,a) = \mathbb{E}_{s,a\sim d_\mu^{\pi^\star}}\phi(s,a)$$

Introduce the Lagrange multiplier $w \in \mathbb{R}^d$ (we have d many constraints), consider the max-min dual version:

$$\max_{w\in\mathbb{R}^d} \min_{\pi} \mathbb{E}_{s,a\sim d_\mu^\pi} \ln \pi(a\,|\,s) + w^\top \left(\mathbb{E}_{s,a\sim d_\mu^\pi}\phi(s,a) - \mathbb{E}_{s,a\sim d_\mu^{\pi^\star}}\phi(s,a) \right)$$

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Next lecture,
we will design algorithm (in high level, it is the iterative algorithm framework)
for this $\max - \min$ problem