Markov Decision Process

Announcements

TA office hours are posted

HW0 is due Wednesday

Programming assignment 1 will be out on Wednesday

Reading Materials:
Reinforcement Learning: Theory & Algorithms

https://rltheorybook.github.io/

This is an extremely advanced RL book, so we will pick **specific subsections** for you to read

Please let us know if you find any typos or mistakes in the book

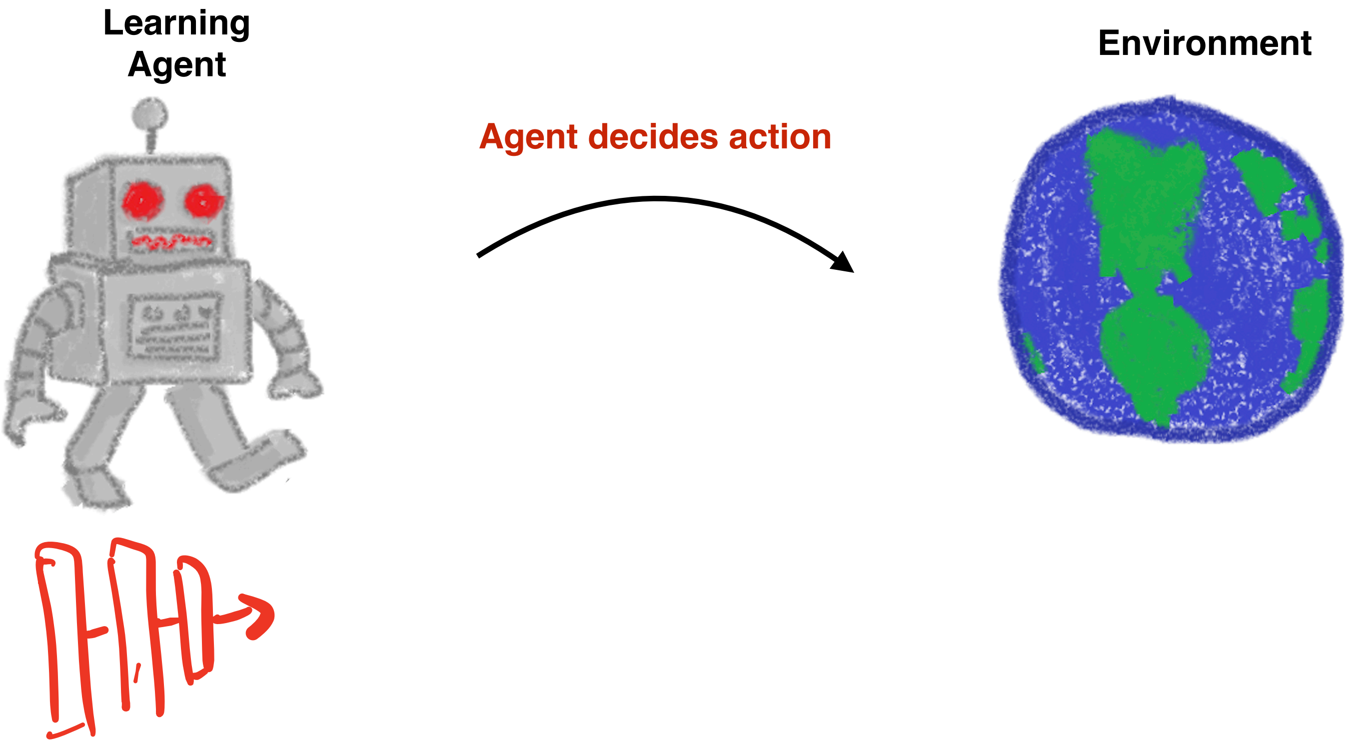
Outlines:

1. Definitions of Markov Decision Process

2. Value functions (V and Q functions)

3. Bellman equations

The Mathematical framework:
Infinite horizon **Markov Decision Process**



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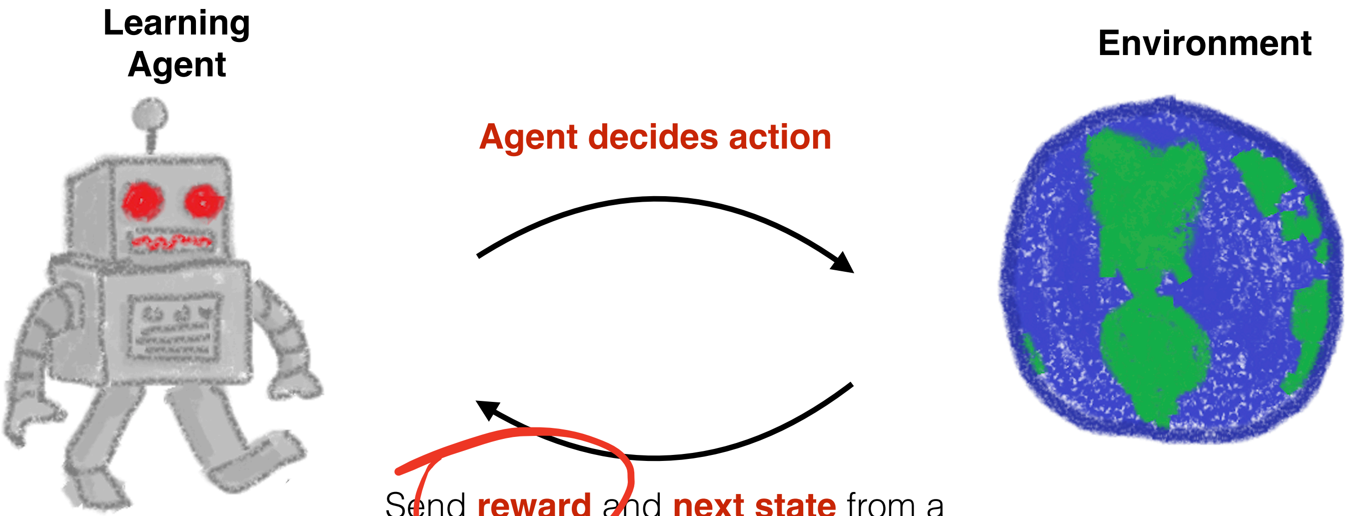

Send **reward** and **next state** from a Markovian transition dynamics

$$r(s,a), s' \sim P(\cdot \mid s,a)$$

$P(. \mid s,a)$: distribution over the next state

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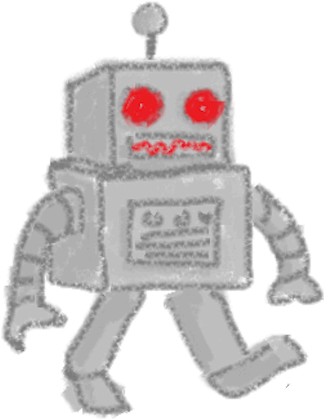
Learning Agent

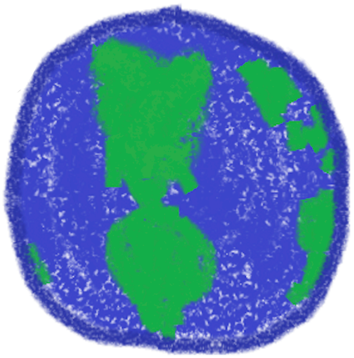


Agent decides action

Infinity many Steps

Send **reward** and **next state** from a Markovian transition dynamics

Environment



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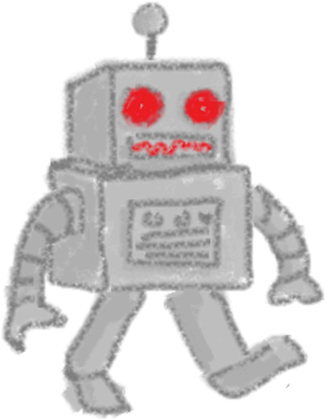

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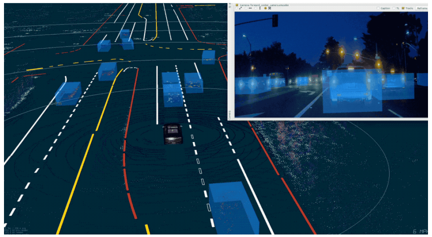

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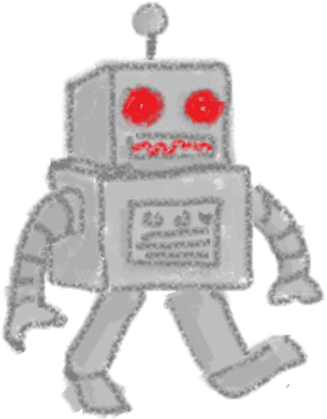

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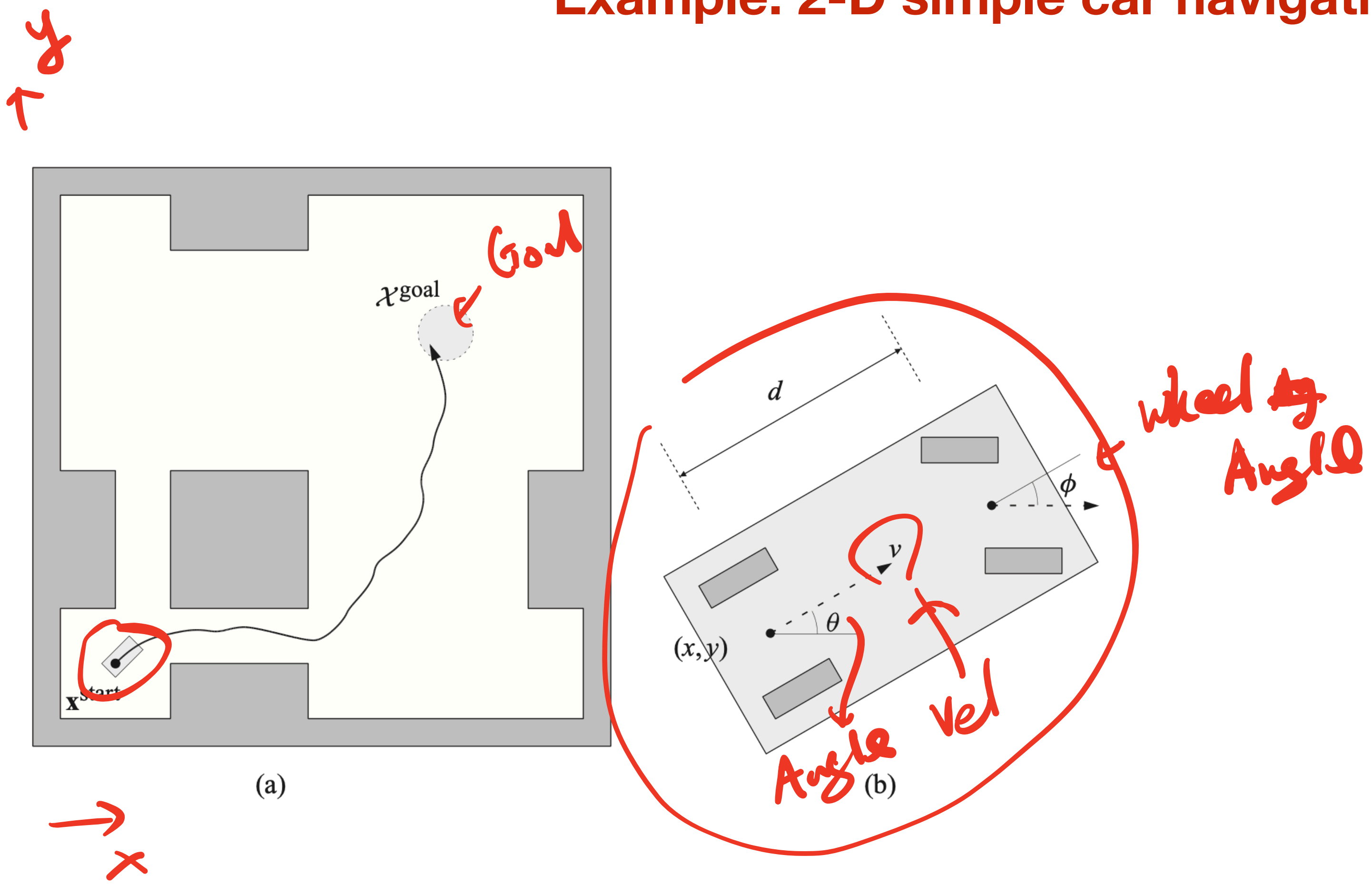

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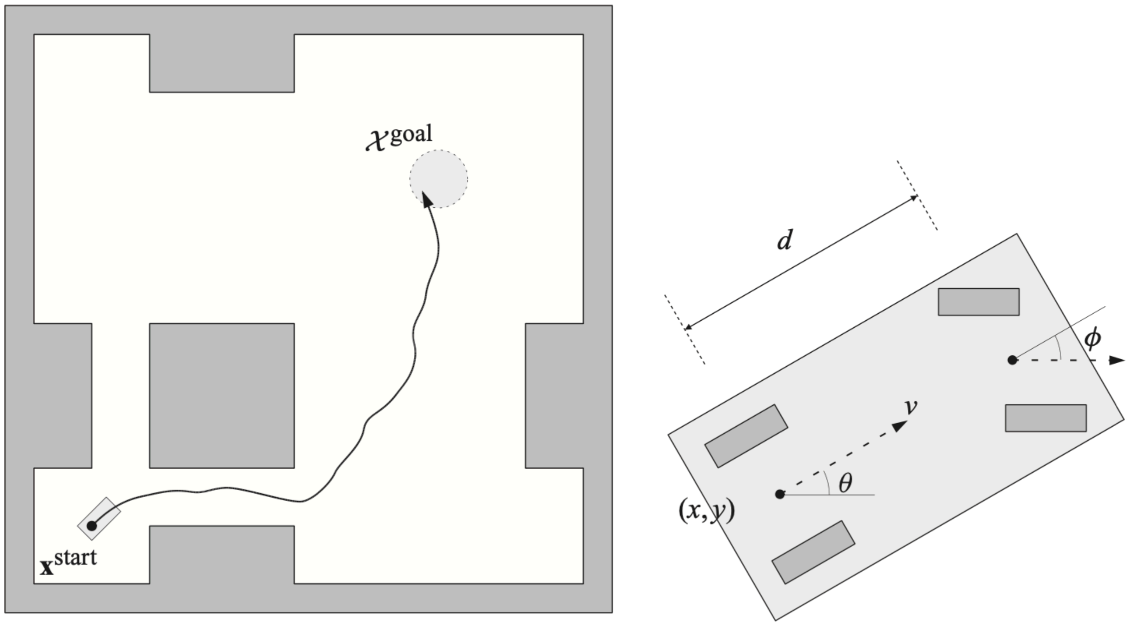
$$s = [x, y, \theta, v]^\top \in \mathbb{R}^4$$



(a) (b)

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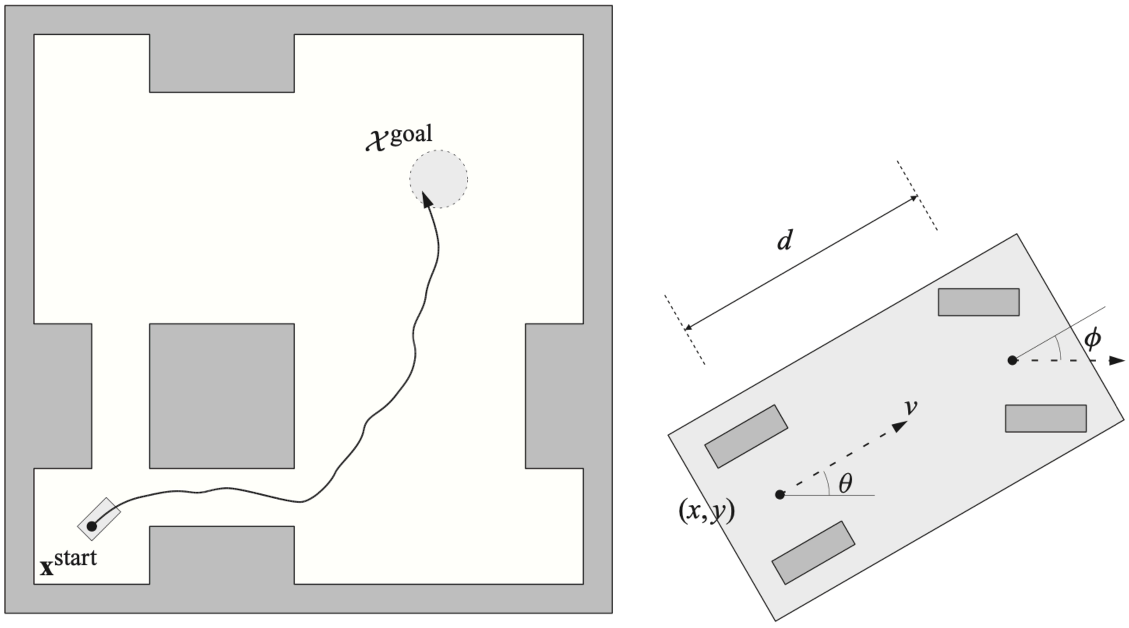

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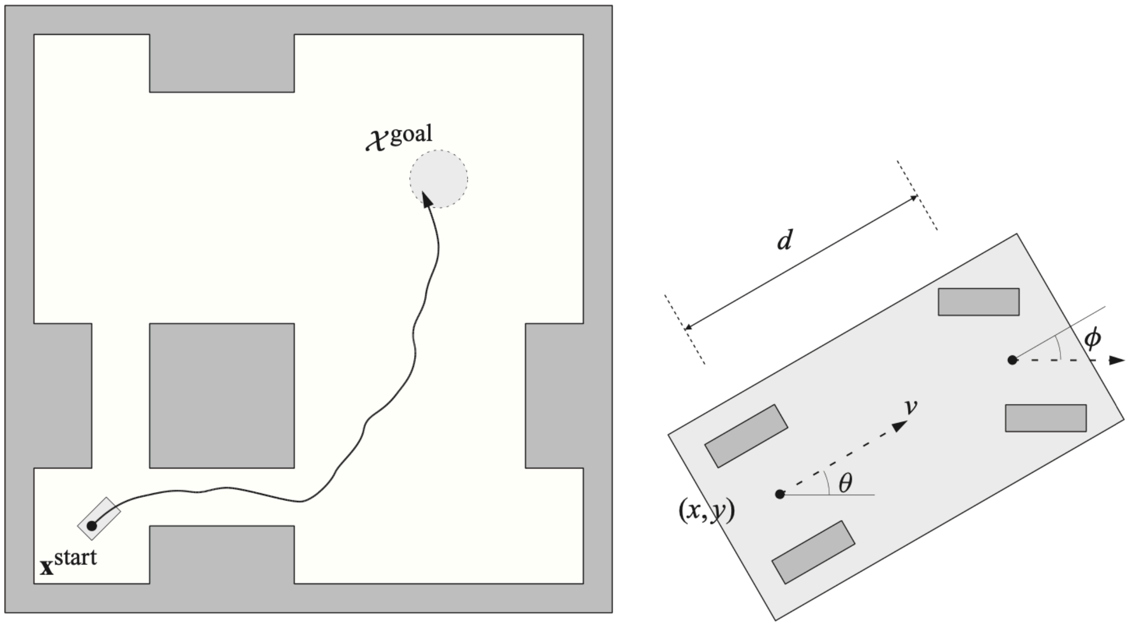
Acc

Wheel Angle

Example: 2-D simple car navigation



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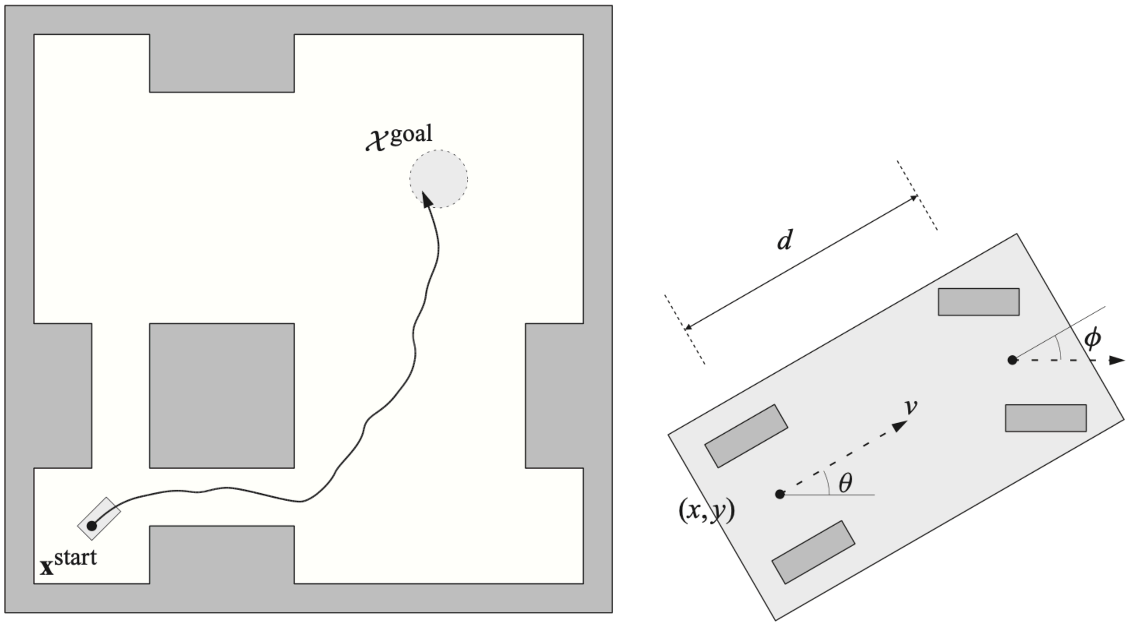

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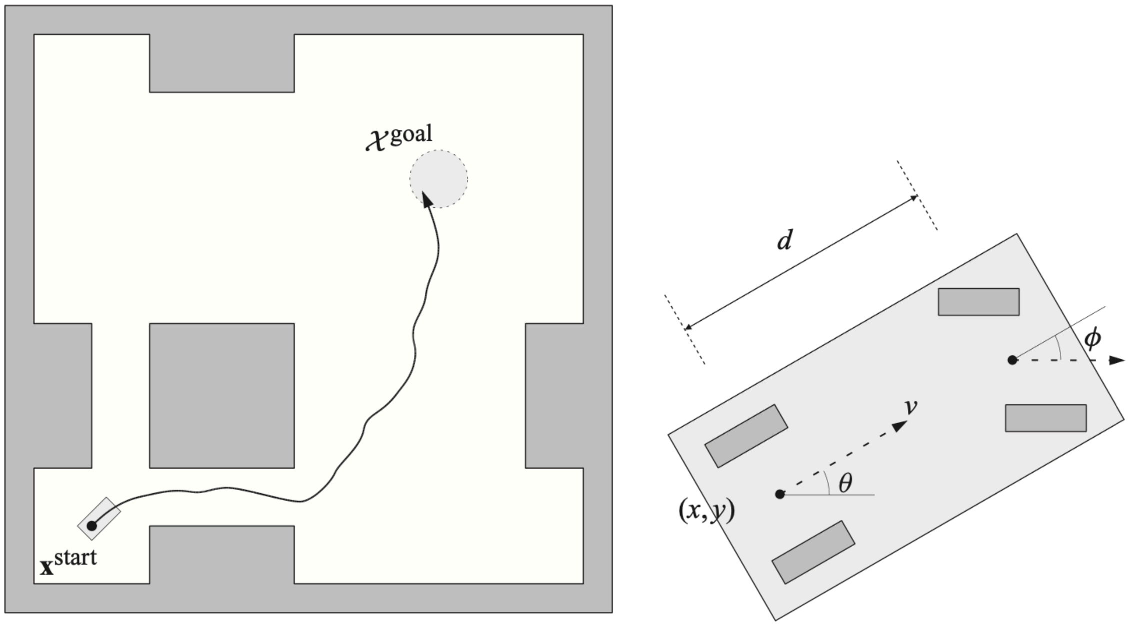

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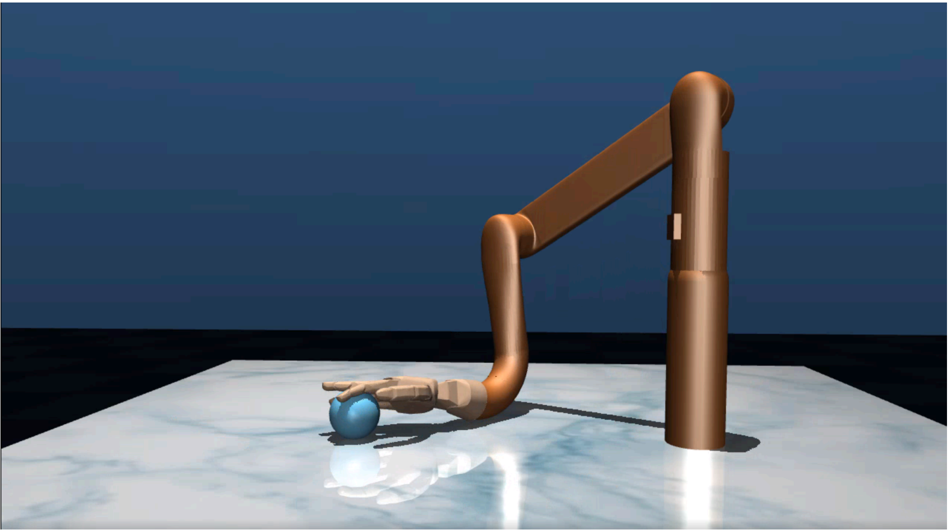
$$f(s,a) = \begin{bmatrix} x + \tau v \cos\theta \\ y + \tau v \sin\theta \\ \theta + \tau v \tan(\phi)/d \\ v + \tau\alpha \end{bmatrix}$$

Example:
robot hand needs to pick the ball and hold it in a goal (x,y,z) position



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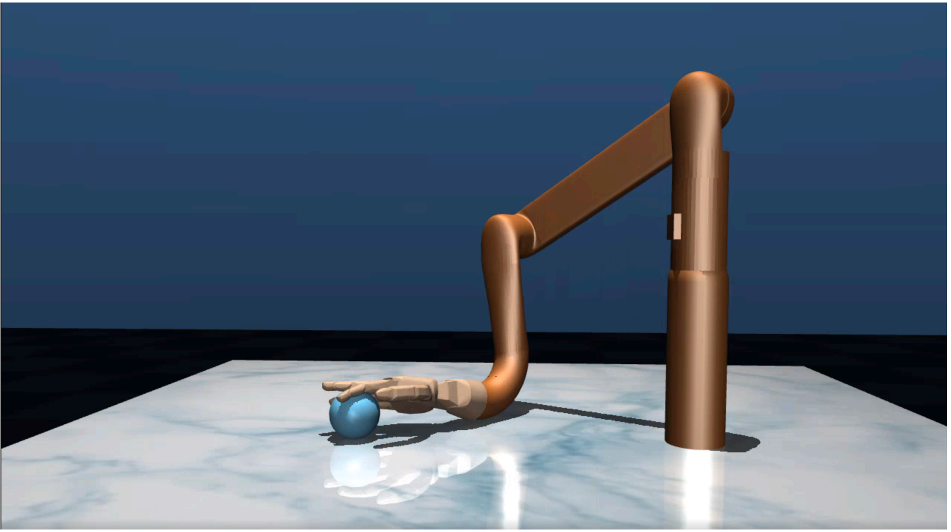

State s: robot configuration (e.g., joint angles) and the ball's position

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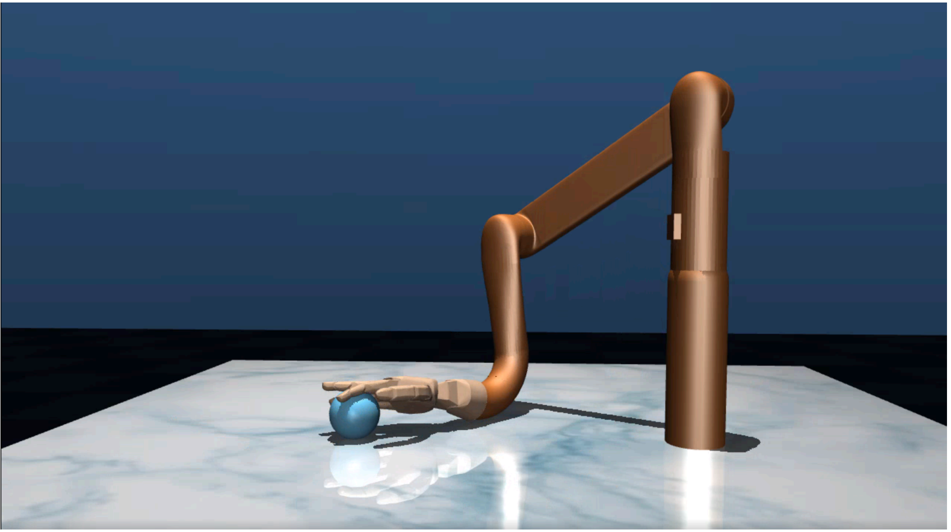

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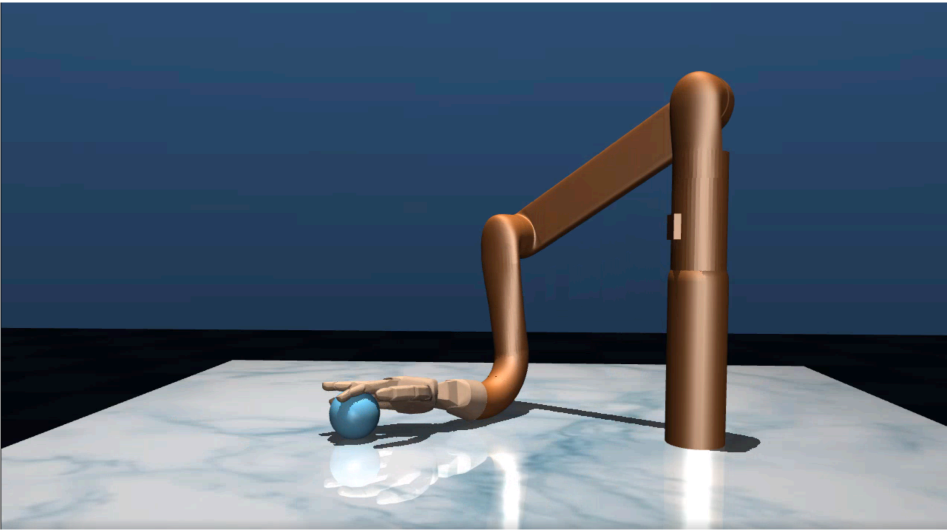

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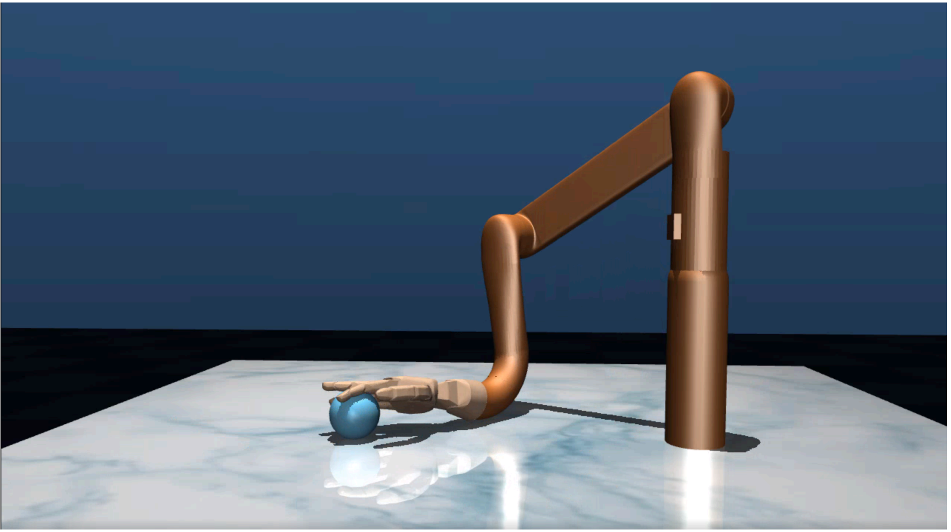

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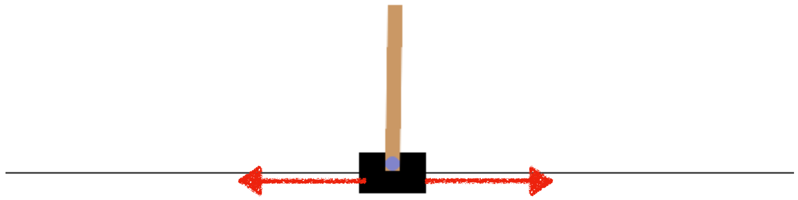
Cost $c(s, a)$: torque magnitude + dist to goal

Example: OpenAI Gym demonstrations

cart pole



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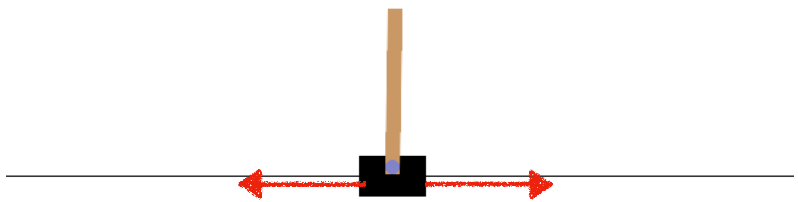

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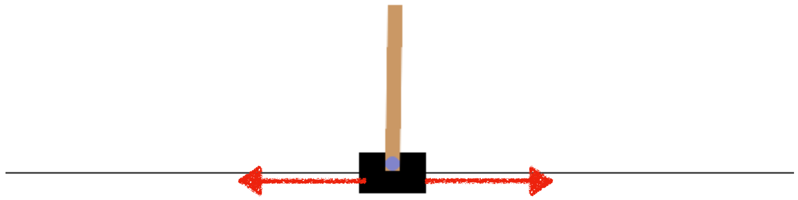
State = [cart pos, cart velocity, pole angle, pole angular velocity] $\in \mathbb{R}^4$



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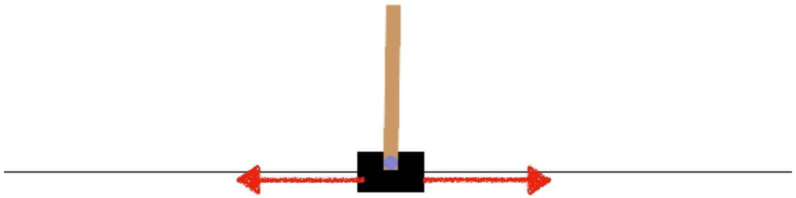

Push left

$a = 0$

Push right

$a = 1$

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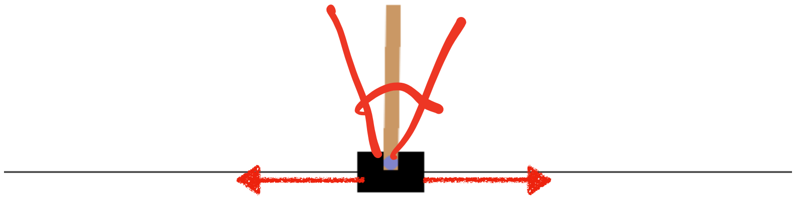

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$$r(s, a) = \begin{cases} 1 & \text{pole angle} \in [-12^o, 12^o], \\ 0 & \text{else} \end{cases}$$

Policy

$$\mathscr{S} \rightarrow \mathscr{A}$$

A mapping from state to action (what action should I take if I'm in this state…)

$$a \sim \pi(\cdot \mid s)$$

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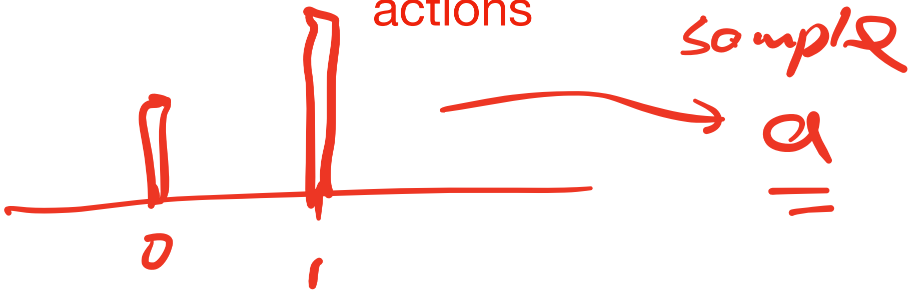
conditional distribution over all actions



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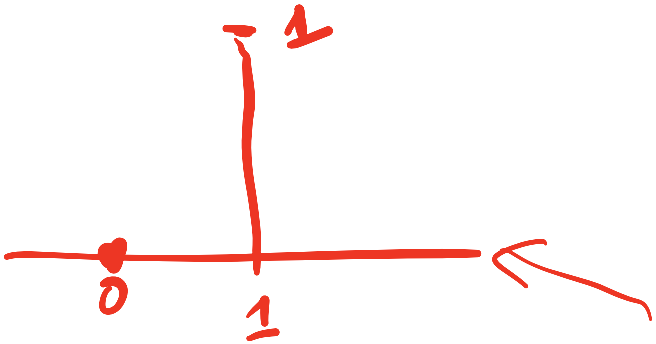

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Deterministic vs stochastic?

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Deterministic vs stochastic?

Q: Assume S state and A actions, how many different deterministic policies we can have?

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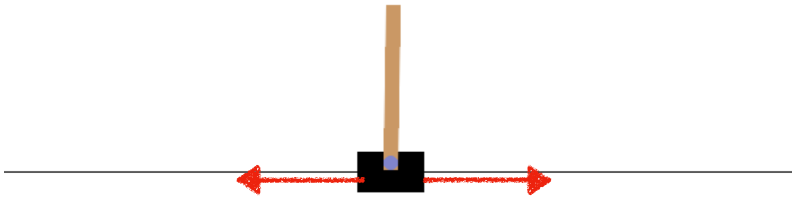

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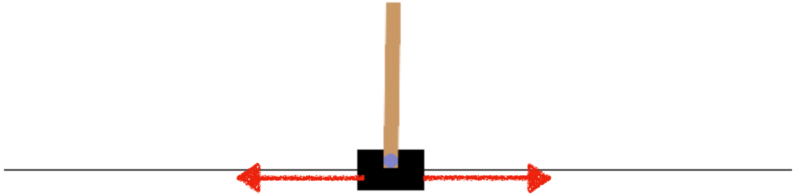

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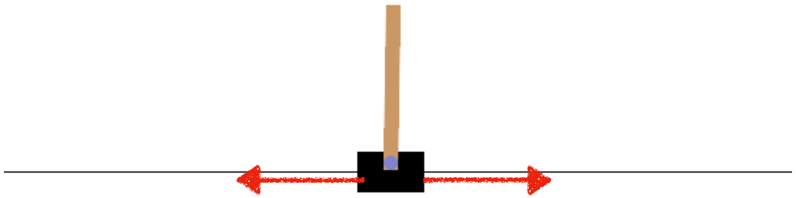

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Policy 2: adaptive

$$\pi(s) = \begin{cases} 0(\text{left}) & \text{if pole angle} < 0 \\ 1(\text{right}) & \text{else} \end{cases}$$

Outlines:

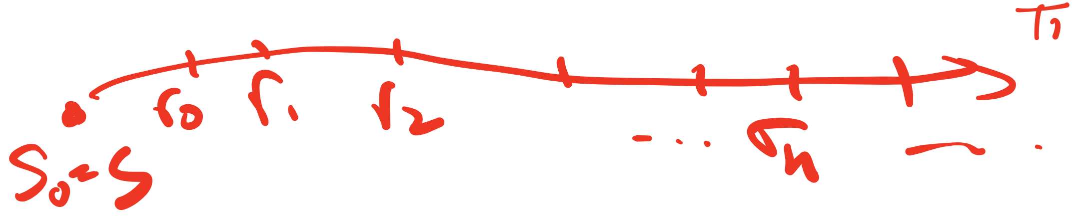
1. Definitions of Markov Decision Process
2. Value functions (V and Q functions)
3. Bellman equations

Performance of a policy π

Expected total reward of a policy π:

$$V^\pi(s) = \mathbb{E}\left[r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots + \gamma^h r_h + \ldots \middle| s_0 = s, \pi \right]$$

$\gamma \in [0,1)$: discount factor (value future reward less and less)



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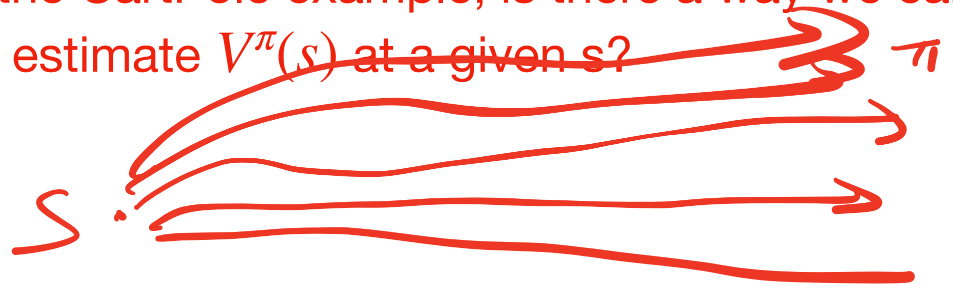
Q: think about the CartPole example, is there a way we can estimate $V^{\pi}(s)$ at a given s?



Optimal policy

$\pi^\star$: the policy that maximizes expected future reward at all states

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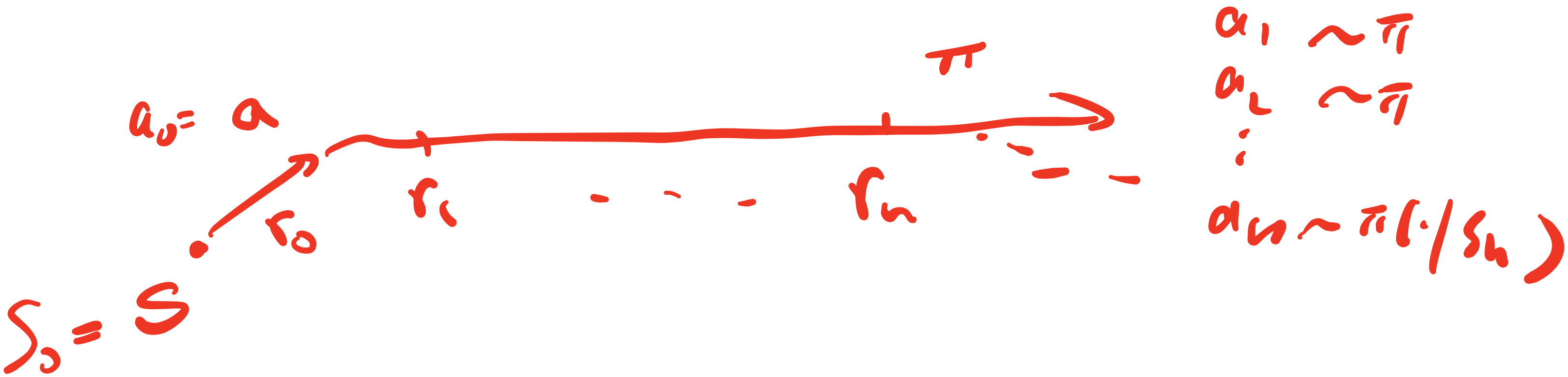
Q: what is the optimal policy when $\gamma = 0$?

State-action Q function

$$Q^{\pi}(s, a) = \mathbb{E}\left[r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots + \gamma^h r_h + \ldots \mid s_0 = s, a_0 = a, \pi\right]$$



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Can we quantify V / Q using one-step transition?

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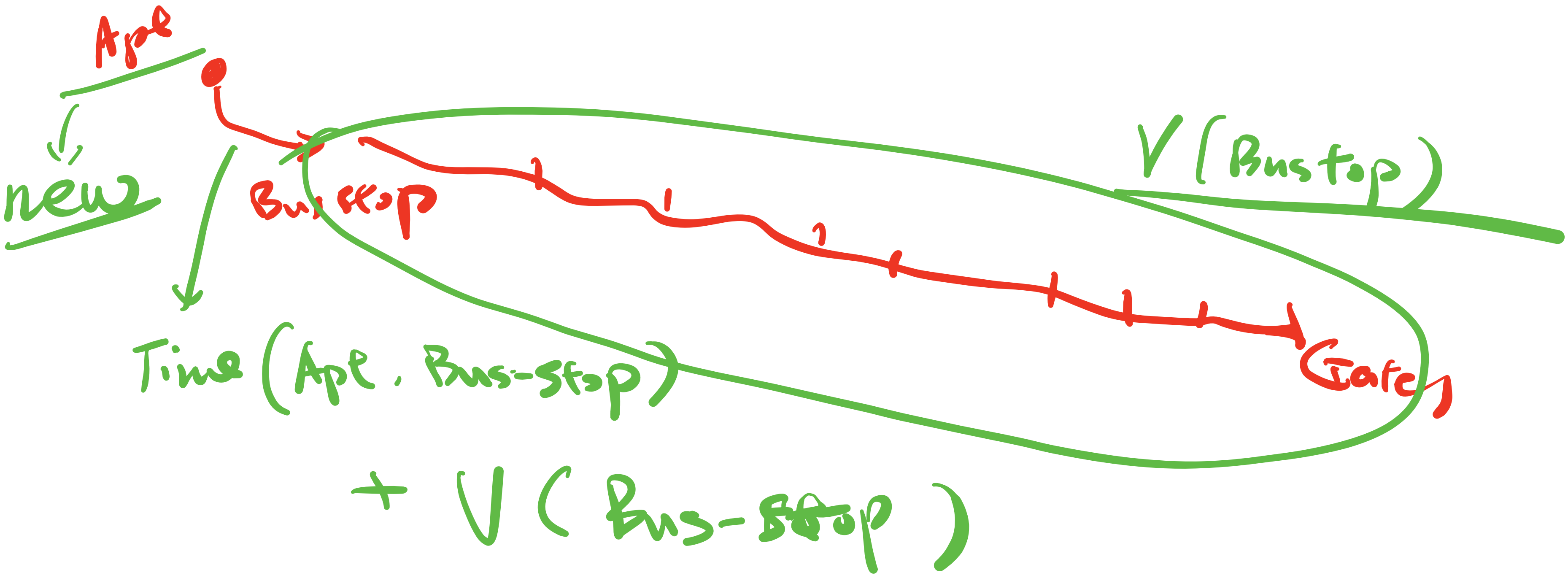

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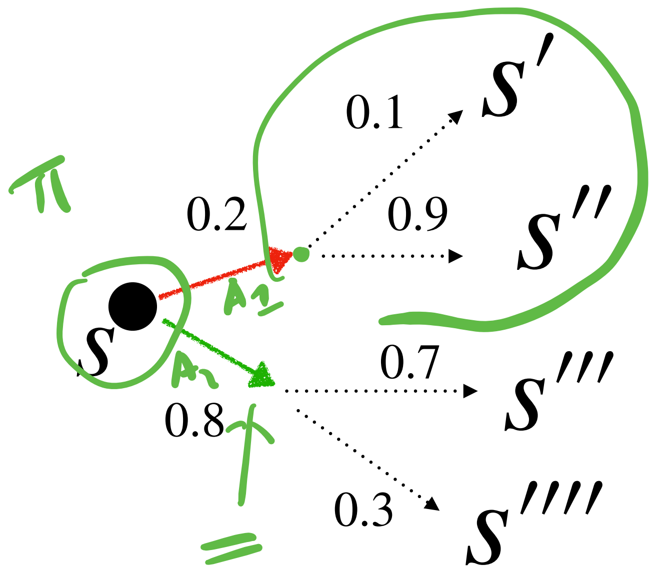

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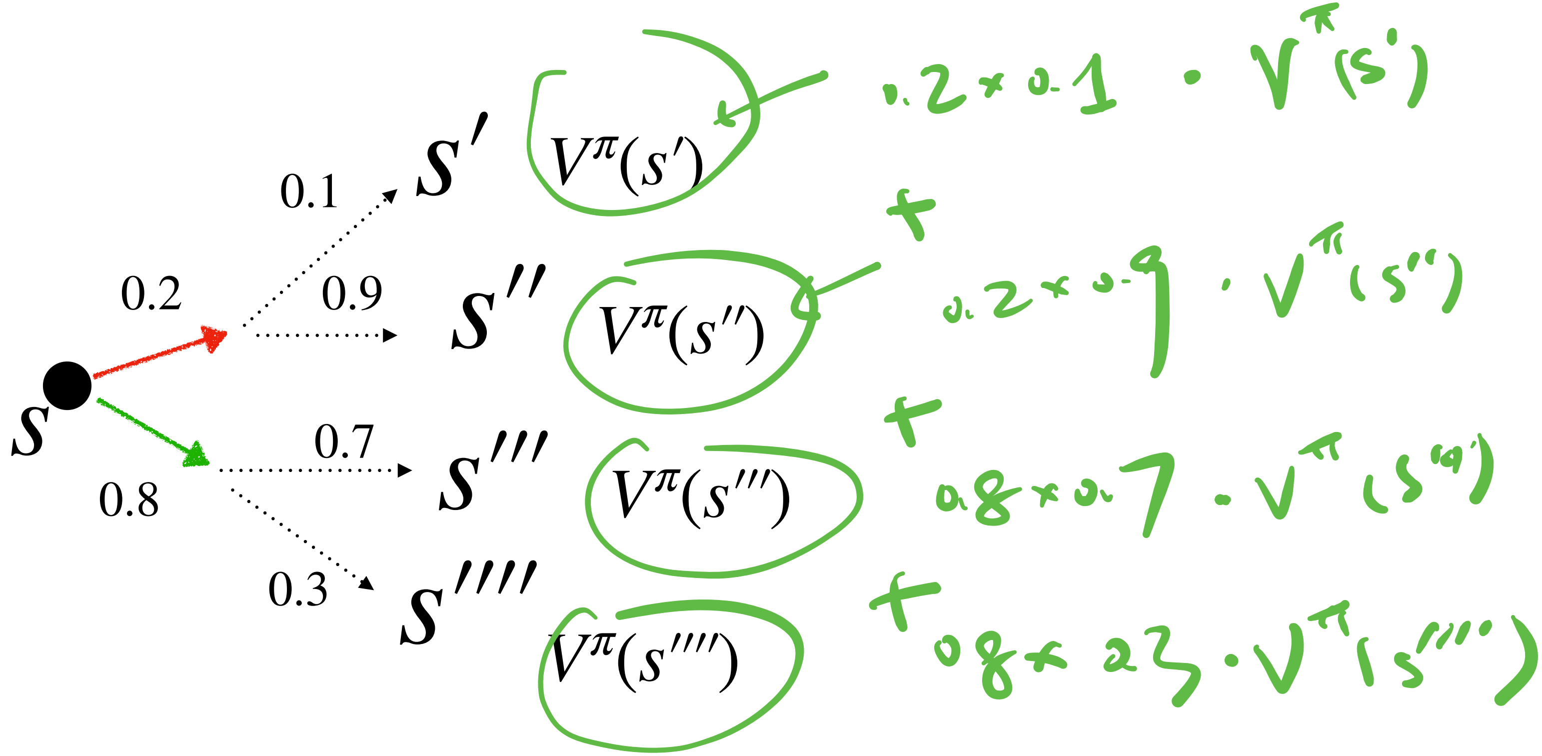

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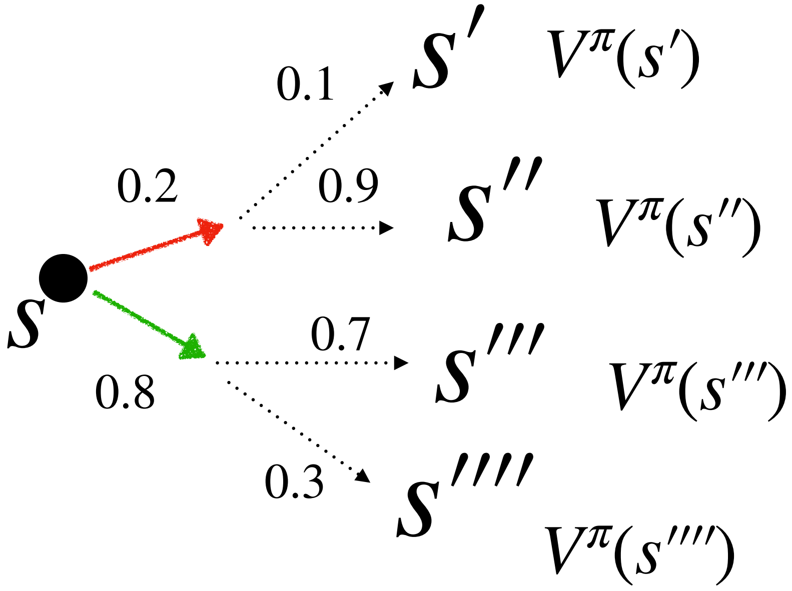

$$V^\pi(s) = \mathbb{E}_{a\sim\pi(\cdot|s)}\left[r(s,a) + \gamma \mathbb{E}_{s'\sim P(.|s,a)}\left[V^\pi(s')\right]\right]$$

Bellman equation for value function

Can we quantify V / Q using one-step transition?

Your homework: understand the one-step relationship between V and Q

$$Q^\pi(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(.|s,a)} \left[V^\pi(s') \right]$$

$$V^\pi(s) = \mathbb{E}_{a \sim \pi(\cdot|s)} Q^\pi(s,a)$$

Summary:

- **Discounted infinite horizon MDP:**
  - State, action, policy, transition, reward (or cost), discount factor
  - **V function and Q function**
  - Key concept: **Bellman equation**