HW2 due ~~Fo~~ Friday

Approximate Policy Iteration
& Conservative Policy Iteration

Recap

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$\max_a A^\pi(s,a) \geq 0$

$$= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_{s_0}^{\pi'}}\left[\underbrace{\max_{a\in A} A^\pi(s,a)}\right] \geq 0 \quad \checkmark$$

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Setting and Notation

Discounted infinite horizon MDP:

$$\mathcal{M} = \{S, A, \gamma, r, P, \mu\}$$

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$s \; a \quad \pi \quad Q^\pi(s,a)$

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Note that the optimal policy $\pi^\star$ may not be in Π $\qquad \pi^\star \notin \Pi$

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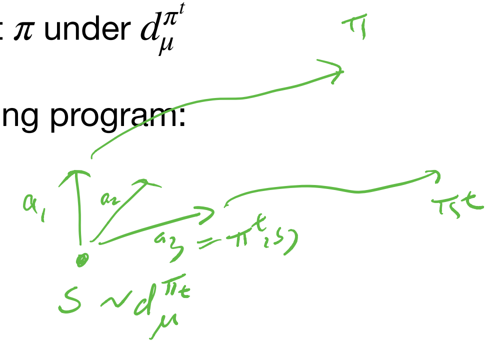

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We can hope for an Approximate Greedy Policy Selector a reduction to Regression

Implementing Approximate Greedy Policy Selector via Regression

We can do a **reduction to Regression** via Advantage function approximation

$$\mathscr{F} = \{f : S \times A \mapsto \mathbb{R}\} \quad (\approx A^{\pi^t})$$

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$f \approx A^{\pi^t}, \forall s, a$

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unbiased from MC rollout

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Do **finite sample analysis for Regression** first, and then transfer the guarantee to greedy policy selection

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Analyzing Approximation error via Regression

Greedy Policy Selector

$$\widetilde{\pi} := \arg\max_{\pi \in \Pi} \mathbb{E}_{s\sim d_\mu^{\pi^t}} \left[A^{\pi^t}(s, \pi(s)) \right]$$

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$$\boxed{\mathbb{E}_{s\sim d_\mu^{\pi^t}}[A^{\pi^t}(s, \hat{\pi}(s))]} = \mathbb{E}_{s\sim d_\mu^{\pi^t}} \left[\hat{f}(s, \hat{\pi}(s)) + A^{\pi^t}(s, \hat{\pi}(s)) - \hat{f}(s, \hat{\pi}(s)) \right]$$

$$\geq \mathbb{E}_{s\sim d_\mu^{\pi^t}} \left[\hat{f}(s, \widetilde{\pi}(s)) + A^{\pi^t}(s, \hat{\pi}(s)) - \hat{f}(s, \hat{\pi}(s)) \right]$$

$$\geq \mathbb{E}_{s\sim d_\mu^{\pi^t}} \left[A^{\pi^t}(s, \widetilde{\pi}(s)) + \hat{f}(s, \widetilde{\pi}(s)) - A^{\pi^t}(s, \widetilde{\pi}(s)) + A^{\pi^t}(s, \hat{\pi}(s)) - \hat{f}(s, \hat{\pi}(s)) \right]$$

$$\geq \mathbb{E}_{s\sim d_\mu^{\pi^t}} \left[A^{\pi^t}(s, \widetilde{\pi}(s)) \right] + \mathbb{E}_{s\sim d_\mu^{\pi^t}} \left[\hat{f}(s, \widetilde{\pi}(s)) - A^{\pi^t}(s, \widetilde{\pi}(s)) + A^{\pi^t}(s, \hat{\pi}(s)) - \hat{f}(s, \hat{\pi}(s)) \right]$$

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$\uparrow$

$\to 0$

N # of Training samples

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i.e., we assume we can do the exact greedy policy selector: $\arg\max_{\pi\in\Pi} \mathbb{E}_{s\sim d_\mu^{\pi^t}}\left[A^{\pi^t}(s, \pi(s))\right]$

Algorithm: Approximate Policy Iteration (API)

Iterate: π_t

API: $$\underset{A}{\pi^{t+1}} \in \underbrace{\arg\max_{\pi \in \Pi} \mathbb{E}_{s,a \sim d_\mu^{\pi^t}} \left[A^{\pi_t}(s, \pi(s)) \right]}$$

← reduce Regression

$N \to \infty$

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Question:

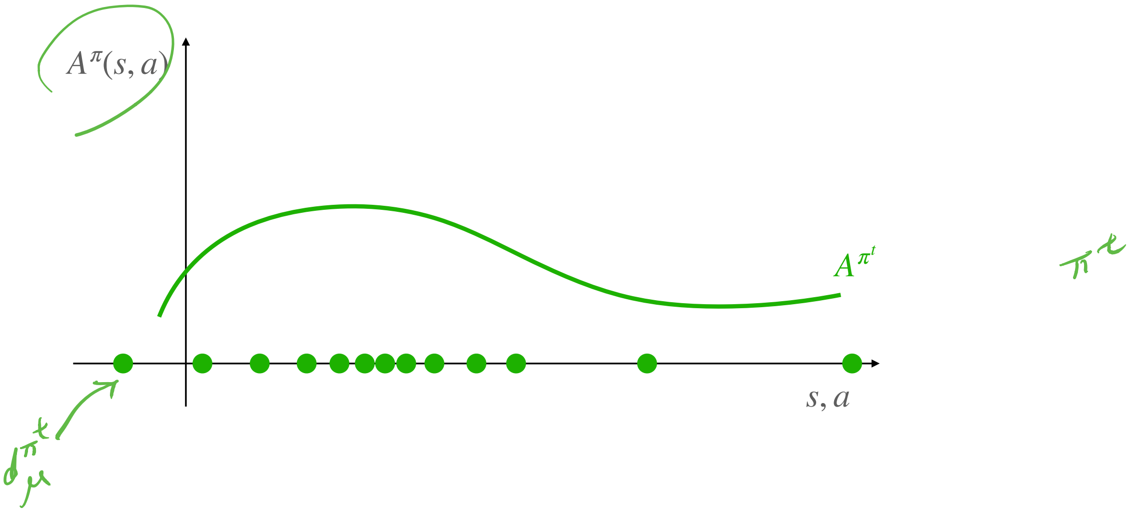
(1) Does API has monotonic improvement?

(2) Does it convergence?

The Oscillation of API from Abrupt Distribution Change

API cannot guarantee to succeed (let's think about advantage function approximation setting)

Concrete example in Chapter 3



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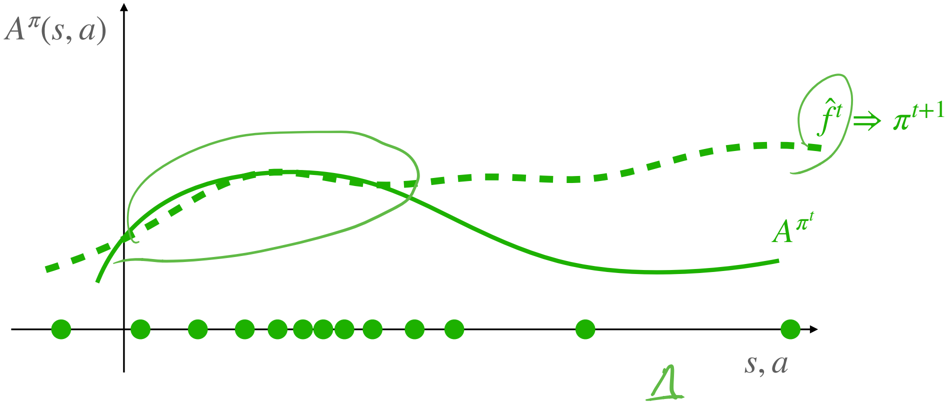

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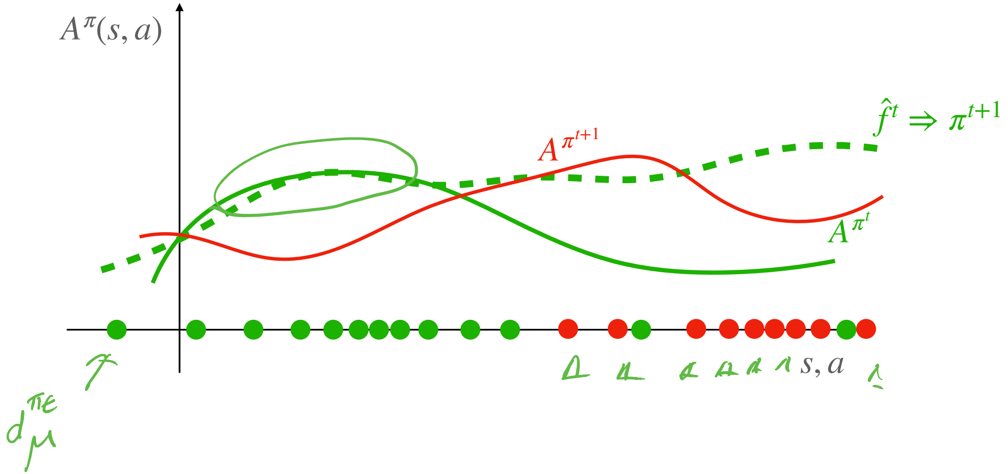

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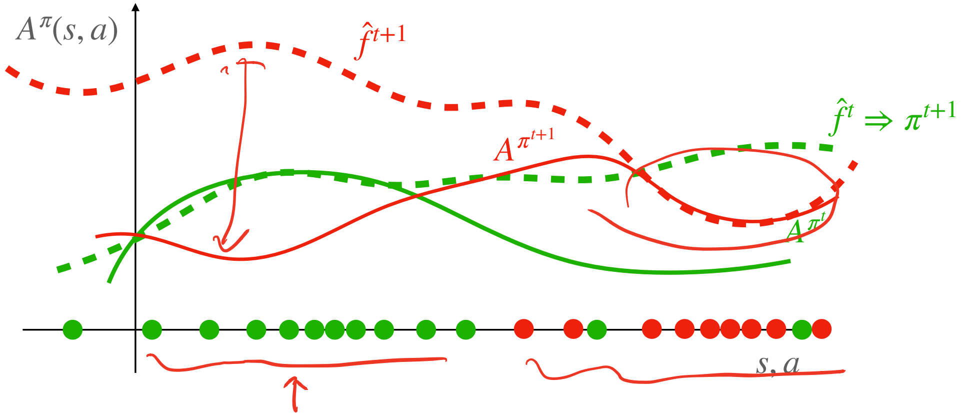

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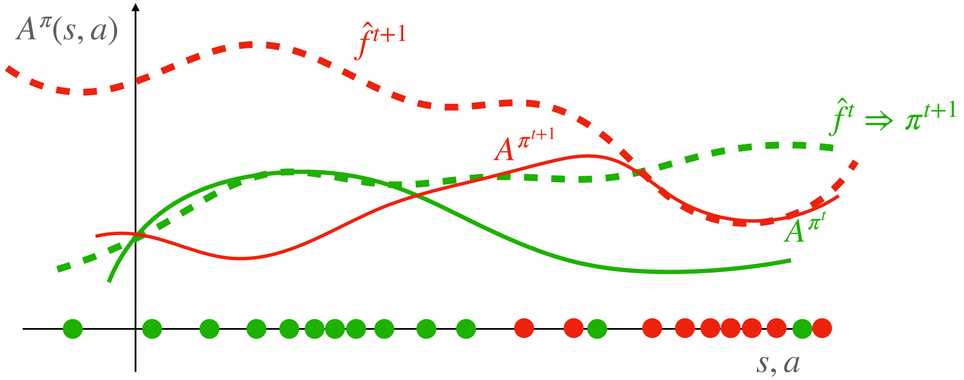

Oscillation between two updates: No monotonic improvement

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$\mathbb{E}_{s\sim d_\mu^{\pi^t}, a\sim U(A)}(\hat{f}^t(s,a) - A^{\pi^t}(s,a))^2$ small,

but $\mathbb{E}_{s\sim d_\mu^{\pi^{t+1}}, a\sim U(A)}(\hat{f}^t(s,a) - A^{\pi^t}(s,a))^2$ **might be arbitrarily big**

$d_\mu^{\pi^t} \neq d_\mu^{\pi^{t+1}}$

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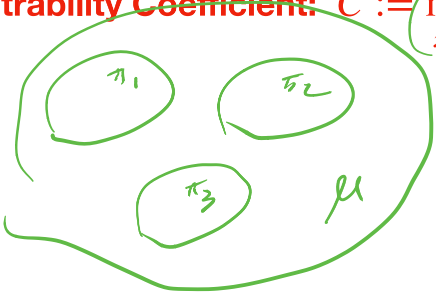

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If $C < \infty$, i.e., μ covers **all** d_μ^π, then we can expect error

$\mathbb{E}_{s\sim d_\mu^{\pi^{t+1}}, a\sim U(A)}(\hat f^t(s,a) - A^{\pi^t}(s,a))^2$ is reasonably under control;

① $d_\mu^{\pi^t} \approx d_\mu^{\pi^{t+1}}$

$d^{\pi^{t+1}}$ μ

Conservative Policy Iteration—An Incremental Policy Optimization Approach

(And the benefit of being incremental)

NPG & Trust Region

$\Uparrow$

$KL\left(\Pr^{\pi^{t+1}} \,\|\, \Pr^{\pi^t}\right) \leq \delta$

Key Idea of CPI: Incremental Update—No Abrupt Distribution Change

Let's design policy update rule such that $d^{\pi^{t+1}}$ and d^{π^t} are not that different!

Δ Δ

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$$d^{\pi^t} \approx d^{\pi^{t+1}}$$

$$\text{s.t., } \mathbb{E}_{s \sim d^{\pi^t}} \left[A^{\pi^t}(s, \pi^{t+1}(s)) \right] \approx \mathbb{E}_{s \sim d^{\pi^{t+1}}} \left[A^{\pi^t}(s, \pi^{t+1}(s)) \right]$$

≥ 0 ≥ 0

This we know how to optimize: the Greedy Policy Selector

Regression, $s \sim d^{\pi^t}$, $a \sim U(A)$, $\hat{A}^{\pi^t}(s,a)$

CPI Algorithm:

CPI Algorithm:

π^t

1. Greedy Policy Selector:

$$\pi' \in \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^{\pi^t}} \left[A^{\pi^t}(s, \pi(s)) \right]$$

(Regression)

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Return π^t

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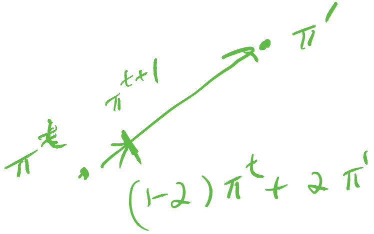
Greedy

3. Incremental Update:

$$\pi^{t+1}(\cdot \mid s) = (1-\alpha)\underset{\Delta}{\pi^t(\cdot \mid s)} + \alpha \underset{\Delta}{\pi'(\cdot \mid s)}, \forall s$$

$$\pi^{t+1}(\cdot \mid s) = \begin{cases} \pi^t(\cdot \mid s) & \text{wp } 1-\alpha \\ \pi'(\cdot \mid s) & \text{wp } \alpha \end{cases}$$

$\alpha \to 0$



$\alpha \in (0,1)$

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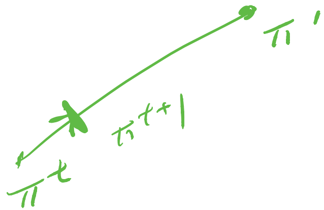
Q: Why this is incremental? In what sense?

Q: Can we get monotonic policy improvement?

The incremental Nature of CPI:

$$\pi^{t+1}(\cdot \mid s) = (1-\alpha)\pi^{t}(\cdot \mid s) + \alpha\pi'(\cdot \mid s), \forall s$$



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Key observation 1: $\forall s$

For any state s, we have $\|\pi^{t+1}(\cdot \mid s) - \pi^{t}(\cdot \mid s)\|_1 \leq 2\alpha$

$$\pi^{t+1}(\cdot \mid s) - \pi^{t}(\cdot \mid s)$$

$$= \alpha\left(\pi^{t}(\cdot \mid s) - \pi'(\cdot \mid s)\right)$$

small

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Key observation 2:

For any two policies π and π', if $\|\pi(\cdot \mid s) - \pi'(\cdot \mid s)\|_1 \leq \delta$, $\forall s$ then $\|d_\mu^{\pi} - d_\mu^{\pi'}\|_1 \leq \dfrac{\gamma\delta}{1-\gamma}$

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CPI ensures incremental update, i.e., $\|d_\mu^{\pi^{t+1}} - d_\mu^{\pi^{t}}\|_1 \leq \dfrac{2\gamma\alpha}{1-\gamma}$

Policy Improvement before Termination:

Before terminate, we have non-trivial avg **local advantage**:

$$\mathbb{A} := \mathbb{E}_{d_\mu^{\pi^t}}\left[A^{\pi^t}(s, \pi'(s))\right] \geq \varepsilon$$

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$$\mathbb{E}_{a\sim\pi^t(\cdot|s)} A^{\pi^t}(s,a) = 0$$

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$$\left|\mathbb{E}_P(f(x)) - \mathbb{E}_Q(f(x))\right| \le \sup_x \|f(x)\| \, \|P-Q\|_{TV}$$

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(Set $\alpha = \dfrac{(1-\gamma)^2\mathbb{A}}{4\gamma}$)

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$$V^{\pi^{t+1}} - V^{\pi^t} \geq \frac{\epsilon^2(1-\gamma)}{8\gamma}$$

Upon Termination we get a locally optimal solution (or globally optimal if μ is nice)

2. If $\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^{\pi^t}}[A^{\pi^t}(s, \pi(s))] \leq \varepsilon$

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1. No more positive advantage by one-step deviation from π^t's own states

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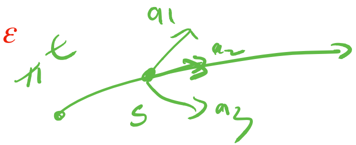

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2. Indeed, we can say more if μ covers $d_\mu^{\pi^\star}$, i.e., $C^\star := \sup_s \frac{d_\mu^{\pi^\star}(s)}{\mu(s)} < \infty$

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$\arg\max_a A^{\pi^t}(s,a) \notin \Pi$

$\Pi \uparrow \quad \epsilon_\Pi \searrow$

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Upon Termination we get a locally optimal solution (or globally optimal if μ is nice)

2. If $\max_{\pi \in \Pi} \mathbb{E}_{s\sim d_\mu^{\pi^t}}[A^{\pi^t}(s,\pi(s))] \le \varepsilon$

Return π^t

1. No more positive advantage by one-step deviation from π^t's own states

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2. Indeed, we can say more if μ covers $d_\mu^{\pi^\star}$, i.e., $C^\star := \sup_s \dfrac{d_\mu^{\pi^\star}(s)}{\mu(s)} < \infty$

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$\Leftarrow$ PDL +

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Connection to Agnostic Guarantees in Supervised Learning

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$$s \sim \nu, y \sim \pi^\star(s), y \in [A]$$

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N: # of samples

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stat

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1. Multi-step prediction (not i.i.d), 2. We don't get to see samples from $d^\star$

Compare the two Concentrability Coefficients from CPI and API:

$$\pi^{t+1} \Leftarrow \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^{\pi_t}} \left(A^{\pi_t}(s, \pi(s)) \right)$$

API: $\max_{\pi \in \Pi} \sup_s \frac{d^\pi(s)}{\mu(s)} < \infty$

Wide enough to cover all policies,

i.e.,making sure f $\widehat{A}$ is accurate at all places where any policy would go

Just need to cover the best in Π,

steady improvement via incremental update

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expert's distribution

1. Prior knowledge of how the optimal trajectories look like

2. Expert demonstrations (Imitation + RL)

$\sim \pi^{*}$

Summary of Policy Gradient Learning

1. PG formulation: $\mathbb{E}_{s,a\sim d^{\pi_\theta}} \nabla \ln \pi_\theta(a \mid s) A^{\pi_\theta}(s, a)$

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$$V^{\pi_\theta} + \lambda \sum_s \sum_a \ln \pi_\theta(a \mid s), \text{ where } \pi_\theta(a \mid s) \propto \exp(\theta_{s,a})$$

Δ

$\ln \pi_\theta(a|s) \to -\infty \quad \pi_\theta(a|s) \to 0$

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$KL(\Pr^{\pi^{t+1}} \| \Pr^{\pi^t}) \le \delta$

3. Natural Policy Gradient (trust region optimization) and its convergence (tabular, linear, & neural)

$$\widehat{w} \in \arg\min_w \mathbb{E}_{s,a\sim d_\nu^{\pi_\theta}}\left[\left(w^\top \nabla_\theta \ln \pi_\theta(a \mid s) - A^{\pi_\theta}(a \mid s)\right)^2\right], \quad \theta' = \theta + \eta \underset{A}{\widehat{w}}$$

$\sigma_{\min}\left(\mathbb{E}_{s,a\sim\mu_0} \phi(s,a)\phi(s,a)^\top\right) > c \in \mathbb{R}^+$

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4. The incremental nature of NPG/CPI/PPO and its advantage comparing to naive API

CPI (TRPO): $V^{\pi^{t+1}} > V^{\pi^t} > V^{\pi^{t-1}}$, thanks to $\| d_\mu^{\pi^{t+1}} - d_\mu^{\pi^t} \|_1$ is small (i.e., incremental)

and API could oscillate and never converges

Next week on Control Theory:

$x_{t+1} = A x_t + B u_t$

Basics of Optimal Control on Linear Quadratic Regulators
(no learning, just planning/control)

$$\max_{\pi} \mathbb{E}_{s \sim d_{\mu}^{\pi^t}} \left[A^{\pi^t}(s, \pi(s)) \right]$$

s.t. $\forall s \in S$

$$\| \pi(\cdot|s) - \pi^t(\cdot|s) \|_{TV} \le \delta$$

① CPI : $\| \pi^{t+1}(\cdot|s) - \pi^t(\cdot|s) \|_{TV} \le \alpha$