Approximate Policy Iteration
& Conservative Policy Iteration

HW2 due Fri'day
Recap

Recall Policy Iteration (PI):
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However, for large scale, unknown MDP there is no way we will be able to know $A^\pi(s, a)$ at all $s, a$, so how can we do policy update?
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= \frac{1}{1 - \gamma} \mathbb{E}_{s \sim d_{s_0}'} \left[ \max_{a \in A} A^\pi(s, a) \right] \geq 0
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However, for large scale, unknown MDP

there is no way we will be able to know \( A^\pi(s, a) \) at all \( s, a \),

so how can we do policy update?
Setting and Notation

Discounted infinite horizon MDP:

$$\mathcal{M} = \{S, A, \gamma, r, P, \mu\}$$
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State visitation:

\[ d^\pi_\mu(s) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h P^\pi_h(s; \mu) \]
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Unbiased estimate of \( A^\pi(s, a) = Q^\pi(s, a) - V^\pi(s) \)
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As we will consider large scale unknown MDP here, we start with a (restricted) function class \( \Pi \):

\[ \Pi = \{ \pi : S \mapsto \Delta(A) \} \]
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As we will consider large scale unknown MDP here, we start with a (restricted) function class \( \Pi \):

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Note that the optimal policy \( \pi^* \) may not be in \( \Pi \)
Attempt One: Approximate Policy Iteration (API)
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Given the current policy $\pi^t$, let’s act greedily wrt $\pi$ under $d_{\mu}^{\pi^t}$.
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i.e., let’s aim to (approximately) solve the following program:

$$\arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^{\pi^t}} \left[ A^{\pi^t}(s, \pi(s)) \right]$$
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Greedy Policy Selector
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But we can only sample from $d_{\mu}^{\pi^t}$, and we can only get an approximation of $A^{\pi^t}(s, a)$
Attempt One: **Approximate Policy Iteration (API)**

Given the current policy $\pi^t$, let’s act greedily wrt $\pi$ under $d^\pi_t$

i.e., let’s aim to (approximately) solve the following program:

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Greedy Policy Selector

But we can only sample from $d^\pi_t$, and we can only get an approximation of $A^{\pi^t}(s, a)$

We can hope for an Approximate Greedy Policy Selector a reduction to Regression
Implementing Approximate Greedy Policy Selector via Regression

We can do a **reduction to Regression** via Advantage function approximation

\[ \mathcal{F} = \{ f : S \times A \mapsto \mathbb{R} \} \]

\( \approx A^{\pi'} \)
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\[ \{ s_i, a_i, \tilde{A}_i \}, s_i \sim d^\pi_\mu, a_i \sim U(A), \mathbb{E} \left[ \tilde{A}_i \right] = A^{\pi'}(s_i, a_i) \]

\[ \Delta \]

unbiased from MC rollout
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Regression oracle:

\[ \hat{f} = \arg \min_{f \in \mathcal{F}} \sum_i \left( f(s_i, a_i) - \tilde{A}_i \right)^2 \]
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Act greedily wrt the estimator \( \hat{f} \) (as we hope \( \hat{f} \approx A^{\pi'} \)): 
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Analyzing Approximation error via Regression

Greedy Policy Selector

\[ \tilde{\pi} := \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu}[A^\pi(s, \pi(s))] \]

\[ \mathbb{E}_{s \sim d_\mu}[A^\pi(s, \hat{\pi}(s))] = \mathbb{E}_{s \sim d_\mu}[\hat{f}(s, \hat{\pi}(s)) + A^\pi(s, \hat{\pi}(s)) - \hat{f}(s, \hat{\pi}(s))] \]

\[ \geq \mathbb{E}_{s \sim d_\mu}[\hat{f}(s, \tilde{\pi}(s)) + A^\pi(s, \tilde{\pi}(s)) - \hat{f}(s, \tilde{\pi}(s))] \]

\[ \geq \mathbb{E}_{s \sim d_\mu}[A^\pi(s, \tilde{\pi}(s)) + \hat{f}(s, \tilde{\pi}(s)) - A^\pi(s, \tilde{\pi}(s)) + A^\pi(s, \hat{\pi}(s)) - \hat{f}(s, \hat{\pi}(s))] \]

\[ \geq \mathbb{E}_{s \sim d_\mu}[A^\pi(s, \tilde{\pi}(s))] + \mathbb{E}_{s \sim d_\mu}[\hat{f}(s, \tilde{\pi}(s)) - A^\pi(s, \tilde{\pi}(s)) + A^\pi(s, \hat{\pi}(s)) - \hat{f}(s, \hat{\pi}(s))] \]

\[ \hat{f} = \arg \min_{f \in \mathcal{F}} \sum_i \left( f(s_i, a_i) - \bar{A}_i \right)^2 \]

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\[ \{s_i, a_i, \bar{A}_i\}, s_i \sim d_\mu, a \sim U(A), \mathbb{E} \left[ \bar{A}_i \right] = A^\pi(s_i, a_i) \]
Summary So Far:

By reduction to Supervised Learning (i.e., classification using $\Pi$ or Regression using $\mathcal{F}$), with high probability, we get:
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\mathbb{E}_{s \sim d_\mu^s} \left[ A^\pi(s, \hat{\pi}(s)) \right] \geq \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^s} \left[ A^\pi(s, \pi(s)) \right] - \frac{A}{1 - \gamma} \sqrt{\frac{\ln(|\mathcal{F}|/\delta)}{N}}
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**statistical error:** $e$
Summary So Far:

By reduction to Supervised Learning (i.e., classification using $\Pi$ or Regression using $\mathcal{F}$), with high probability, we get:

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\mathbb{E}_{s \sim d^t_\mu} \left[ A^\pi(s, \hat{\pi}(s)) \right] \geq \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^t_\mu} \left[ A^\pi(s, \pi(s)) \right] - \frac{A}{1 - \gamma} \sqrt{\frac{\ln(|\mathcal{F}|/\delta)}{N}} \quad \text{statistical error: } e
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In the rest of the lecture, as we will focus on convergence rather than sample complexity, we ignore the statistical error (goes to zero as $N$ increases),
Summary So Far:

By reduction to Supervised Learning (i.e., classification using $\Pi$ or Regression using $\mathcal{F}$), with high probability, we get:

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\mathbb{E}_{s \sim d^\mu_t} \left[ A^\pi_t(s, \hat{\pi}(s)) \right] \geq \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\mu_t} \left[ A^\pi_t(s, \pi(s)) \right] - \frac{A}{1 - \gamma} \sqrt{\frac{\ln(| \mathcal{F} | / \delta)}{N}}
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statistical error:

In the rest of the lecture, as we will focus on convergence rather than sample complexity, we ignore the statistical error (goes to zero as $N$ increases),

i.e., we assume we can do the exact greedy policy selector: arg max $\pi \in \Pi \mathbb{E}_{s \sim d^\mu_t} \left[ A^\pi_t(s, \pi(s)) \right]$
Algorithm: Approximate Policy Iteration (API)

Iterate: \( \pi_t \)

API: \( \pi^{t+1} \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s, a \sim d_{\pi}^{t}} [A^{\pi}(s, \pi(s))] \)

\[ A \xrightarrow{\text{Reduce Regression}} N \rightarrow \infty \]
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API: \( \pi^{t+1} \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s,a \sim d^t_\pi} [A^{\pi_t}(s, \pi(s))] \)

Question:
(1) Does API has monotonic improvement?
(2) Does it convergence?
The Oscillation of API from Abrupt Distribution Change

API cannot guarantee to succeed (let’s think about advantage function approximation setting)

Concrete example in Chapter 3
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\[ A^\pi(s, a) \]
\[ \hat{f}^t \Rightarrow \pi^{t+1} \]
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Oscillation between two updates: No monotonic improvement
Key Issue: Abrupt Policy Change, i.e., $d_{\mu}^{\pi_{t}^{t+1}}$ and $d_{\mu}^{\pi_{t}^{t}}$ could be widely different
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Our estimator $\hat{f}_t$ is only good under $d_{\mu}^{\pi_{t}}$, i.e.,

$$\mathbb{E}_{s \sim d_{\mu}^{\pi_{t}}, a \sim U(A)}(\hat{f}_t(s, a) - A^{\pi_t}(s, a))^2$$ small,

**but** $\mathbb{E}_{s \sim d_{\mu}^{\pi_{t+1}}, a \sim U(A)}(\hat{f}_t(s, a) - A^{\pi_t}(s, a))^2$ might be arbitrarily big
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To make API to make improvement, we need a much stronger coverage assumption:

A strong Concentrability Coefficient: $C := \max_{\pi \in \Pi} \sup_{s} \frac{d_{\mu}^{\pi}(s)}{\mu(s)} < \infty$
Key Issue: Abrupt Policy Change, i.e., $d_{\mu}^{\pi t+1}$ and $d_{\mu}^{\pi t}$ could be widely different

Our estimator $\hat{f}^t$ is only good under $d_{\mu}^{\pi t}$, i.e.,

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To make API to make improvement, we need a much stronger coverage assumption:

A strong Concentrability Coefficient: $C := \max_{\Delta} \sup_{\pi \in \Pi} \frac{d_{\mu}^{\pi}(s)}{\mu(s)} < \infty$

If $C < \infty$, i.e., $\mu$ covers all $d_{\mu}^{\pi}$, then we can expect error

$$\mathbb{E}_{s \sim d_{\mu}^{\pi t+1}, a \sim U(A)}(\hat{f}^t(s, a) - A^\pi(s, a))^2 \text{ is reasonably under control;}$$
Conservative Policy Iteration—An Incremental Policy Optimization Approach

(And the benefit of being incremental)

\[
\text{NPG} \quad \text{Trust Region} \\
T \\
\text{KL}(p_{\pi_{t+1}} \parallel p_{\pi_t}) \leq \delta
\]
Key Idea of CPI: Incremental Update—No Abrupt Distribution Change

Let’s design policy update rule such that $d^{\pi^{i+1}}$ and $d^{\pi^i}$ are not that different!
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Recall Performance Difference Lemma:

$$V^{\pi^{t+1}} - V^{\pi^t} = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim d^{\pi^{t+1}}} \left[ A^{\pi^t}(s, \pi^{t+1}(s)) \right]$$
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$$d^{π_t} \approx d^{π_{t+1}}$$
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s.t., $\mathbb{E}_{s \sim d^{\pi_t}} \left[ A^{\pi_t}(s, \pi^{t+1}(s)) \right] \approx \mathbb{E}_{s \sim d^{\pi_{t+1}}} \left[ A^{\pi_t}(s, \pi^{t+1}(s)) \right]$
Key Idea of CPI: Incremental Update—No Abrupt Distribution Change

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$$d^t \approx d^{t+1}$$

s.t.,

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This we know how to optimize: the Greedy Policy Selector
CPI Algorithm:
CPI Algorithm:

1. Greedy Policy Selector:

\[ \pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi_t} \left[ A^{\pi'}(s, \pi(s)) \right] \]
CPI Algorithm:

1. Greedy Policy Selector:

\[ \pi' \in \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^t} \left[ A_{\pi'}(s, \pi(s)) \right] \]

2. If \( \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^t}[A_{\pi'}(s, \pi(s))] \leq \varepsilon \)

Return \( \pi^t \)
CPI Algorithm:

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   Return \( \pi^t \)

3. Incremental Update:
   \[ \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha \pi'(\cdot | s), \forall s \]

\[ (1 - 2 \pi^t + 2 \pi', \pi^t) \xi \in (0, 1) \]
CPI Algorithm:

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   \[ \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha \pi'(\cdot | s), \forall s \]

Q: Why this is incremental? In what sense?
Q: Can we get monotonic policy improvement?
The incremental Nature of CPI:

\[ \pi^{t+1}(\cdot \mid s) = (1 - \alpha)\pi^t(\cdot \mid s) + \alpha \pi'(\cdot \mid s), \forall s \]
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Key observation 1:

For any state \( s \), we have \( \|\pi^{t+1}( \cdot | s) - \pi^t( \cdot | s)\|_1 \leq 2\alpha \)

\[ \pi^{t+1}( \cdot | s) - \pi^t( \cdot | s) = 2 \left( \pi^t( \cdot | s) - \pi'( \cdot | s) \right) \]

Small
The incremental Nature of CPI:

\[
\pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha \pi'(\cdot | s), \forall s
\]

Key observation 1:

For any state \(s\), we have \(\|\pi^{t+1}(\cdot | s) - \pi^t(\cdot | s)\|_1 \leq 2\alpha\)

Key observation 2:

For any two policies \(\pi\) and \(\pi'\), if \(\|\pi(\cdot | s) - \pi'(\cdot | s)\|_1 \leq \delta\), then \(\|d^\pi_\mu - d'^\pi_\mu\|_1 \leq \frac{\gamma\delta}{1 - \gamma}\)
The incremental Nature of CPI:

\[ \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha\pi'(\cdot | s), \forall s \]

Key observation 1:
For any state \( s \), we have \( \|\pi^{t+1}(\cdot | s) - \pi^t(\cdot | s)\|_1 \leq 2\alpha \)

Key observation 2:
For any two policies \( \pi \) and \( \pi' \), if \( \|\pi(\cdot | s) - \pi'(\cdot | s)\|_1 \leq \delta \), then \( \|d^\pi_{\mu} - d^{\pi'}_{\mu}\|_1 \leq \frac{\gamma \delta}{1 - \gamma} \)

CPI ensures incremental update, i.e., \( \|d^{\pi^{t+1}}_{\mu} - d^{\pi'}_{\mu}\|_1 \leq \frac{2\gamma \alpha}{1 - \gamma} \)
Policy Improvement before Termination:

Before terminate, we have non-trivial avg **local advantage**:

\[ A := \mathbb{E}_{d_{\mu}^t} \left[ A^{\pi'}(s, \pi'(s)) \right] \geq \varepsilon \]

Recall CPI:

1. Greedy Policy Selector:
   \[ \pi' \in \arg\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^t} \left[ A^{\pi'}(s, \pi(s)) \right] \]

2. If \[ \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^t} \left[ A^{\pi'}(s, \pi(s)) \right] \leq \varepsilon \]
   **Return** \( \pi'^t \)

3. Incremental Update:
   \[ \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha\pi'(\cdot | s), \forall s \]
Policy Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

\[ \mathbb{A} := \mathbb{E}_{\mu_t} \left[ A^{\pi'}(s, \pi'(s)) \right] \geq \varepsilon \]

Can we translate local advantage \( \mathbb{A} \) to \( V^{\pi_{t+1}} - V^{\pi_t} \)?

Recall CPI:

1. Greedy Policy Selector:
   \[ \pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi_t} \left[ A^{\pi'}(s, \pi(s)) \right] \]

2. If \( \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi_t} [A^{\pi'}(s, \pi(s))] \leq \varepsilon \)
   Return \( \pi' \)

3. Incremental Update:
   \[ \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha \pi'(\cdot | s), \forall s \]
Policy Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

$$A := \mathbb{E}_{d_{\mu}^t} \left[ A^\pi'(s, \pi'(s)) \right] \geq \epsilon$$

Can we translate local advantage $A$ to $V^{\pi^{t+1}} - V^{\pi^t}$?

$$(1 - \gamma) \left( V^{\pi^{t+1}}_\mu - V^{\pi^t}_\mu \right) = \mathbb{E}_{s \sim d_{\mu}^{t+1}} \left[ \mathbb{E}_{a \sim \pi^{t+1}(\cdot | s)} A^\pi'(s, a) \right]$$

Recall CPI:

1. Greedy Policy Selector:
   $$\pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^t} \left[ A^\pi'(s, \pi(s)) \right]$$

2. If $\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^t} [A^{\pi'}(s, \pi(s))] \leq \epsilon$
   Return $\pi'$

3. Incremental Update:
   $$\pi^{t+1}(\cdot | s) = (1 - \alpha) \pi^t(\cdot | s) + \alpha \pi'(\cdot | s), \forall s$$
Policy Improvement before Termination:

Before terminate, we have non-trivial avg **local advantage**:

$$A := \mathbb{E}_{d^t_\mu} \left[ A^{\pi'}(s, \pi'(s)) \right] \geq \epsilon$$

Can we translate local advantage $A$ to $V^{\pi'+1} - V^{\pi'}$?

$$(1 - \gamma)(V^{\pi'+1}_\mu - V^{\pi'}_\mu) = \mathbb{E}_{s \sim d^{\pi'+1}_\mu} \left[ \mathbb{E}_{a \sim \pi^{i+1}(s)} A^{\pi'}(s,a) \right]$$

$$= \mathbb{E}_{s \sim d^{\pi'+1}_\mu} \left[ \alpha A^{\pi'}(s, \pi'(s)) \right] := \alpha A$$

Recall CPI:

1. Greedy Policy Selector:
   $$\pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^t_\mu} \left[ A^{\pi'}(s, \pi(s)) \right]$$

2. If max $\mathbb{E}_{s \sim d^t_\mu} \left[ A^{\pi'}(s, \pi(s)) \right] \leq \epsilon$
   
   **Return** $\pi'$

3. Incremental Update:
   $$\pi^{i+1}(\cdot | s) = (1 - \alpha)\pi^i(\cdot | s) + \alpha \pi'(\cdot | s), \forall s$$
Policy Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

$$A := \mathbb{E}_{d_{t}^{\pi}} \left[ A^{\pi}(s, \pi'(s)) \right] \geq \varepsilon$$

Can we translate local advantage $A$ to $V^{\pi_{t}+1} - V^{\pi_{t}}$?

$$(1 - \gamma) \left( V_{t}^{\pi_{t}+1} - V_{t}^{\pi_{t}} \right) = \mathbb{E}_{s \sim d_{t}^{\pi_{t}+1}} \left[ \mathbb{E}_{a \sim \pi_{t+1}^{\pi}(s)} A^{\pi}(s, a) \right]$$

$$= \mathbb{E}_{s \sim d_{t}^{\pi_{t}+1}} \left[ \alpha A^{\pi}(s, \pi'(s)) \right] \quad (:= \alpha A)$$

$$= \mathbb{E}_{s \sim d_{t}^{\pi_{t}}} \left[ \alpha A^{\pi}(s, \pi'(s)) \right] + \mathbb{E}_{s \sim d_{t}^{\pi_{t}}} \left[ \alpha A^{\pi}(s, \pi'(s)) \right] - \mathbb{E}_{s \sim d_{t}^{\pi_{t}}} \left[ \alpha A^{\pi}(s, \pi'(s)) \right]$$
Policy Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

\[
\mathbb{A} := \mathbb{E}_{d^t_\mu} \left[ A^{\pi^t}(s, \pi^t(s)) \right] \geq \varepsilon
\]

Can we translate local advantage \( \mathbb{A} \) to \( V^{\pi^t+1} - V^{\pi^t} \)?

\[
(1 - \gamma) \left( V^{\pi^t+1}_\mu - V^{\pi^t}_\mu \right) = \mathbb{E}_{s \sim d^{\pi^t+1}_\mu} \left[ \mathbb{E}_{a \sim \pi^t(s)} A^{\pi^t}(s, a) \right]
\]

\[
= \mathbb{E}_{s \sim d^{\pi^t+1}_\mu} \left[ \alpha A^{\pi^t}(s, \pi^t(s)) \right] (:= \alpha \mathbb{A})
\]

\[
= \mathbb{E}_{s \sim d^{\pi^t}_\mu} \left[ \alpha A^{\pi^t}(s, \pi^t(s)) \right] + \mathbb{E}_{s \sim d^{\pi^t+1}_\mu} \left[ \alpha A^{\pi^t}(s, \pi^t(s)) \right] - \mathbb{E}_{s \sim d^{\pi^t}_\mu} \left[ \alpha A^{\pi^t}(s, \pi^t(s)) \right]
\]

\[
\geq \mathbb{E}_{s \sim d^{\pi^t}_\mu} \left[ \alpha A^{\pi^t}(s, \pi^t(s)) \right] - \frac{\alpha}{1 - \gamma} \| d^{\pi^t+1}_\mu - d^{\pi^t}_\mu \|_1
\]

Recall CPI:

1. Greedy Policy Selector:
   \[
   \pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^t_\mu} \left[ A^{\pi'}(s, \pi(s)) \right]
   \]

2. If \( \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^t_\mu} [A^{\pi'}(s, \pi(s))] \leq \varepsilon \)
   
   Return \( \pi' \)

3. Incremental Update:
   \[
   \pi^{t+1}(\cdot | s) = (1 - \alpha) \pi^t(\cdot | s) + \alpha \pi'(\cdot | s), \forall s
   \]
Policy Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

$$\bar{A} := \mathbb{E}_{d^t_{\mu}} \left[ A^\pi(s, \pi'(s)) \right] \geq \varepsilon$$

Can we translate local advantage $\bar{A}$ to $V^{\pi'+1} - V^\pi$?

$$(1 - \gamma) \left( V^{\pi'+1}_\mu - V^\pi_\mu \right) = \mathbb{E}_{s \sim d^{t+1}_{\mu}} \left[ \mathbb{E}_{a \sim \pi_{t+1}(\cdot | s)} A^\pi(s, a) \right]$$

$$= \mathbb{E}_{s \sim d^{t+1}_{\mu}} \left[ \alpha A^\pi(s, \pi'(s)) \right] \quad (:= \alpha \bar{A})$$

$$= \mathbb{E}_{s \sim d^{t}_{\mu}} \left[ \alpha A^\pi(s, \pi'(s)) \right] + \mathbb{E}_{s \sim d^{t+1}_{\mu}} \left[ \alpha A^\pi(s, \pi'(s)) \right] - \mathbb{E}_{s \sim d^{t}_{\mu}} \left[ \alpha A^\pi(s, \pi'(s)) \right]$$

$$\geq \mathbb{E}_{s \sim d^{t}_{\mu}} \left[ \alpha A^\pi(s, \pi'(s)) \right] - \frac{\alpha}{1 - \gamma} \|d^{t+1}_{\mu} - d^{t}_{\mu}\|_1$$

$$\geq \frac{\alpha \bar{A} - 2\gamma \alpha^2}{(1 - \gamma)^2} \leq \frac{2\alpha}{1 - \gamma}$$
Policy Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

\[ A := \mathbb{E}_{s \sim d_t} \left[ A^\pi'(s, \pi'(s)) \right] \geq \epsilon \]

Can we translate local advantage \( A \) to \( V^{\pi'^{t+1}} - V^{\pi^t} \)?

\[
\begin{align*}
(1 - \gamma) \left( V^{\pi'^{t+1}} - V^{\pi^t} \right) &= \mathbb{E}_{s \sim d_{\pi'^{t+1}}} \left[ \mathbb{E}_{a \sim \pi^{t+1}(\cdot \mid s)} A^\pi'(s, a) \right] \\
&= \mathbb{E}_{s \sim d_{\pi'^{t+1}}} \left[ \alpha A^\pi'(s, \pi'(s)) \right] \\
&= \mathbb{E}_{s \sim d_{\pi'^{t+1}}} \left[ \alpha A^\pi'(s, \pi'(s)) \right] + \mathbb{E}_{s \sim d_{\pi^t}} \left[ \alpha A^\pi'(s, \pi'(s)) \right] - \mathbb{E}_{s \sim d_{\pi'^{t}}} \left[ \alpha A^\pi'(s, \pi'(s)) \right] \\
&\geq \mathbb{E}_{s \sim d_{\pi'^{t+1}}} \left[ \alpha A^\pi'(s, \pi'(s)) \right] - \frac{\alpha}{1 - \gamma} \left\| d_{\pi'^{t+1}} - d_{\pi^t} \right\|_1 \\
&\geq \alpha A - \frac{2\gamma \alpha^2}{(1 - \gamma)^2} \\
\end{align*}
\]

\( \text{(Set } \alpha = \frac{(1 - \gamma)^2 A}{4\gamma} \text{)} \)

Recall CPI:

1. Greedy Policy Selector:
   \[ \pi' \in \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\pi'}} \left[ A^\pi'(s, \pi(s)) \right] \]

2. If \( \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\pi}} [A^\pi'(s, \pi(s))] \leq \epsilon \)
   
   \text{Return } \pi'

3. Incremental Update:
   \[ \pi^{t+1}(\cdot \mid s) = (1 - \alpha) \pi^t(\cdot \mid s) + \alpha \pi'(\cdot \mid s), \forall s \]
Policy Improvement before Termination:

Before terminate, we have non-trivial avg local advantage:

\[ A := \mathbb{E}_{d_{\mu}^{t}} \left[ A^{\pi^t}(s, \pi'(s)) \right] \geq \varepsilon \]

Can we translate local advantage \( A \) to \( V^{\pi^{t+1}} - V^{\pi^t} \)?

\[
(1 - \gamma)(V^{\pi^{t+1}} - V^{\pi^t}) = \mathbb{E}_{s \sim d_{\mu}^{t+1}} \left[ \mathbb{E}_{a \sim \pi^{t+1}(\cdot | s)} A^{\pi^t}(s, a) \right] = \mathbb{E}_{s \sim d_{\mu}^{t+1}} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right]
\]

\[
= \mathbb{E}_{s \sim d_{\mu}^{t+1}} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right] + \mathbb{E}_{s \sim d_{\mu}^{t+1}} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right] - \mathbb{E}_{s \sim d_{\mu}^{t}} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right]
\]

\[
\geq \mathbb{E}_{s \sim d_{\mu}^{t+1}} \left[ \alpha A^{\pi^t}(s, \pi'(s)) \right] - \frac{\alpha}{1 - \gamma} \|d_{\mu}^{t+1} - d_{\mu}^{t} \|_1
\]

\[
\geq 2\gamma^2 \alpha^2 - \frac{\alpha A}{(1 - \gamma)^2} \geq \frac{A^2(1 - \gamma)}{8 \gamma} \quad \text{(Set } \alpha = \frac{(1 - \gamma)^2 A}{4 \gamma})
\]

Recall CPI:

1. Greedy Policy Selector:
   \[ \pi' \in \text{arg max} \mathbb{E}_{s \sim d_{\mu}^{t}} \left[ A^{\pi^t}(s, \pi(s)) \right] \]

2. If \[ \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^{t}} \left[ A^{\pi^t}(s, \pi(s)) \right] \leq \varepsilon \]
   Return \( \pi' \)

3. Incremental Update:
   \[ \pi^{t+1}(\cdot | s) = (1 - \alpha)\pi^t(\cdot | s) + \alpha \pi'(\cdot | s), \forall s \]
Upon Termination we get a locally optimal solution (or globally optimal if $\mu$ is nice)

1. No more positive advantage by one-step deviation from $\pi^t$'s own states

$$\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\pi}^t}[A^{\pi^t}(s, \pi(s))] \leq \varepsilon$$

2. If $\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\pi}^t}[A^{\pi^t}(s, \pi(s))] \leq \varepsilon$

Return $\pi^t$
Upon Termination we get a locally optimal solution
(or globally optimal if $\mu$ is nice)

1. No more positive advantage by one-step deviation from $\pi^f$'s own states

$$\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^{\pi^f}}[A^{\pi^f}(s, \pi(s))] \leq \varepsilon$$

2. Indeed, we can say more if $\mu$ covers $d^{\pi^*}_{\mu}$, i.e., $C^* := \sup_{s} \frac{d^{\pi^*}_{\mu}(s)}{\mu(s)} < \infty$

2. If $\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^{\pi^f}}[A^{\pi^f}(s, \pi(s))] \leq \varepsilon$

Return $\pi^f$
Upon Termination we get a locally optimal solution
(or globally optimal if $\mu$ is nice)

1. No more positive advantage by one-step deviation from $\pi^t$’s own states

$$\max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi \mu}[A^{\pi^t}(s, \pi(s))] \leq \varepsilon$$

2. Indeed, we can say more if $\mu$ covers $d^\pi \mu^*$, i.e., $C^* := \sup_s \frac{d^\pi \mu^*(s)}{\mu(s)} < \infty$

Recall $\Pi$ is restricted, denote $\varepsilon_{\Pi} = \mathbb{E}_{s \sim d^\pi \mu} \left[ \max_a A^{\pi^t}(s, a) \right] - \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi \mu} \left[ A^{\pi^t}(s, \pi(s)) \right]$
Upon Termination we get a locally optimal solution
(or globally optimal if $\mu$ is nice)

1. No more positive advantage by one-step deviation from $\pi^t$'s own states

$$\max_{\pi \in \Pi} \mathbb{E}_{s \sim d^t_{\pi}}[A^{\pi^t}(s, \pi(s))] \leq \varepsilon$$

2. Indeed, we can say more if $\mu$ covers $d^*_{\pi^t}$, i.e., $C^* := \sup_s \frac{d^*_{\mu}(s)}{\mu(s)} < \infty$

Recall $\Pi$ is restricted, denote $\epsilon_\Pi = \mathbb{E}_{s \sim d^t_{\pi}} \left[ \max_a A^{\pi^t}(s, a) \right] - \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^t_{\pi}} \left[ A^{\pi'(s, \pi(s))} \right]$

$$\mathbb{E}_{s \sim d^t_{\pi}} \left[ A^{\pi'(s, \pi(s))} \right] + \epsilon_\Pi \geq \mathbb{E}_{s \sim d^t_{\pi}} \left[ \max_a A^{\pi^t}(s, a) \right] = \mathbb{E}_{s \sim d^t_{\pi}} \left( \frac{d^*_{\mu}(s)}{d^*(s)} \right) \max_a A^{\pi^t}(s, a)$$
Upon Termination we get a locally optimal solution (or globally optimal if $\mu$ is nice)

1. No more positive advantage by one-step deviation from $\pi^*$'s own states

$$\max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi} [A^\pi(s, \pi(s))] \leq \varepsilon$$

2. Indeed, we can say more if $\mu$ covers $d^\pi_\mu$, i.e., $C^* := \sup_s \frac{d^\pi_\mu(s)}{\mu(s)} < \infty$

Recall $\Pi$ is restricted, denote $\varepsilon_\Pi = \mathbb{E}_{s \sim d^\pi} \left[ \max_a A^\pi(s, a) \right] - \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi} \left[ A^\pi(s, \pi(s)) \right]$

$$\mathbb{E}_{s \sim d^\pi} \left[ A^\pi(s, \pi'(s)) \right] + \varepsilon_\Pi \geq \mathbb{E}_{s \sim d^\pi} \left[ \max_a A^\pi'(s, a) \right] = \mathbb{E}_{s \sim d^*} \left( \frac{d^\pi_\mu(s)}{d^*(s)} \right) \max_a A^\pi'(s, a)$$

$$\geq \inf_s \left( \frac{d^\pi_\mu(s)}{d^*(s)} \right) \mathbb{E}_{s \sim d^*} \max_a A^\pi'(s, a)$$
Upon Termination we get a locally optimal solution
(or globally optimal if $\mu$ is nice)

1. No more positive advantage by one-step deviation from $\pi^t$’s own states

$$\max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi_t}[A^\pi_t(s, \pi(s))] \leq \varepsilon$$

2. Indeed, we can say more if $\mu$ covers $d^\pi_\mu$, i.e., $C^* := \sup_s \frac{d^\pi_\mu(s)}{\mu(s)} < \infty$

Recall $\Pi$ is restricted, denote

$$\epsilon_\Pi = \mathbb{E}_{s \sim d^\pi_t} \left[ \max_a A^\pi_t(s, a) \right] - \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi_t} \left[ A^\pi_t(s, \pi(s)) \right]$$

$$\mathbb{E}_{s \sim d^\pi_t} \left[ A^\pi_t(s, \pi'(s)) \right] + \epsilon_\Pi \geq \mathbb{E}_{s \sim d^\pi_t} \left[ \max_a A^{\pi'}(s, a) \right] = \mathbb{E}_{s \sim d^*} \left( \frac{d^\pi_\mu(s)}{d^*(s)} \right) \max_a A^{\pi'}(s, a)$$

$$\geq \inf_s \left( \frac{d^\pi_\mu(s)}{d^*(s)} \right) \mathbb{E}_{s \sim d^*} \max_a A^{\pi'}(s, a) \geq \inf_s \left( \frac{d^\pi_\mu(s)}{d^*(s)} \right) \mathbb{E}_{s \sim d^*} A^{\pi'}(s, \pi^*(s))$$
Upon Termination we get a locally optimal solution (or globally optimal if $\mu$ is nice)

1. No more positive advantage by one-step deviation from $\pi^t$'s own states

$$\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^{\pi^*}}[A^{\pi^*}(s, \pi(s))] \leq \varepsilon$$

2. Indeed, we can say more if $\mu$ covers $d_{\pi^*}^{\pi^*}$, i.e., $C^* := \sup_{s} \frac{d_{\mu}^{\pi^*}(s)}{\mu(s)} < \infty$

Recall $\Pi$ is restricted, denote $\epsilon_\Pi = \mathbb{E}_{s \sim d_{\mu}^{\pi^*}} \left[ \max_a A^{\pi^*}(s, a) \right] - \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_{\mu}^{\pi}} [A^{\pi}(s, \pi(s))]$

$$\mathbb{E}_{s \sim d_{\mu}^{\pi}} [A^{\pi}(s, \pi^t(s))] + \epsilon_\Pi \geq \mathbb{E}_{s \sim d_{\mu}^{\pi}} \left[ \max_a A^{\pi^*}(s, a) \right] = \mathbb{E}_{s \sim d^*} \left( \frac{d_{\mu}^{\pi^*}(s)}{d^*(s)} \right) \max_a A^{\pi^*}(s, a)$$

$$\geq \inf_{s} \left( \frac{d_{\mu}^{\pi^*}(s)}{d^*(s)} \right) \mathbb{E}_{s \sim d^*} A^{\pi^*}(s, \pi^*(s)) \geq \inf_{s} \left( \frac{d_{\mu}^{\pi^*}(s)}{d^*(s)} \right) (V^* - V^{\pi^*})(1 - \gamma)$$
Upon Termination we get a locally optimal solution
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1. No more positive advantage by one-step deviation from $\pi^t$'s own states

$$\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^\pi}[A^\pi(s, \pi(s))] \leq \varepsilon$$

2. Indeed, we can say more if $\mu$ covers $d_\mu^{\pi^*}$, i.e., $C^* := \sup_s \frac{d_\mu^{\pi^*}(s)}{\mu(s)} < \infty$

Recall $\Pi$ is restricted, denote $\epsilon_{\Pi} = \mathbb{E}_{s \sim d_\mu^\pi} \left[ \max_a A^\pi(s, a) \right] - \max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^\pi}[A^\pi(s, \pi(s))]$

$$\mathbb{E}_{s \sim d_\mu^\pi}[A^\pi(s, \pi^*(s))] + \epsilon_{\Pi} \geq \mathbb{E}_{s \sim d_\mu^\pi} \left[ \max_a A^{\pi^*}(s, a) \right] \geq \mathbb{E}_{s \sim d^*} \left( \frac{d_\mu^{\pi^*}(s)}{d^*(s)} \right) \max_a A^\pi(s, a)$$

$$\geq \inf_s \left( \frac{d_\mu^{\pi^*}(s)}{d^*(s)} \right) \mathbb{E}_{s \sim d^*} \max_a A^{\pi^*}(s, a) \geq \inf_s \left( \frac{d_\mu^{\pi^*}(s)}{d^*(s)} \right) \mathbb{E}_{s \sim d^*} A^{\pi^*}(s, \pi^*(s)) \geq \inf_s \left( \frac{d_\mu^{\pi^*}(s)}{d^*(s)} \right) (V^* - V^{\pi^*})(1 - \gamma)$$

$$V^* - V^{\pi^*} \leq \sup_s \left( \frac{d^*(s)}{d_\mu^{\pi^*}(s)} \right) \frac{\epsilon + \epsilon_{\Pi}}{1 - \gamma}$$
Upon Termination we get a locally optimal solution (or globally optimal if $\mu$ is nice)

1. No more positive advantage by one-step deviation from $\pi^t$'s own states

\[
\max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi_\mu}[A^\pi(s, \pi(s))] \leq \varepsilon
\]

2. Indeed, we can say more if $\mu$ covers $d^\pi_\mu$, i.e., $C^* := \sup_s \frac{d^\pi_\mu(s)}{\mu(s)} < \infty$

Recall $\Pi$ is restricted, denote $\varepsilon_\Pi = \mathbb{E}_{s \sim d^\pi_\mu} \left[ \max_a A^\pi(s, a) \right] - \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi_\mu}[A^\pi(s, \pi(s))]$

\[
\mathbb{E}_{s \sim d^\pi_\mu}[A^\pi(s, \pi(s))] + \varepsilon_\Pi \geq \mathbb{E}_{s \sim d^\pi_\mu} \left[ \max_a A^\pi(s, a) \right] = \mathbb{E}_{s \sim \pi^*} \left( \frac{d^\pi_\mu(s)}{d^*(s)} \right) \max_a A^\pi(s, a)
\]

\[
\geq \inf_s \left( \frac{d^\pi_\mu(s)}{d^*(s)} \right) \mathbb{E}_{s \sim \pi^*} \max_a A^\pi(s, a) \geq \inf_s \left( \frac{d^\pi_\mu(s)}{d^*(s)} \right) \mathbb{E}_{s \sim \pi^*} A^\pi(s, \pi^*(s)) \geq \inf_s \left( \frac{d^\pi_\mu(s)}{d^*(s)} \right) (V^* - V^\pi)(1 - \gamma)
\]

\[
V^* - V^\pi \leq \sup_s \left( \frac{d^*(s)}{d^\pi_\mu(s)} \right) \frac{\varepsilon + \varepsilon_\Pi}{1 - \gamma} \leq C^* \left( \frac{\varepsilon + \varepsilon_\Pi}{(1 - \gamma)^2} \right)
\]
Connection to Agnostic Guarantees in Supervised Learning

Multi-class Classification (A many classes):

\[ s \sim \nu, y \sim \pi^*(s), y \in [A] \]

↑ ↑
Connection to Agnostic Guarantees in Supervised Learning

Multi-class Classification (A many classes):

$s \sim \nu, y \sim \pi^*(s), y \in [A]$

We start with a set of classifiers $\Pi$; but we cannot guarantee $\pi^* \in \Pi$;
Connection to Agnostic Guarantees in Supervised Learning

Multi-class Classification (A many classes):

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We start with a set of classifiers \( \Pi \); but we cannot guarantee \( \pi^* \in \Pi \);

What we can hope is that we can find the best classifier in the class \( \Pi \)
Connection to Agnostic Guarantees in Supervised Learning

Multi-class Classification (A many classes):

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What we can hope is that we can find the best classifier in the class \( \Pi \)

\[
\mathbb{E}_{s \sim \nu, y = \pi^*(s)} \ell(\hat{\pi}(s), y) \leq \min_{\pi \in \Pi} \mathbb{E}_{s \sim \nu, y = \pi^*(s)} \ell(\pi(s), y) + \sqrt{\frac{\ln(|\Pi|/\delta)}{N}} \]

statistical error: \( \epsilon \)

\( N \) : # of samples
Connection to Agnostic Guarantees in Supervised Learning

Multi-class Classification (A many classes):

\[ s \sim \nu, y \sim \pi^*(s), y \in [A] \]

We start with a set of classifiers \( \Pi \); but we cannot guarantee \( \pi^* \in \Pi \);
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\]

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In RL:
Connection to Agnostic Guarantees in Supervised Learning

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\]

In RL:

\[
V^* - V^{\hat{\pi}} \leq \sup_s \left( \frac{d^*(s)}{\mu(s)} \right) \frac{\epsilon_{\Pi} + \epsilon}{(1 - \gamma)^2}
\]
Connection to Agnostic Guarantees in Supervised Learning

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In RL: \( V^* - V^{\hat{\pi}} \leq \sup_s \left( \frac{d^*(s)}{\mu(s)} \right) \frac{\epsilon_{\Pi} + \epsilon}{(1 - \gamma)^2} \)

1. Multi-step prediction (not i.i.d), 2. We don’t get to see samples from \( d^* \)
Compare the two Concentrability Coefficients from CPI and API:

API: \( \max_{\pi \in \Pi} \sup_s \frac{d^\pi(s)}{\mu(s)} < \infty \)

CPI: \( \sup_s \frac{d^*(s)}{\mu(s)} < \infty \)

Wide enough to cover all policies, i.e., making sure \( \mu \) is accurate at all places where any policy would go.

Just need to cover the best in \( \Pi \), steady improvement via incremental update.
Compare the two Concentrability Coefficients from CPI and API:

API: \[
\max_{\pi \in \Pi} \sup_s \frac{d^\pi(s)}{\mu(s)} < \infty
\]

Wide enough to cover all policies, i.e., making sure \( \hat{A} \) is accurate at all places where any policy would go

CPI: \[
\sup_s \frac{d^*(s)}{\mu(s)} < \infty
\]

1. Prior knowledge of how the optimal trajectories look like
2. Expert demonstrations (Imitation + RL)

Just need to cover the best in \( \Pi \), steady improvement via incremental update
Summary of Policy Gradient Learning

1. PG formulation: \( \mathbb{E}_{s,a \sim \pi_\theta} \nabla \ln \pi_\theta(a \mid s) A_\pi_\theta(s, a) \)
Summary of Policy Gradient Learning

1. PG formulation: $\mathbb{E}_{s,a \sim d^\pi_\theta} \nabla \ln \pi_\theta(a \mid s) A^\pi_\theta(s, a)$

2. For tabular MDP, gradient ascent on KL-regularized objective converges to global optimality:

$$V^{\pi_\theta} + \lambda \sum_s \sum_a \ln \pi_\theta(a \mid s), \text{ where } \pi_\theta(a \mid s) \propto \exp(\theta_{s,a})$$

\[ \ln \pi_\theta(a \mid s) \to -\infty \quad \pi_\theta(a \mid s) \to 0 \]
Summary of Policy Gradient Learning

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3. Natural Policy Gradient (trust region optimization) and its convergence (tabular, linear, & neural)

$$\hat{w} \in \arg \min_w \mathbb{E}_{s,a \sim d^{\pi_\theta}} \left[ (w^T \nabla \ln \pi_\theta(a \mid s) - A^{\pi_\theta}(a \mid s))^2 \right], \quad \theta' = \theta + \eta \hat{w}$$
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\]

4. The incremental nature of NPG/CPI/PPO and its advantage comparing to naive API

\[
\text{CPI (TRPO): } V^{\pi_{t+1}} > V^{\pi_t} > V^{\pi_{t-1}}, \text{ thanks to } \|d_{\mu}^{\pi_{t+1}} - d_{\mu}^{\pi_t}\|_1 \text{ is small (i.e., incremental)}
\]

\[\Delta\]

and API could oscillate and never converges
Next week on Control Theory:

Basics of Optimal Control on Linear Quadratic Regulators
(no learning, just planning/control)
\[
\max \mathbb{E}_{\pi \sim \pi_t} \left[ A^T_t(s, \pi_t(s)) \right]
\]

s.t. \forall s \in S

\[
\| \pi(\cdot|s) - \pi^c(\cdot|s) \|_W \leq \delta
\]

\[\circ\quad \text{CP}\quad \| \pi^{tu}(\cdot|s) - \pi^c(\cdot|s) \|_W \leq 2\]