Optimal Control Theory and Linear Quadratic Regulators

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CS 6789: Foundations of Reinforcement Learning

Today

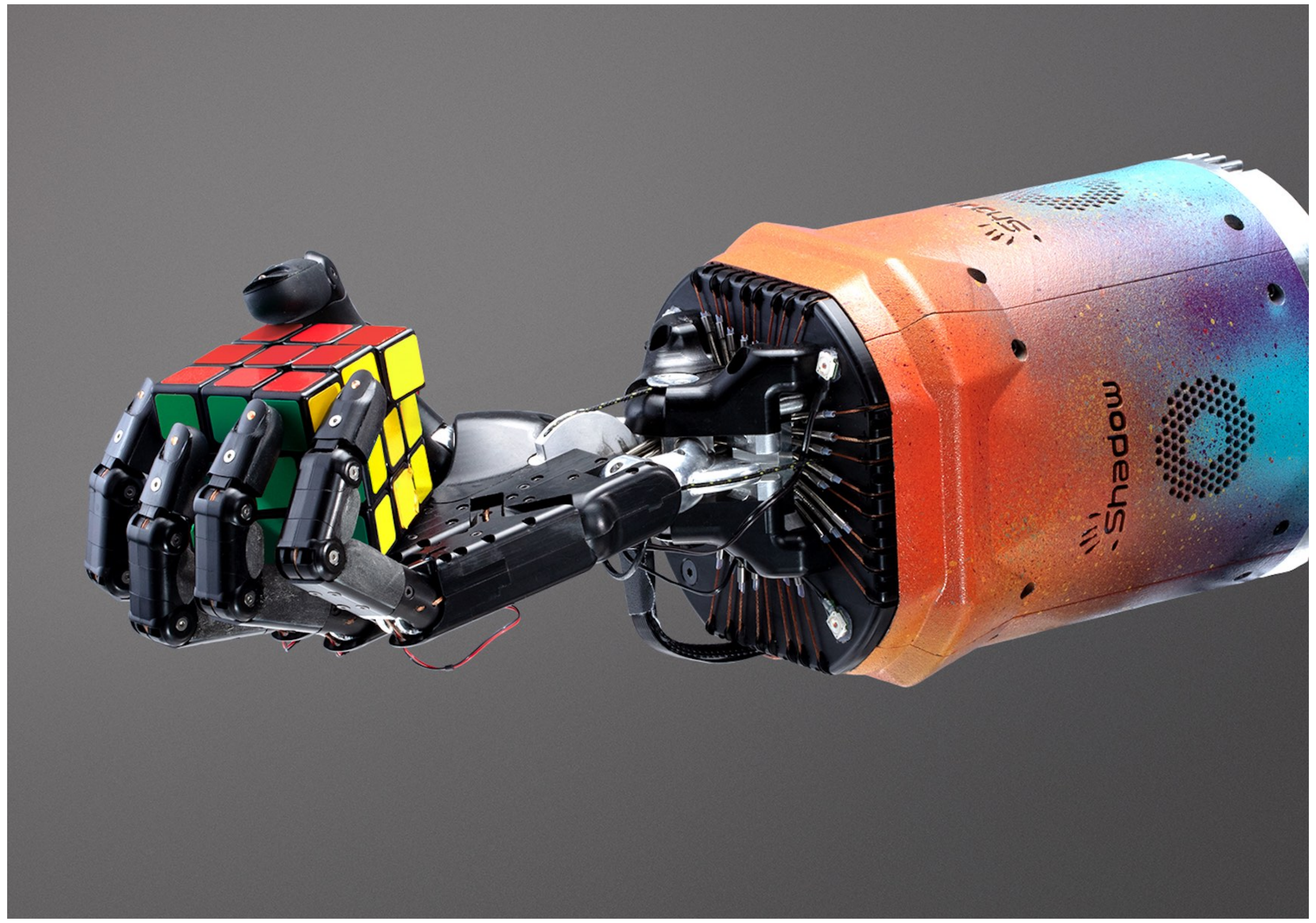
- Recap:
  - LQR model + Ricatti equations

- Today: LQRs
  - Infinite horizon model + SDP formulations
  - (convex) SLS parameterization

Recap

Robotics and Controls

Dexterous Robotic Hand Manipulation
OpenAI, 2019



Optimal Control

- f_t maps a state $x_t \in R^d$, a control (the action) $u_t \in R^k$, and a disturbance w_t, to the next state $x_{t+1} \in R^d$, starting from an initial state x_0.
  $x_{t+1} = f_t(x_t, u_t, w_t)$

- The objective is to find the control policy π which minimizes the long term cost,

  minimize $\quad E_\pi\left[\sum_{t=0}^{H-1} c_t(x_t, u_t)\right]$

  such that $\quad x_{t+1} = f_t(x_t, u_t, w_t)$

  where H is the time horizon (which can be finite or infinite)
- often solved by considering the linearized control (sub-)problem where the dynamics are approximated by
  $x_{t+1} = A_t x_t + B_t u_t + w_t,$
  with the matrices A_t and B_t are derivatives of the dynamics f (around some trajectory) and where the costs are approximated by a quadratic function in x_t and u_t.

The Linear Quadratic Regulator (LQR)

(finite horizon case)

- Let's suppose this local approximation to a non-linear model is globally valid. (clearly false but this is an effective approach once when we 'close').

- The finite horizon LQR problem is given by

$$\text{minimize} \quad E\left[x_H^\top Q x_H + \sum_{t=0}^{H-1} (x_t^\top Q x_t + u_t^\top R u_t)\right]$$

$$\text{such that} \quad x_{t+1} = A_t x_t + B_t u_t + w_t\,, \quad x_0 \sim D,\ w_t \sim N(0, \sigma^2 I)\,,$$

where initial state $x_0 \sim D$ is randomly distributed according D;
the disturbance $w_t \in R^d$ is multi-variate normal, with covariance $\sigma^2 I$;
$A_t \in R^{d \times d}$ and $B_t \in R^{d \times k}$ are referred to as system (or transition) matrices;
$Q \in R^{d \times d}$ and $R \in R^{k \times k}$ are psd matrices that parameterize the quadratic costs.

- Note that this model is a finite horizon MDP, where the $S = R^d$ and $A = R^k$.

Same defs (but for costs)

- define the value function $V_h^\pi : R^d \to R$ as

$$V_h^\pi(x) = E\Big[x_H^\top Q x_H + \sum_{t=h}^{H-1} (x_t^\top Q x_t + u_t^\top R u_t) \;\Big|\; \pi, x_h = x\Big],$$

- and the state-action value $Q_h^\pi : R^d \times R^k \to R$ as:

$$Q_h^\pi(x, u) = E\Big[x_H^\top Q x_H + \sum_{t=h}^{H-1} (x_t^\top Q x_t + u_t^\top R u_t) \;\Big|\; \pi, x_h = x, u_h = u\Big],$$

Value Iteration and the Ricatti Equations

Theorem: (for the finite horizon case, with time homogenous $A_t = A, B_t = B$)

The optimal policy is a linear controller specified by:

$$\pi^\star(x_t) = -K_t^\star x_t \text{ where } K_t^\star = (B^\top P_{t+1} B + R)^{-1} B^\top P_{t+1} A$$

where P_t can be computed iteratively, in a backwards manner, using the following algebraic Ricatti equations, where for $t \in [H]$,

$$\begin{aligned} P_t &= A^\top P_{t+1} A + Q - A^\top P_{t+1} B (B^\top P_{t+1} B + R)^{-1} B^\top P_{t+1} A \\ &= A^\top P_{t+1} A + Q - (K_{t+1}^\star)^\top (B^\top P_{t+1} B + R) K_{t+1}^\star \end{aligned}$$

and where $P_H = Q$.

The above equation is simply the value iteration algorithm.

Furthermore, for $t \in [H]$, we have that:

$$V_t^\star(x) = x^\top P_t x + \sigma^2 \sum_{h=t+1}^{H} \text{Trace}(P_h)$$

Proof: optimal control at $h = H - 1$

- Bellman equations ⇒ there is an optimal policy which is deterministic + only a function of x_t and t.
- Due to that $x_H = Ax + Bu + w_{H-1}$, we have:

$$Q_{H-1}(x, u) = E\left[(Ax + Bu + w_{H-1})^\top Q(Ax + Bu + w_{H-1})\right] + x^\top Qx + u^\top Ru$$
$$= (Ax + Bu)^\top Q(Ax + Bu) + \sigma^2 \text{Trace}(Q) + x^\top Qx + u^\top Ru$$

- This is a quadratic function of u. Solving for the optimal control at x, gives:

$$\pi^\star_{H-1}(x) = -(B^\top QB + R)^{-1} B^\top QAx = -K^\star_{H-1}x,$$

where the last step uses that $P_H := Q$.

Proof: optimal value at $h = H - 1$

- (shorthand $K^\star_{H-1} = K$). using the optimal control at:

$$V^\star_{H-1}(x) = Q_{H-1}(x, -K^\star_{H-1}x)$$
$$= x^\top (A - BK)^\top Q(A - BK)x + x^\top Qx + x^\top K^\top RKx - \sigma^2 \text{Trace}(Q)$$

- Continuing

$$V^\star_{H-1}(x) - \sigma^2 \text{Trace}(Q) = x^\top \Big((A - BK)^\top Q(A - BK) + Q + K^\top RK \Big) x$$
$$= x^\top \Big(AQA + Q - 2K^\top B^\top QA + K^\top (B^\top QB + R)K \Big) x$$
$$= x^\top \Big(AQA + Q - 2K^\top (B^\top QB + R)K + K^\top (B^\top QB + R)K \Big) x$$
$$= x^\top \Big(AQA + Q - K^\top (B^\top QB + R)K \Big) x$$
$$= x^\top P_{H-1} x\,.$$

where the fourth step uses our expression for $K = K^\star_{H-1}$.

Proof: wrapping up…

- This implies that:

$$
\begin{aligned}
Q^{\star}_{H-2}(x,u) &= E[V^{\star}_{H-1}(Ax + Bu + w_{H-2})] + x^{\top}Qx + u^{\top}Ru \\
&= (Ax+Bu)^{\top}P_{H-1}(Ax+Bu) + \sigma^2\Big(\text{Trace}(P_{H-1}) + \text{Trace}(Q)\Big) + x^{\top}Qx + u^{\top}Ru\,.
\end{aligned}
$$

- The remainder of the proof follows from a recursive argument, which can be verified along identical lines to the $t = H - 1$ case.

Today

The Linear Quadratic Regulator (LQR)

(infinite horizon case)

- The infinite horizon LQR problem is given by

minimize $\quad \lim_{H\to\infty} \frac{1}{H} E\left[\sum_{t=0}^{H}(x_t^\top Q x_t + u_t^\top R u_t)\right]$

such that $\quad x_{t+1} = Ax_t + Bu_t + w_t\,, \quad x_0 \sim D,\ w_t \sim N(0,\sigma^2 I)\,.$

where A and B are time homogenous.

- Studied often in theory, but less relevant in practice (?)
(largely due to that time homogenous, globally linear models are rarely good approximations)
- Discounted case never studied.
(discounting doesn't necessarily make costs finite)
- Note that we can have 'unbounded' average cost.

Infinite horizon case

Theorem:

Suppose that the optimal average cost is finite.

Let P be a solution to the following algebraic Riccati equation:

$$P = A^T PA + Q - A^T PB(B^T PB + R)^{-1} B^T PA\,.$$

(Note that P is a positive definite matrix).

Infinite horizon case

Theorem:

Suppose that the optimal average cost is finite.

Let P be a solution to the following algebraic Riccati equation:

$$P = A^T PA + Q - A^T PB(B^T PB + R)^{-1} B^T PA \, .$$

(Note that P is a positive definite matrix).

We have that the optimal policy is:

$$\pi^\star(x) = - K^\star x$$

where the optimal control gain is:

$$K^* = - (B^T PB + R)^{-1} B^T PA$$

We have that P is unique and that the optimal average cost is $\sigma^2 \mathrm{Trace}(P)$.

Semidefinite Programs to find P

The Primal SDP:

(for the infinite horizon LQR)

- The primal optimization problem is given as:

$$\text{maximize} \quad \sigma^2 \text{Trace}(P)$$

$$\text{subject to} \quad \begin{bmatrix} A^T P A + Q - I & A^\top P B \\ B^T P A & B^\top P B + R \end{bmatrix} \succeq 0, \quad P \succeq 0$$

where the optimization variable is P.

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  where the optimization variable is P.

- This SDP has a unique solution, $P^\star$, which implies:
  - $P^\star$ satisfies the Ricatti equations.
  - The optimal average cost of the infinite horizon LQR is $\sigma^2 \text{Trace}(P^\star)$
  - The optimal policy use the gain matrix: $K^* = -(B^T P B + R)^{-1} B^T P A$

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- Proof idea: Following from the Ricatti equation,
  we have the relaxation that for all matrices K, the matrix P must satisfy:

$$P \succeq A^T PA + Q - A^T PB(B^T PB + R)^{-1} B^T PA$$

The Dual SDP:

- The dual optimization problem is:

$$\text{minimize} \quad \operatorname{Trace}\left(\Sigma \cdot \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}\right)$$

$$\text{subject to} \quad \Sigma_{xx} = (A \quad B)\Sigma(A \quad B)^\top + \sigma^2 I, \quad \Sigma \succeq 0$$

where the optimization variable is Σ, a $(d+k) \times (d+k)$ matrix, with the block structure:

$$\Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xu} \\ \Sigma_{ux} & \Sigma_{uu} \end{bmatrix}$$

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- This SDP has a unique solution, say $\Sigma^\star$. The optimal gain matrix is then given by: $K^\star = -\Sigma^\star_{ux}(\Sigma^\star_{xx})^{-1}$