Optimal Control Theory and Linear Quadratic Regulators

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CS 6789: Foundations of Reinforcement Learning

Today

- Recap:
  - TRPO/PPO

- Today: LQRs
  - The model + planning + SDP formulations
  - LQRs are MDPs with special structure

Recap

TRPO: second order Taylor's expansion

$\propto \delta$

TRPO:

$\delta \to 0$

equivalent

$$\max_{\pi_\theta} V^{\pi_\theta}(\rho)$$

$$\text{s.t., } KL\left(\Pr^{\pi_{\theta_0}} || \Pr^{\pi_\theta}\right) \leq \delta$$

$$\max_{\theta} \nabla V^{\pi_{\theta_0}}(\rho)^\top \left(\theta - \theta_0\right)$$

$$\text{s.t. } \left(\theta - \theta_0\right)^\top F_{\theta_0}(\theta - \theta_0) \leq \delta$$

We have a closed form solution:

$$\theta = \theta_0 + \sqrt{\frac{\delta}{(\nabla V^{\pi_{\theta_0}})^\top F_{\theta_0}^{-1} \nabla V^{\pi_{\theta_0}}}} \cdot F_{\theta_0}^{-1} \nabla V^{\pi_{\theta_0}}$$

- Self-normalized step-size (Learning rate is adaptive)
- Solve with CG

PPO

θ_{t+1} θ_t

$\pi_{t+1} \leftarrow \pi_t$

- To find the next policy π_{t+1}, use objective:

$$\max_{\theta} \quad E_{s \sim d^{\pi_t}} E_{a \sim \pi^\theta(\cdot|s)} A^{\pi_t}(s, a)$$

linearized objective.

subject to $\quad \sup_s \left\| \pi^\theta(\cdot \mid s) - \pi_t(\cdot \mid s) \right\|_{TV} \leq \delta,$

This is like the CPI greedy policy chooser.

PPO

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This is like the CPI greedy policy chooser.

same

(imp. sampling)

- We can do multiple gradient steps by rewriting the objective function using importance weighting:

$$\max_{\theta} \ \hat{E}_{s\sim d^{\pi_t}} \hat{E}_{a\sim \pi_t(\cdot|s)} \left[\frac{\pi^{\theta}(a\,|\,s)}{\pi_t(\cdot\,|\,s)} \hat{A}^{\pi_t}(s,a) \right]$$

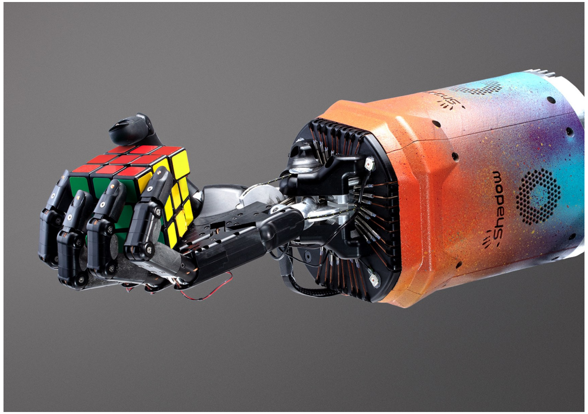
practice: enforce constraint by just changing θ a “little” (say with a few gradient steps)

Today:

~~Natural Policy Gradient and Approximation~~

Robotics and Controls

Dexterous Robotic Hand Manipulation
OpenAI, 2019



The LQR Model

Optimal Control

- a dynamical system is described as
  $x_{t+1} = f_t(x_t, u_t, w_t)$
  where f_t maps a state $x_t \in R^d$, a control (the action) $u_t \in R^k$, and a disturbance w_t, to the next state $x_{t+1} \in R^d$, starting from an initial state x_0.

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- The objective is to find the control policy π which minimizes the long term cost,

  minimize $\quad E_\pi\left[\sum_{t=0}^{H-1} c_t(x_t, u_t)\right]$

  such that $\quad x_{t+1} = f_t(x_t, u_t, w_t)$

  where H is the time horizon (which can be finite or infinite) and where w_t is either statistical or constrained in some way.

Linearization Approach

Linearization Approach

- In practice, this is often solved by considering the linearized control (sub-)problem where the dynamics are approximated by

  $B_t \approx \frac{\partial f(x_t, u, w_t)}{\partial u}\Big|_{u=u_t}$

  $x_{t+1} = A_t x_t + B_t u_t + w_t,$

  with the matrices A_t and B_t are derivatives of the dynamics f (around some trajectory) and where the costs are approximated by a quadratic function in x_t and u_t.

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- This linearization is often accurate provided the noise is 'small' and the dynamics are 'smooth'. (The details are important). + u_t has small changes

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- This approach does not capture global information.

The Linear Quadratic Regulator (LQR)

(finite horizon case)

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quadratic costs
in state & control.

- The finite horizon LQR problem is given by

minimize $\quad E\left[x_H^\top Q x_H + \sum_{t=0}^{H-1} (x_t^\top Q x_t + u_t^\top R u_t)\right]$

such that $\quad x_{t+1} = A_t x_t + B_t u_t + w_t\,, \quad x_0 \sim D,\, w_t \sim N(0,\sigma^2 I)\,,$

where initial state $x_0 \sim D$ is randomly distributed according D;

the disturbance $w_t \in R^d$ is multi-variate normal, with covariance $\sigma^2 I$;

$A_t \in R^{d\times d}$ and $B_t \in R^{d\times k}$ are referred to as system (or transition) matrices;

$Q \in R^{d\times d}$ and $R \in R^{k\times k}$ are psd matrices that parameterize the quadratic costs.

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- Note that this model is a finite horizon MDP, where the $S = R^d$ and $A = R^k$.

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- The infinite horizon LQR problem is given by

minimize $\lim_{H \to \infty} \frac{1}{H} E \left[\sum_{t=0}^{H} (x_t^\top Q x_t + u_t^\top R u_t) \right]$

such that $x_{t+1} = A x_t + B u_t + w_t\,, \quad x_0 \sim D,\ w_t \sim N(0, \sigma^2 I)\,.$

where A and B are time homogenous.

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- Discounted case never studied.
(discounting doesn't necessarily make costs finite)

$x \in \mathbb{R}$

$x_0 = 1$ and $w_t = 0$

suppose $u_t = 0$

$\Downarrow$

$x_t = 2^t$

$x_{t+1} = 2x_t + u_t + w_t$

The Linear Quadratic Regulator (LQR)

(infinite horizon case)

assume that the optimal av. cost is finite.

- The infinite horizon LQR problem is given by

$$\text{minimize} \quad \lim_{H\to\infty} \frac{1}{H} E\left[\sum_{t=0}^{H} (x_t^\top Q x_t + u_t^\top R u_t)\right]$$

$$\text{such that} \quad x_{t+1} = Ax_t + Bu_t + w_t\,, \quad x_0 \sim D,\ w_t \sim N(0,\sigma^2 I)\,.$$

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(controllability assumption)

- Discounted case never studied. (discounting doesn't necessarily make costs finite)
- Note that we can have 'unbounded' average cost. for some policies

Bellman Optimality:

Value Iteration and the Ricatti Equations

What do the values look like in an LQR??

Lin MDP

$\Downarrow$

$Q^{\pi}(s,a) = \vec{w}^{\pi} \cdot \vec{\Phi}(s,a)$

same as P is low rank.

LQR vs Lin MDP

$x_t \in \mathbb{R}^d$

$u \in \mathbb{R}^k$

lin dyn.

$\left.\begin{array}{l} s \in \mathcal{S} \\ a \in \mathcal{A} \end{array}\right\}$ arbitrary

$\phi(s,a) \in \mathbb{R}^d$.

$s' \sim P(\cdot \mid s,a)$

$P(s' \mid s,a) = \vec{\mu}(s') \cdot \vec{\phi}(s,a)$

or

$|\mathcal{S}| \times |\mathcal{S} \times \mathcal{A}|$ $P = \mu \, \Phi$

matrices

Same defs (but for costs)

- define the value function $V_h^\pi : R^d \to R$ as

$$V_h^\pi(x) = E\Big[x_H^\top Q x_H + \sum_{t=h}^{H-1} (x_t^\top Q x_t + u_t^\top R u_t) \Big| \pi, x_h = x\Big],$$

- and the state-action value $Q_h^\pi : R^d \times R^k \to R$ as:

$$Q_h^\pi(x, u) = E\Big[x_H^\top Q x_H + \sum_{t=h}^{H-1} (x_t^\top Q x_t + u_t^\top R u_t) \Big| \pi, x_h = x, u_h = u\Big],$$

Value Iteration and the Ricatti Equations

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Theorem: (for the finite horizon case, with time homogenous $A_t = A, B_t = B$)

The optimal policy is a linear controller specified by:

$$\pi^\star(x_t) = -K_t^\star x_t \text{ where } K_t^\star = (B^\top P_{t+1} B + R)^{-1} B^\top P_{t+1} A$$

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where P_t can be computed iteratively, in a backwards manner, using the following algebraic Ricatti equations, where for $t \in [H]$,

$$\begin{aligned} P_t &= A^\top P_{t+1} A + Q - A^\top P_{t+1} B (B^\top P_{t+1} B + R)^{-1} B^\top P_{t+1} A \\ &= A^\top P_{t+1} A + Q - (K_{t+1}^\star)^\top (B^\top P_{t+1} B + R) K_{t+1}^\star \end{aligned}$$

and where $P_H = Q$.

consider
$t = H-1$
$K^\star_{H-1}$
(need to know
P_H)

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$$P_t = A^\top P_{t+1} A + Q - A^\top P_{t+1} B (B^\top P_{t+1} B + R)^{-1} B^\top P_{t+1} A$$
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and where $P_H = Q$.

The above equation is simply the value iteration algorithm.
Furthermore, for $t \in [H]$, we have that:

$$V_t^\star(x) = x^\top P_t x + \sigma^2 \mathrm{Trace}(P_{t+1})$$

quadratic in state.

Proof: optimal control at $h = H - 1$

Value iteration
compute V^*_{H-1}
and go backwards

- Bellman equations $\Rightarrow$ there is an optimal policy which is deterministic+ ~~stationary~~.

+ function only of
current (x, h)

Proof: optimal control at $h = H - 1$

$E\left[\underbrace{\text{cost}_H}_{x_H^\top Q x_H}\right] + \text{cost}_{H-1} = x_{H-1}^\top Q x_{H-1} + u_{H-1}^\top R u_{H-1}$

- Bellman equations $\Rightarrow$ there is an optimal policy which is deterministic+ stationary.
- Due to that $x_H = Ax + Bu + w_{H-1}$, we have:

$$Q_{H-1}(x, u) = E\left[(Ax + Bu + w_{H-1})^\top Q(Ax + Bu + w_{H-1})\right] + x^\top Q x + u^\top R u$$

$$= (Ax + Bu)^\top Q(Ax + Bu) + \sigma^2 \text{Trace}(Q) + x^\top Q x + u^\top R u$$

Proof: optimal control at $h = H - 1$

$2 B^\top Q(Ax+Bu) + 2Ru = 0$

$\Rightarrow B^\top QAx + (B^\top QB + R)u$

- Bellman equations ⇒ there is an optimal policy which is deterministic+ stationary.
- Due to that $x_H = Ax + Bu + w_{H-1}$, we have:

$$Q_{H-1}(x,u) = E\left[(Ax + Bu + w_{H-1})^\top Q(Ax + Bu + w_{H-1})\right] + x^\top Qx + u^\top Ru$$

$$= (Ax + Bu)^\top Q(Ax + Bu) + \sigma^2 \text{Trace}(Q) + x^\top Qx + u^\top Ru$$

use Q

- This is a quadratic function of u. Solving for the optimal control at x, gives:

$u = -Kx$

$$\pi^\star_{H-1}(x) = -(B^\top QB + R)^{-1} B^\top QAx = -K^\star_{H-1} x,$$

where the last step uses that $P_H := Q$.

$= \arg\min_u Q_{H-1}(x,u)$

Proof: optimal value at $h = H - 1$

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claim for $V^\star_{H-1}(x) = x^\top P_{H-1} x + \sigma^2 Tr(Q)$

- (shorthand $K^\star_{H-1} = K$). using the optimal control at:

$$V^\star_{H-1}(x) = Q_{H-1}(x, -K^\star_{H-1}x) \quad = Q_{H-1}(x, \pi^\star(x))$$

$$= x^\top (A - BK)^\top Q(A - BK)x + x^\top Qx + x^\top K^\top RKx - \sigma^2 \text{Trace}(Q)$$

Proof: optimal value at $h = H - 1$

$K = (B^\top QB + R)^{-1} B^\top QA$

$(BQB + R)K = BQA$

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- Continuing

$$V^\star_{H-1}(x) - \sigma^2 \text{Trace}(Q) = x^\top \Big((A - BK)^\top Q(A - BK) + Q + K^\top RK \Big) x$$

algebra

$$= x^\top \Big(AQA + Q - 2K^\top B^\top QA + K^\top (B^\top QB + R)K \Big) x$$

using

$$= x^\top \Big(AQA + Q - 2K^\top (B^\top QB + R)K + K^\top (B^\top QB + R)K \Big) x$$
$$= x^\top \Big(AQA + Q - K^\top (B^\top QB + R)K \Big) x$$
$$= x^\top P_{H-1} x \,.$$

where the fourth step uses our expression for $K = K^\star_{H-1}$.

Proof: wrapping up…

Proof: wrapping up…

$x_{H-1} = Ax + Bu + w_{H-2}$

typo.

- This implies that:

$$Q^\star_{H-2}(x,u) = E[\boxed{V^\star_{H-1}}(Ax + Bu + w_{H-2})] + x^\top Qx + u^\top Ru$$

$$= (Ax+Bu)^\top P_{H-1}(Ax+Bu) + \sigma^2 \text{Trace}(P_{H-1}) + x^\top Qx + u^\top Ru \,. + \sigma^2 Tr(Q)$$

$$\Rightarrow E\left[(Ax+Bu+w_{H-2}) P_{H-1} (Ax+Bu+w_{H-2})\right]$$

$$+ \sigma^2 Tr(Q)$$

Proof: wrapping up…

- This implies that:

$$
\begin{aligned}
Q^\star_{H-2}(x,u) &= E[V^\star_H(Ax+Bu+w_{H-2})] + x^\top Qx + u^\top Ru \\
&= (Ax+Bu)^\top P_{H-1}(Ax+Bu) + \sigma^2 \text{Trace}(P_{H-1}) + x^\top Qx + u^\top Ru\,.
\end{aligned}
$$

- The remainder of the proof follows from a recursive argument, which can be verified along identical lines to the $t = H-1$ case.

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Let P be a solution to the following algebraic Riccati equation:

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(Note that P is a positive definite matrix).

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We have that the optimal policy is:

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where the optimal control gain is:

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We have that P is unique and that the optimal average cost is $\sigma^2 \mathrm{Trace}(P)$.

Semidefinite Programs to find P

The Primal SDP:

(for the infinite horizon LQR)

- The primal optimization problem is given as:

$$\text{maximize} \quad \sigma^2 \text{Trace}(P)$$

$$\text{subject to} \quad \begin{bmatrix} A^T PA + Q - I & A^\top PB \\ B^T PA & B^\top PB + R \end{bmatrix} \succeq 0, \quad P \succeq 0$$

where the optimization variable is P.

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  where the optimization variable is P.

- This SDP has a unique solution, $P^\star$, which implies:
  - $P^\star$ satisfies the Ricatti equations.
  - The optimal average cost of the infinite horizon LQR is $\sigma^2 \text{Trace}(P^\star)$
  - The optimal policy use the gain matrix: $K^* = -(B^T P B + R)^{-1} B^T P A$

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- Proof idea: Following from the Ricatti equation,

  we have the relaxation that for all matrices K, the matrix P must satisfy:

$$P \succeq (A - BK)^T P(A - BK) + Q - K^\top RK\,.$$

The Dual SDP:

- The dual optimization problem is:

$$\text{minimize} \quad \text{Trace}\left(\Sigma \cdot \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}\right)$$

$$\text{subject to} \quad \Sigma_{xx} = (A \quad B)\Sigma(A \quad B)^\top + \sigma^2 I, \quad \Sigma \succeq 0$$

where the optimization variable is Σ, a $(d+k) \times (d+k)$ matrix, with the block structure:

$$\Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xu} \\ \Sigma_{ux} & \Sigma_{uu} \end{bmatrix}$$

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- This SDP has a unique solution, say $\Sigma^\star$. The optimal gain matrix is then given by: $K^\star = -\Sigma^\star_{ux}(\Sigma^\star_{xx})^{-1}$