Lecture 5:
Multi-Armed Bandits (MAB)

<u>Guest Lecturer</u>

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Learning objective: Intro to exploration

Previously on CS 6789

- *Planning via Bellman equations:* known underlying MDP known
- *Generative model:* ability to reset from anywhere

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- Maximize expected reward w/o known underlying MDP or ability to reset!

Focus: Multi-Armed Bandits

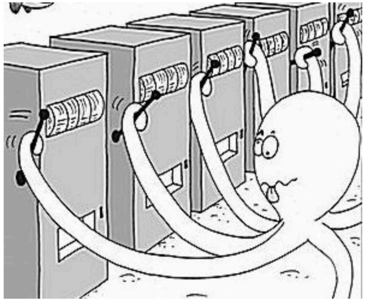
- Simplest setting capturing *explore*-*exploit* trade-off
- Key ideas extend to richer RL settings

Multi-Armed-Bandits: High-level picture

<u>Setting</u>

- Set of alternatives (arms)
- Each arm has a reward distribution

- Learner adaptively selects arms
- Challenge: Distributions not known



Images from:
https://towardsdatascience.com/beyond-a-b-testing-multi-armed-bandit-experiments-1493f709f804
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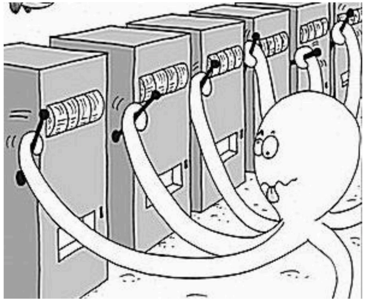

Application: Online advertising

- Arms are advertisers
- Each arm has click-through-rate (CTR)
  probability of getting clicked
- Platform adaptively selects ads
- Challenge: CTRs are not known

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MAB Protocol

Arm $a \in [k]$ has **distribution $\boldsymbol{F(a)}$ with mean $\boldsymbol{\mu(a)}$** and support $[0,1]$

<u>At round $t = 1 \ldots T$:</u>

1. Learner selects arm a^t (possibly in randomized manner)
2. Reward for arm a: $r^t(a) \sim F(a)$
3. Learner earns (and only observes) reward $r^t(a^t)$

Probabilistic Approximate Correct (PAC)

Benchmark: Best arm had we known the distributions: $a^\star = \max_a \mu(a)$

Fix $\epsilon, \delta > 0$

How many samples to identify an **ϵ-optimal arm a** w.p. $1 - \delta$?

$$\boldsymbol{\mu(a^\star) - \mu(a) < \epsilon}$$

Regret Objective

Explore-exploit version: Average cumulative mean: $\mathbf{ALG} = \frac{1}{\mathrm{T}}\sum_t \boldsymbol{\mu}(\boldsymbol{a}^t)$

Benchmark (no exploration): Mean of best arm: $\mathbf{OPT} = \boldsymbol{\mu}(\boldsymbol{a}^\star)$

$$\boxed{\boldsymbol{Regret} = \boldsymbol{OPT} - \boldsymbol{ALG}}$$

Greedy algorithm

Pick each arm once; then highest empirical mean

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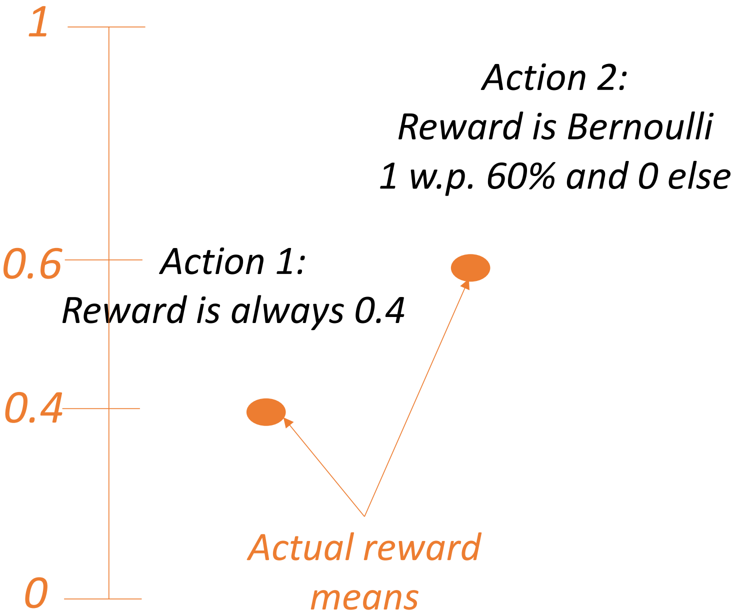

Greedy algorithm

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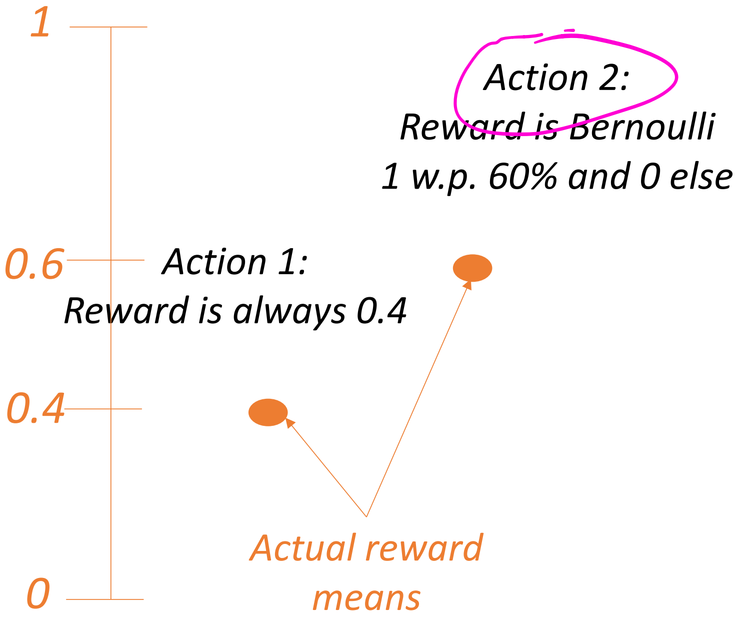
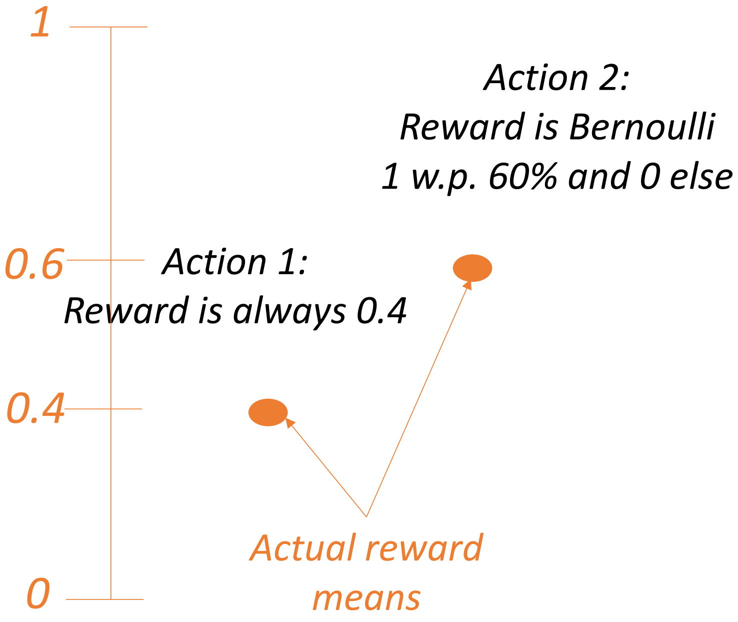


$\underline{\boldsymbol{\epsilon < 0.\cancel{4}^{2}, \delta < 0.\cancel{2}^{4}}}$:

Greedy does not achieve PAC

Greedy algorithm

Pick each arm once; then highest empirical mean



$\underline{\boldsymbol{\epsilon < 0.4, \delta < 0.2}}$**:**

Greedy does not achieve PAC

$\underline{\boldsymbol{Regret = 0.4 \cdot 0.2 \cdot T = 0.08 \cdot T}}$

Regret linear in time-horizon

Explore-Then-Commit (ETC)

Pick each arm $\mathrm{N}(\epsilon) = \frac{log\ (kT/\delta)}{\epsilon^2}$ times; then highest empirical mean $\widetilde{\mu}(a)$

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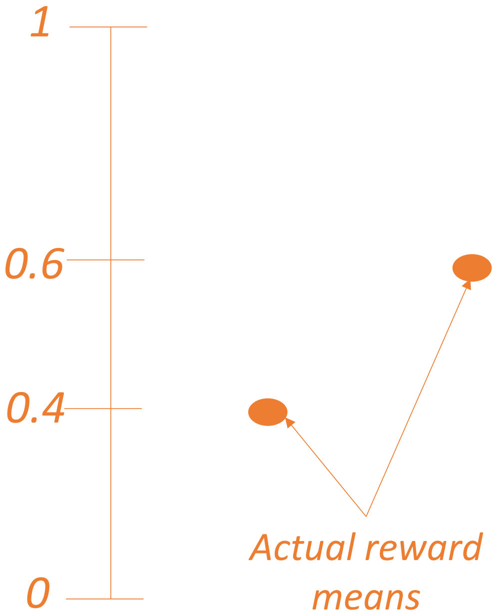
Pick each arm $\mathrm{N}(\epsilon) = \frac{4 log\,(k/\delta)}{\epsilon^2}$ times; then highest empirical mean $\tilde{\mu}(a)$



Hoeffding inequality:

$X_1, X_2, \dots, X_n$ r.v. in $[0,1]$ with mean μ

$$Pr\left[\left|\frac{1}{n}\sum_i X_i - \mu\right| \geq \rho\right] \leq 2 \cdot exp(-2n\rho^2)$$

$\frac{\delta}{k}$

Explore-Then-Commit (ETC)

Pick each arm $\boxed{N(\epsilon) = \frac{4\log(k/\delta)}{\epsilon^2}}$ times; then highest empirical mean $\tilde{\mu}(a)$

By Hoeffding, $\forall a \in [k]$ after $N(\epsilon)$ plays of a, with probability $\geq 1 - \delta/k$, it holds:

$$|\tilde{\mu}(a) - \mu(a)| \leq \epsilon/2$$

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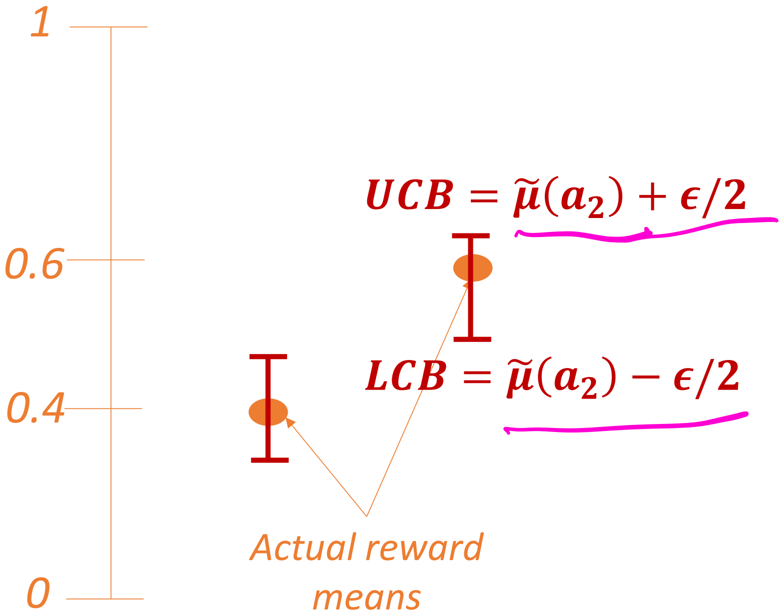

By Hoeffding, $\forall a \in [k]$ after $N(\epsilon)$ plays of a, with probability $\geq 1 - \delta/k$, it holds:

$$|\tilde{\mu}(a) - \mu(a)| \leq \epsilon/2$$

By union bound, after $N(\epsilon)$ plays of every arm, with probability $\geq 1 - \delta$, it holds $\forall a \in [k]$:

$$\boxed{|\tilde{\mu}(a) - \mu(a)| \leq \epsilon/2}$$

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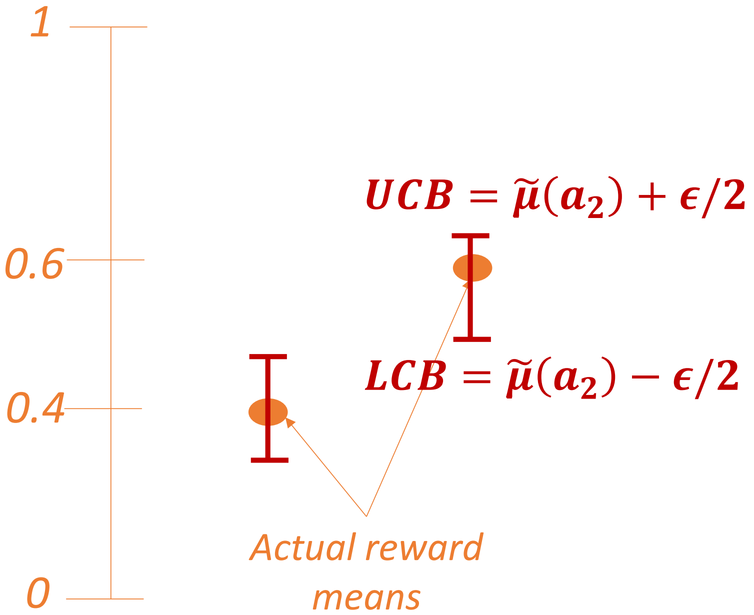
Pick each arm $N(\epsilon) = \frac{4log\,(k/\delta)}{\epsilon^2}$ times; then highest empirical mean $\widetilde{\mu}(a)$

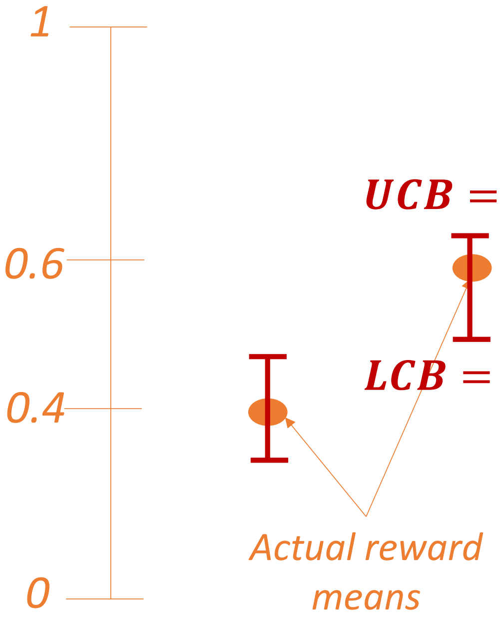


$$\boxed{\textbf{PAC bound:}\quad k \cdot N(\epsilon) = k \cdot \frac{4log\,(k/\delta)}{\epsilon^2}}$$

Explore-Then-Commit (ETC)

How many samples do I need so that I output arm a s.t. w.p. $1-\delta$ $\mu(a^*) - \mu(a) \le \epsilon$

Pick each arm $N(\epsilon) = \frac{4\log(k/\delta)}{\epsilon^2}$ times; then highest empirical mean $\tilde{\mu}(a)$



$$UCB = \tilde{\mu}(a_2) + \epsilon/2$$

$$LCB = \tilde{\mu}(a_2) - \epsilon/2$$

$$\boxed{\textbf{PAC bound: } k \cdot N(\epsilon) = k \cdot \frac{4\log(k/\delta)}{\epsilon^2}}$$

Proof: For selected arm a : $\tilde{\mu}(a) \ge \tilde{\mu}(a^\star)$ and

$$\underline{\mu(a^\star) - \mu(a)} \le \underbrace{\left(\tilde{\mu}(a^\star) + \frac{\epsilon}{2}\right)}_{UCB(a^*)} - \underbrace{\left(\tilde{\mu}(a) - \frac{\epsilon}{2}\right)}_{LCB(a)} \le \epsilon$$

$\frac{1}{2}\epsilon$ — — $\frac{1}{2}$ $\mu(a^*) - \mu(a) = \epsilon$ loose:

Explore-Then-Commit (ETC)

$\times\epsilon$ → in reality for regret

Pick each arm $N(\epsilon) = \frac{4\log(k/\delta)}{\epsilon^2}$ times; then highest empirical mean $\tilde{\mu}(a)$

$\times 1$

PAC bound: $k \cdot N(\epsilon) = k \cdot \frac{4\log(k/\delta)}{\epsilon^2}$ $\rightarrow 1$

Regret bound: $k \cdot \frac{4\log\left(\frac{k}{\delta}\right)}{\epsilon^2} + \epsilon \cdot T + \delta \cdot T$

$UCB = \tilde{\mu}(a_2) + \epsilon/2$

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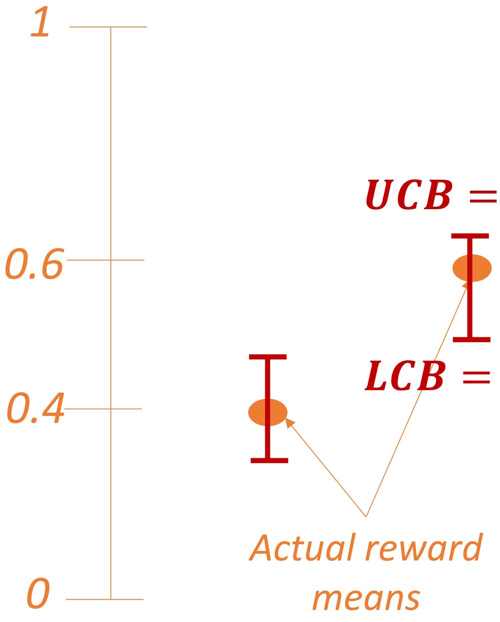

Adaptive $\sum_a N(\epsilon(a))\ \epsilon(a)$ PAC bound $\rightarrow \frac{4\log(\cdot)}{\epsilon(a)^2} \cdot \epsilon(a)$

Regret $\leq k\,N(\epsilon) \cdot 1 \rightarrow \frac{4\log(\cdot\cdot)}{\epsilon(a)^2}\ 1$

T known in advance

$4\log(kT/\delta)$

$\min_{\epsilon} \max(A(\epsilon), B(\epsilon))$

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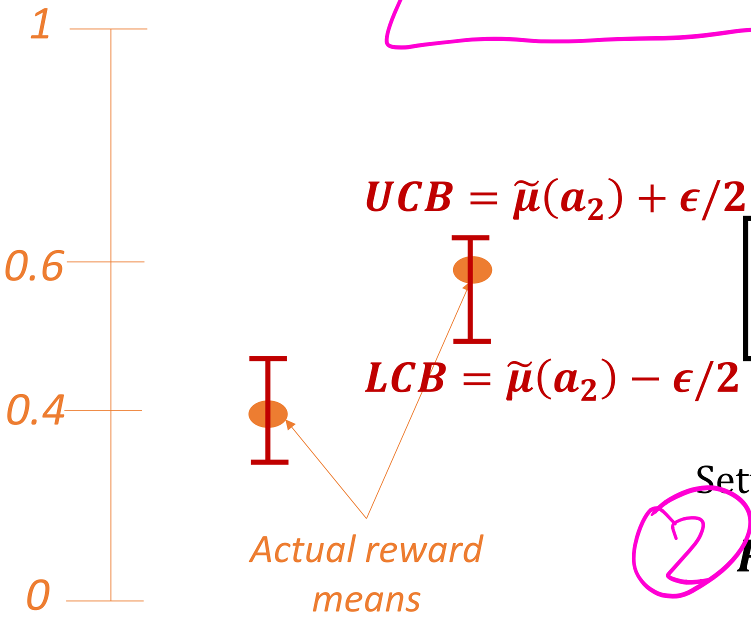

PAC bound: $k \cdot N(\epsilon) = k \cdot \frac{4\log(k/\delta)}{\epsilon^2}$

1

Regret bound: $k \cdot \frac{4\log\left(\frac{k}{\delta}\right)}{\epsilon^2} + \epsilon \cdot T + \delta \cdot T$

$A(\epsilon)$ $B(\epsilon)$

$f(\epsilon) = k \cdot \frac{4\log(kT/\delta)}{\epsilon^2} + \epsilon \cdot T + 1$

Setting $\epsilon = (k \cdot 4\log(kT/\delta))^{1/3} \cdot T^{-1/3}$ and $\delta = 1/T$

2

$$Regret = O\left((k \cdot \log(kT))^{1/3} \cdot T^{2/3}\right)$$

bound

Active Arm Elimination (AAE)

1. Keep adaptive Upper/Lower Confidence Bounds and active set A^t
2. Play round-robin across arms in A^t
3. Remove a from A^t if $UCB^t(a) < \max_{a' \in A^{t-1}} LCB^t(a')$

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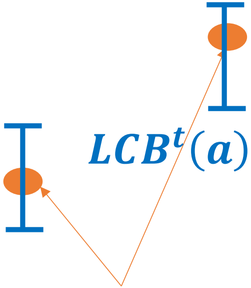
$$\boldsymbol{UCB^t(a) = \tilde{\mu}^t(a) + \sqrt{\frac{\log(2kT/\delta)}{2n^t(a)}}}$$



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Actual reward means

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$\frac{\delta}{kT}$

By Hoeffding and union bound, with probability $\geq 1 - \delta$, it holds $\forall a \in [k], t \in [T]$:

$$\boxed{\mu(a) \in [LCB^t(a), UCB^t(a)]}$$

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Claim A: If confidence intervals hold, i.e. $\forall a, t$: $\boldsymbol{\mu(a) \in [LCB^t(a), UCB^t(a)]}$, the best arm is never eliminated, i.e., $\forall t: a^\star \in A^t$

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Proof:

$$\forall a \neq a^\star:\ \mathrm{UCB}^t(a^\star) \geq \mu(a^\star) \geq \mu(a) \geq LCB(a)$$

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Actual reward means

gap

<u>Claim B:</u> In that event, arm a: $\boxed{\mu(a^\star) - \mu(a) = \epsilon(a)}$ is eliminated after $N(a) = \frac{2\log(2kT/\delta)}{(\epsilon(a))^2}$ plays

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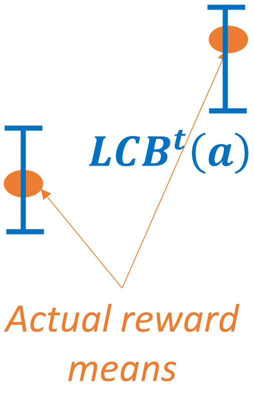

Claim B: In that event, arm a: $\mu(a^\star) - \mu(a) = \epsilon(a)$ is eliminated after $N(a) = \frac{2\log(2kT/\delta)}{(\epsilon(a))^2}$ plays

Proof: Let $\tau(a)$ be that time. By Claim A: $a^\star \in A^{\tau(a)}$.

$$UCB^{\tau(a)}(a) \le \mu(a) + \frac{\epsilon(a)}{2}$$

$$LCB^{\tau(a)}(a^\star) \ge \mu(a^\star) - \frac{\epsilon(a)}{2}$$

$UCB^{\tau(a)}(a) - LCB^{\tau(a^*)}(a^*) \le \mu(a) - \mu(a^*) + \epsilon(a) \le 0$

ETC: $k \cdot \frac{\log(kT/\delta)}{\epsilon^2}$

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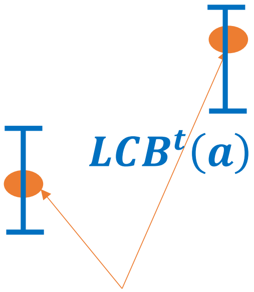

Actual reward means

Claim B: In that event, arm a ($\mu(a^*) - \mu(a) = \epsilon(a)$) is eliminated after $N(a) = \frac{2\log(2kT/\delta)}{(\epsilon(a))^2}$ plays

PAC bound

$$\sum_{a:\epsilon(a)>\epsilon} N(a) + \sum_{a:\epsilon(a)<\epsilon} \frac{2\log(..)}{\epsilon^2}$$

$\epsilon(a)$

Active Arm Elimination (AAE)

a on regret: $N(a) \cdot \epsilon(a) = \frac{2\log(2kT/\delta)}{\epsilon(a)}$

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PAC bound

$\sum_{a:\epsilon(a)>\epsilon} N(\alpha)$ + . .

Regret bound: $\sum_a \min(N(\alpha), T) \cdot \epsilon(a) + \delta \cdot T$

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Setting $\delta = 1/T$

Regret bound: $\sum_a \min\left(\frac{4 \log(kT)}{\epsilon(a)}, \epsilon(a) \cdot T\right) + 1$

$\mu(a^\star) - \mu(a) = \epsilon(a)$

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A(ε) B(ε)

Regret bound: $\sum_a \min\left(\frac{4\log(kT)}{\epsilon(a)}, \epsilon(a)\cdot T\right) + 1$

$\mu(a^\star) - \mu(a) = \epsilon(a)$

For worst-case choice of $\epsilon(a) = \sqrt{\frac{k \cdot \log(kT)}{T}}$:

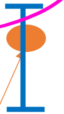
$$Regret = O\left(\sqrt{k \cdot T \cdot \log(kT)}\right)$$

rather than $T^{2/3}$

Upper Confidence Bound (UCB)

Pick arm with highest Upper Confidence Bound

$$UCB^t(a) = \tilde{\mu}^t(a) + \sqrt{\frac{\log(2kT/\delta)}{2n^t(a)}}$$



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Actual reward means

Upper Confidence Bound (UCB)

Pick arm with highest Upper Confidence Bound

By Hoeffding and union bound, with probability $\geq 1 - \delta$, it holds $\forall a \in [k], t \in [T]$:

$$\boldsymbol{\mu(a) \in [LCB^t(a), UCB^t(a)]}$$

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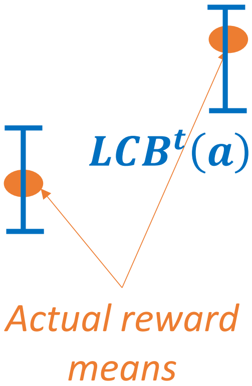

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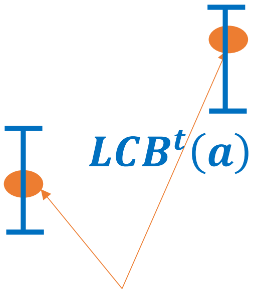

Actual reward means

Claim : In the event that all confidence intervals confidence intervals hold, the regret is at most $\sum_t (UCB^t(a^t) - LCB^t(a^t)) + \delta \cdot T$

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Actual reward means

By Hoeffding and union bound, with probability $\geq 1-\delta$, it holds $\forall a \in [k], t \in [T]$:

$$\boldsymbol{\mu(a) \in [LCB^t(a), UCB^t(a)]}$$

Claim : In the event that all confidence intervals confidence intervals hold, the regret is at most $\sum_t (UCB^t(a^t) - LCB^t(a^t)) + \delta \cdot T$

Proof: $Reg^t = \mu(a^\star) - \mu(a^t)$

$$\leq UCB^t(a^\star) - LCB^t(a^t)$$

$$\leq UCB^t(a^t) - LCB^t(a^t)$$

Upper Confidence Bound (UCB)

$$\boldsymbol{UCB^t(a) = \tilde{\mu}^t(a) + \sqrt{\frac{\log(2kT/\delta)}{2n^t(a)}}} \qquad \boldsymbol{LCB^t(a) = \tilde{\mu}^t(a) - \sqrt{\frac{\log(2kT/\delta)}{2n^t(a)}}}$$

Claim : In the event that all confidence intervals confidence intervals hold, the regret is at most $\sum_t (UCB^t(a^t) - LCB^t(a^t)) + \cancel{+}\delta \cdot T$

$$\sum_{j=1}^{T/k} \sqrt{\frac{1}{j}} \le \int_{j=1}^{T/k+1} \sqrt{\frac{1}{j}}\, dj = \sqrt{T/k+1}$$

Upper Confidence Bound (UCB)

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Regret bound by confidence sum

concave

$$\sum_t (UCB^t(a^t) - LCB^t(a^t)) \le 2 \cdot \sum_t \sqrt{\frac{\log\left(\frac{2kT}{\delta}\right)}{2n^t(a^t)}} = \sum_a \sum_{j=1}^{N(a)} \sqrt{\frac{\log\left(\frac{2kT}{\delta}\right)}{2 \cdot j}}$$

$$\le \sum_a \sum_{j=1}^{\frac{T}{k}} \sqrt{\frac{\log\left(\frac{2kT}{\delta}\right)}{2 \cdot j}} \le k \cdot \sqrt{\log\left(\frac{2kT}{\delta}\right) \cdot \frac{T}{k}} = \boxed{O\left(\sqrt{T \cdot k \cdot \log\left(\frac{kT}{\delta}\right)}\right)}$$

Upper Confidence Bound (UCB)

Resulting guarantee similar to the one of AAE

<u>Confidence sum analysis:</u>

1. Extends to RL (see next lecture)
2. Gap-dependent guarantees
   - Small modification in analysis
3. Allows for anytime guarantees (unknown horizon)
   - Small modification in confidence bounds

Stochastic MAB Protocol

Arm $a \in [k]$ has distribution $F(a)$ with mean $\mu(a)$ and support $[0, 1]$

<u>At round $t = 1 \ldots T$:</u>

1. Learner commits to a distribution p^t across arms
2. **Reward for arm a: $r^t(a) \sim F(a)$**
3. Learner draws arm $a^t \sim p^t$
4. Learner earns (and only observes) reward $r^t(a^t)$

Adversarial MAB Protocol

<u>At round $t = 1 \ldots T$:</u>

1. Learner commits to a distribution p^t across arms
2. **Reward for arm $\boldsymbol{a}$: $\boldsymbol{r^t(a) \in [0, 1]}$ adversarially selected**
3. Learner draws arm $a^t \sim p^t$
4. Learner earns (and only observes) reward $r^t(a^t)$

Stochastic and Adversarial worlds

Stochastic world

- If arms have a gap in their means, i.e., $\mu(a^\star) - \mu(a) = \epsilon(a)$ then regret of the order of:

$$\sum_{a} \min\left(\frac{4\log(kT)}{\epsilon(a)}, \epsilon(a) \cdot T\right)$$

- If not then regret of the order of $\sqrt{kT}$
- **If rewards are not stochastic, stochastic MAB algs: linear regret**

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- Regret of the order of $\sqrt{kT}$ without assuming stochasticity (e.g., EXP3)
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Question: Best of both worlds?

- Single algorithm with logarithmic guarantee when input *stochastic* and square-root when input *adversarial*!

[Bubeck,Slivkins, COLT '12]

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<u>Answer: Yes!</u>

- Approach 1: Start from AAE and test for non-consistency; if identified then switch to EXP3 **[Bubeck, Slivkins, COLT ’12] [Auer, Chiang, ICML’ 16]**
- Approach 2: Start from adversarial with aggressive “learning rate”; adapt it over time **[Seldin, Slivkins, ICML’14] [Seldin, Lugosi, COLT ’17] [Wei, Luo, COLT ’18] [Zimmert, Seldin, AISTATS ’19]**

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RL: Only very preliminary results for known transitions

[Jin, Luo, working' 20]

Corrupted MAB

Arm $\boldsymbol{a} \in [\boldsymbol{k}]$ has distribution $\boldsymbol{F}(\boldsymbol{a})$ with mean $\boldsymbol{\mu}(\boldsymbol{a})$ and support $[\mathbf{0}, \mathbf{1}]$

<u>At round $t = 1 \ldots T$:</u>

1. Learner commits to a distribution p^t across arms
2. **Reward for arm $\boldsymbol{a}$: $\boldsymbol{r}^{\boldsymbol{t}}(\boldsymbol{a}) \sim \boldsymbol{F}(\boldsymbol{a})$**
3. **Adversary corrupts rewards $\boldsymbol{r}^{\boldsymbol{t}}(\boldsymbol{a})$** ***(total corruption budget of $\boldsymbol{C}$)***
4. Learner draws arm $a^t \sim p^t$
5. Learner earns uncorrupted (or corrupted) reward & observes only corrupted

Corrupted MAB

Question

- Algorithm with logarithmic guarantee when stochastic and gracefully degrades with corruption budget **[Lykouris, Mirrokni, Paes Leme, STOC '18]**

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Answer:

- Initial algorithm based on a multi-layering version of AAE and lower bound for high-probability **[Lykouris, Mirrokni, Paes Leme, STOC '18]**
- Improved algorithm using a phase scheme and the Improved-UCB algorithm of Otner and Auer'10 **[Gupta, Koren, Talwar, COLT '19]**
- For expectations and corrupted: Between both wolds **[Zimmert, Seldin '20]**

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RL: Multi-layering version of UCBVI enhanced with appropriate active sets **[Lykouris, Simchowitz, Slivkins, Sun' 19]**

From MAB to episodic RL

Best of both worlds and corrupted MAB: Examples of MAB informing RL

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[Dann, Mansour, Mohri, Sekhari, Sridharan ’20]

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- Key ideas extend to richer RL & tackle complexities not understood in RL

Algorithms

- Greedy: Not PAC / Linear regret
- Explore-Then-Commit: Regret of $T^{2/3}$
- Active Arm Elimination: Regret logarithmic for arms separated and $\sqrt{T}$ else
- Upper Confidence Bound: Same regret; analysis extends to RL