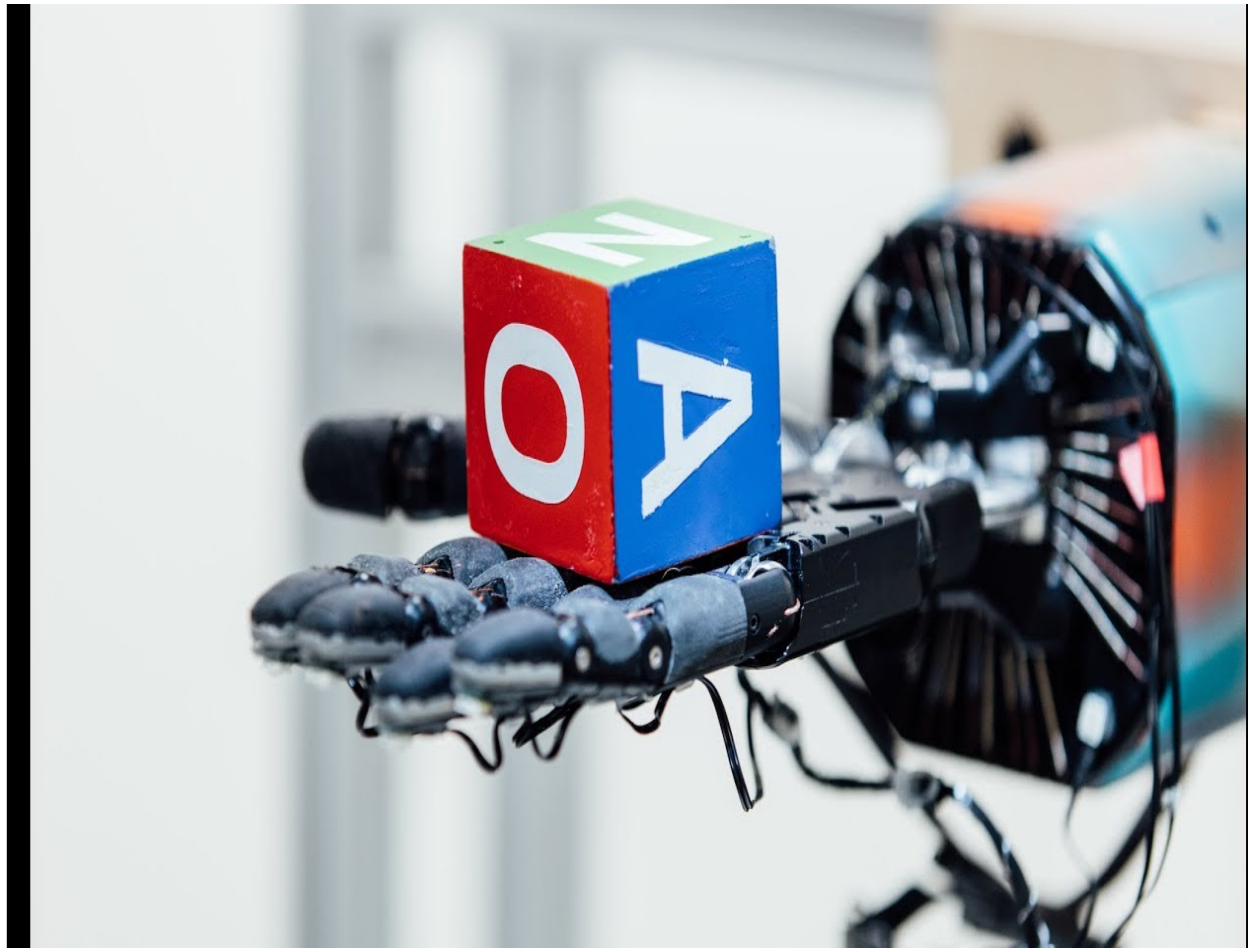
Policy Gradient:
REINFORCE, Variance Reduction, Convergence

Policy Optimization

[AlphaZero, Silver et.al, 17]

[OpenAI Five, 18]



[OpenAI,19]

Recap: Infinite Horizon Discounted MDPs

$$\mathcal{M} = \{P, r, \gamma, \rho, S, A\}$$

where $s_0 \sim \rho$

Objective: $J(\pi) := \mathbb{E}_\pi \left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \middle| s_0 \sim \rho, s_{h+1} \sim P_{s_h, a_h}, a_h \sim \pi(\cdot \mid s_h) \right]$

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Discounted visitation $d^\pi(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h^\pi(s, a)$

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Advantage function: $A^\pi(s, a) = Q^\pi(s, a) - V^\pi(s)$

Today: Policy Gradient Derivation

e.g., Reinforce, Natural Policy Gradient, TRPO, PPO:

(Williams 92, Kakade 02, Schulman et al 15, 17)

$$\pi_\theta(a \mid s) = \pi(a \mid s; \theta)$$

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Main question for today's lecture:
how to compute the gradient?

Policy Gradient: Examples of Policy Parameterization (discrete actions)

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Neural network $f_\theta : S \times A \mapsto \mathbb{R}$

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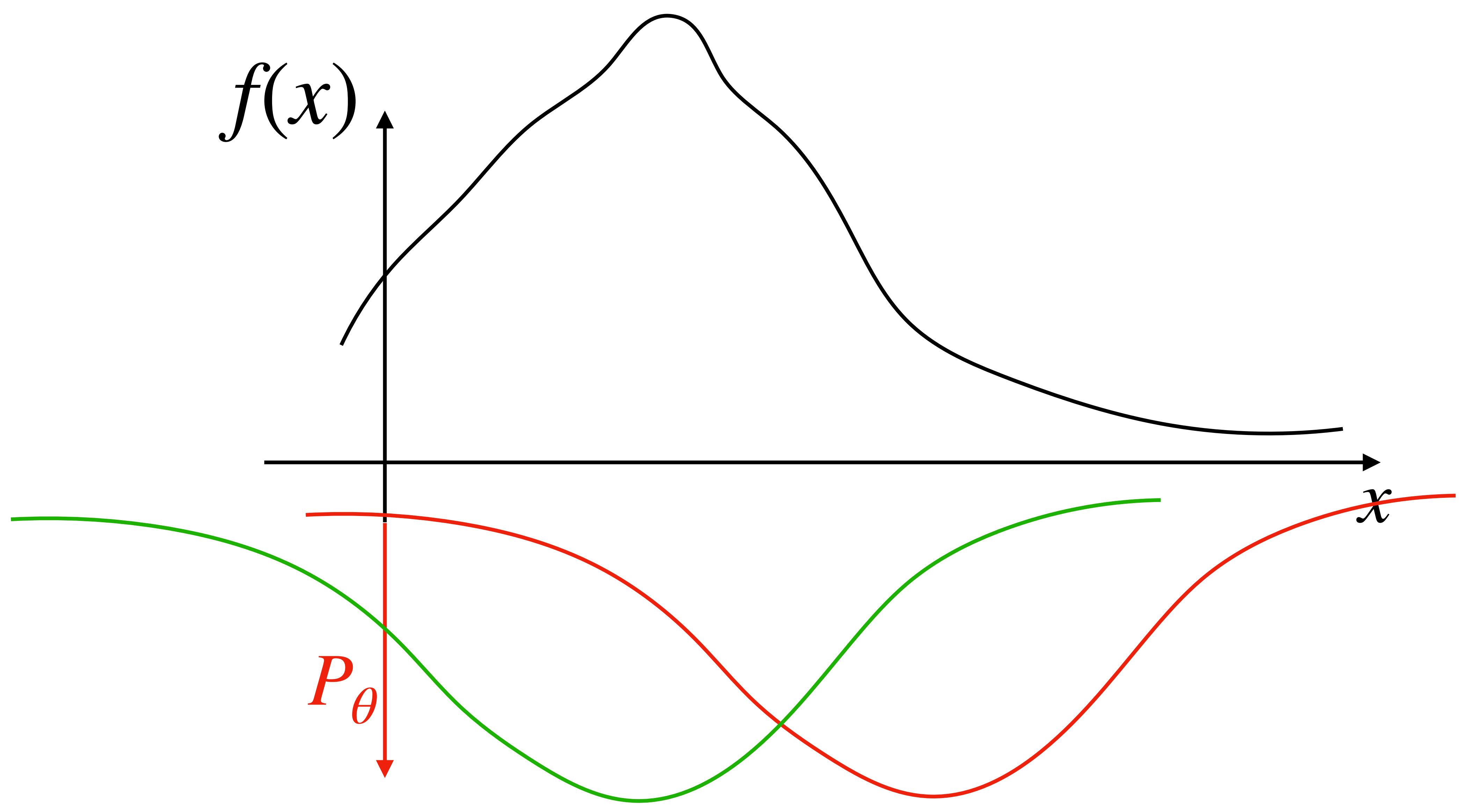

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$$\tau = \{s_0, a_0, s_1, a_1, \ldots\}$$

$$\rho_\theta(\tau) = \rho(s_0)\pi_\theta(a_0 \mid s_0)P(s_1 \mid s_0, a_0)\pi_\theta(a_1 \mid s_1)\ldots$$

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Recall definition of value function $V^{\pi_\theta}(s)$

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$$= \mathbb{E}_{s_0 \sim \rho}\left[\mathbb{E}_{a_0 \sim \pi_\theta(a_0 \mid s_0)} \nabla_\theta \ln \pi_\theta(a_0 \mid s_0) \cdot Q^{\pi_\theta}(s_0, a_0)\right] + \gamma \mathbb{E}_{s_1 \sim \mathbb{P}_1^{\pi_\theta}} \nabla_\theta V^{\pi_\theta}(s_1)$$

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$$= \sum_{h=0}^{\infty} \mathbb{E}_{s_h, a_h \sim \mathbb{P}_h^{\pi_\theta}} \nabla_\theta \ln \pi_\theta(a_h \mid s_h) Q^{\pi_\theta}(s_h, a_h) \quad = \frac{1}{1-\gamma} \mathbb{E}_{s, a \sim d^{\pi_\theta}} \nabla_\theta \ln \pi_\theta(a \mid s) Q^{\pi_\theta}(s, a)$$

Derivation of Policy Gradient w/ Q^π

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Derivation of unbiased Stochastic Policy Gradient

$$\nabla_\theta J(\theta) := \frac{1}{1-\gamma}\mathbb{E}_{s,a\sim d^{\pi_\theta}}\left[\nabla_\theta \ln \pi_{\theta_t}(a\,|\,s) Q^{\pi_\theta}(s,a)\right]$$

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Unbiased estimate: $\nabla_\theta \ln \pi_{\theta_t}(a_h\,|\,s_h)\widetilde{Q}^{\pi_{\theta_t}}(s_h, a_h)$

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Variance Reduction via Action-Independent Baseline

Unbiased Estimate:

$$\nabla_\theta \ln \pi_\theta(a_h \mid s_h)\left(\widetilde{Q}^{\pi_\theta}(s_h, a_h) - b(s_h)\right)$$

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Variance Reduction via Action-Independent Baseline

$$\nabla_\theta \ln \pi_\theta(a_h | s_h)\left(\widetilde{Q}^{\pi_\theta}(s_h, a_h) - b(s_h)\right)$$

The best baseline:

$$\min_b \; \mathbb{E}\left[\left(\nabla_\theta \ln \pi_\theta(a_h | s_h)\left(\widetilde{Q}^{\pi_\theta}(s_h, a_h) - b(s_h)\right)\right)^\top \nabla_\theta \ln \pi_\theta(a_h | s_h)\left(\widetilde{Q}^{\pi_\theta}(s_h, a_h) - b(s_h)\right)\right]$$

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Summary so far:

The most commonly used formulation:
Policy Gradient with V^{π_θ} as a baseline:

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Next: Stochastic Gradient Ascent Converges to Stationary Point

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[**Theorem**] If $J(\theta)$ is β-smooth, and we run SGA: $\theta_{t+1} = \theta_t + \eta \widetilde{\nabla}_\theta J(\theta_t)$

where $\mathbb{E}\left[\widetilde{\nabla}_\theta J(\theta_t)\right] = \nabla_\theta J(\theta_t), \quad \mathbb{E}\left[\|\widetilde{\nabla}_\theta J(\theta_t)\|_2^2\right] \leq \sigma^2,$

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Convergence to Stationary Point

[**Theorem**] If $J(\theta)$ is β-smooth, and we run SGA: $\theta_{t+1} = \theta_t + \eta \widetilde{\nabla}_\theta J(\theta_t)$

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Actor-Critic Algorithm

$$\nabla_\theta J(\theta) = \frac{1}{1-\gamma}\mathbb{E}_{s,a\sim d^{\pi_\theta}}\left[\nabla_\theta \ln \pi_\theta(a\,|\,s)\left(Q^{\pi_{\theta_t}}(s,a) - V^{\pi}_{\theta_t}(s)\right)\right]$$

Critic $Q_w(s,a)$: approximately minimizes $\mathbb{E}_{s,a\sim\mu}\left(Q_w(s,a) - Q^{\pi_\theta}(s,a)\right)^2$

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If compatible, i.e., $\nabla_w \widehat{Q}_w(s,a) = \nabla_\theta \ln \pi_\theta(a\,|\,s)$, then, we get unbiased gradient estimate:

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Another one: $Q_w(s,a) = w^\top \phi(s,a), \quad \pi_\theta(a \mid s) \propto \exp(\theta^\top \phi(s,a))$

Summary

$$\nabla_\theta J(\theta) = \frac{1}{1-\gamma} \mathbb{E}_{s,a \sim d^{\pi_\theta}} \left[\nabla_\theta \ln \pi_{\theta_t}(a \mid s) \left(Q^{\pi_{\theta_t}}(s,a) - V^{\pi}_{\theta_t}(s) \right) \right]$$

Use unbiased estimate of $\nabla_\theta J(\theta)$, SG ascent converges to stationary point

Actor-Critic with Compatible function (warm up for Natural Policy Gradient)