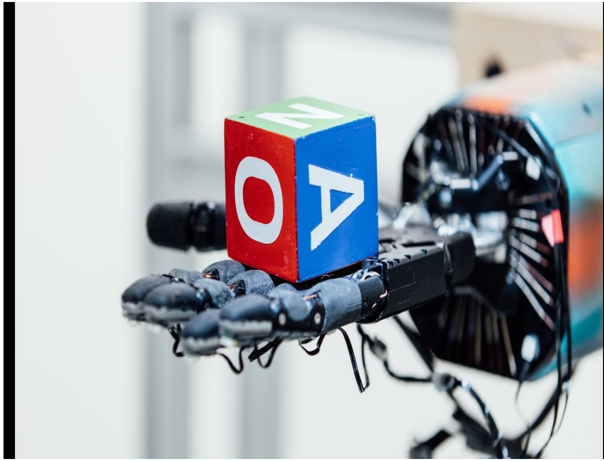
Policy Gradients: Optimality

Recap

Policy Optimization

[AlphaZero, Silver et.al, 17]

[OpenAI Five, 18]



[OpenAI,19]

Today: Policy Gradient Deriviation

e.g., Reinforce, Natural Policy Gradient, TRPO, PPO:

(Williams 92, Kakade 02, Schulman et al 15, 17)

$$\pi_\theta(a \mid s) = \pi(a \mid s; \theta) \qquad J(\pi_\theta) = \mathbb{E}_{\pi_\theta}\left[\sum_{h=0}^{\infty} \gamma^h r_h\right]$$

$$\theta_{t+1} = \theta_t + \eta \nabla_\theta J(\pi_\theta)\,|_{\theta=\theta_t}$$

Main question for today's lecture:
how to compute the gradient?

$$\nabla_\theta J(\theta) := \frac{1}{1-\gamma}\mathbb{E}_{s,a\sim d^{\pi_\theta}}\left[\nabla_\theta \ln \pi_\theta(a \mid s) Q^{\pi_\theta}(s,a)\right]$$

Policy Gradient: Examples of Policy Parameterization (discrete actions)

1. Softmax Policy for Tabular MDPs:

$$\theta_{s,a} \in \mathbb{R}, \forall s, a \in S \times A$$

$$\pi_\theta(a \mid s) = \frac{\exp(\theta_{s,a})}{\sum_{a'} \exp(\theta_{s,a'})}$$

2. Softmax linear Policy (e.g., for linear MDPs):

Feature vector $\phi(s, a) \in \mathbb{R}^d$, and parameter $\theta \in \mathbb{R}^d$

$$\pi_\theta(a \mid s) = \frac{\exp(\theta^\top \phi(s, a))}{\sum_{a'} \exp(\theta^\top \phi(s, a'))}$$

3. Neural Policy:

Neural network $f_\theta : S \times A \mapsto \mathbb{R}$

$$\pi_\theta(a \mid s) = \frac{\exp(f_\theta(s, a))}{\sum_{a'} \exp(f_\theta(s, a'))}$$

Non-Convex Optimization

(review? Or new?)

Convergence to Stationary Points

$J(\pi_\theta)$ is non-convex (see example in the AJKS)

even for softmax case.

Convergence to Stationary Points

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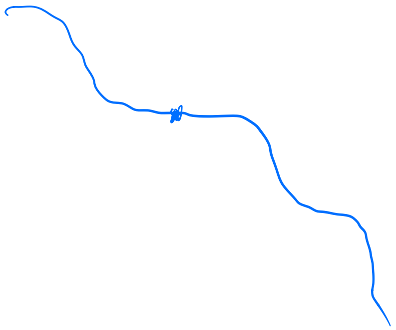
- Def of a β-smooth function F:

$$\|\nabla_\theta F(\theta) - \nabla_\theta F(\theta_0)\|_2 \leq \beta \|\theta - \theta_0\|_2$$

iff.

which implies:

$$\left| F(\theta) - F(\theta_0) - \nabla_\theta F(\theta_0)^\top (\theta - \theta_0) \right| \leq \frac{\beta}{2} \|\theta - \theta_0\|_2^2$$



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- Proposition: (stationary point convergence) Assume $F(\theta)$ is β-smooth.
  Suppose we run gradient ascent: $\theta_{t+1} = \theta_t + \eta\nabla_\theta F(\theta_t)$, with $\eta = 1/(2\beta)$. Then:
  $$\min_{t \leq T} \|\nabla_\theta F(\theta_t)\|_2^2 \leq \frac{2\beta\left(\max_\theta F(\theta) - F(\theta_0)\right)}{T}$$

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set $\eta = \frac{1}{\text{"smoothness"}}$

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$$\frac{2\beta}{T}$$

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Today (+future):

When does small gradients imply a performance bound in RL?

Why are PG methods successful?

globally

- Do they converge? to opt ?
- How do they deal with approximation? (e.g. neural policies?)
- How they compare to approximate value function methods?

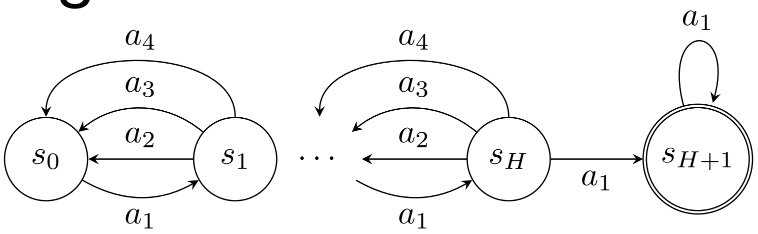
When do PG methods find an optimal solution?

(remember: we have non-convex opt problem)

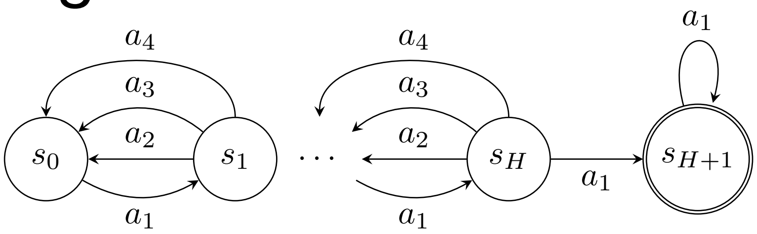
- Let's focus on "complete" parameterizations (e.g. the "tabular" case)
  Π contains all stochastic policies (e.g. softmax)

- today: When do PG methods converge?
  - landscape of the problem
  - what about "exploration"?
  - do small gradients imply good performance?
- Let's consider using exact gradients!

Vanishing Gradients and Saddle Points



Vanishing Gradients and Saddle Points



Set $\gamma = H/(H+1)$. Policy param:

for $a = a_1, a_2, a_3,\ \ \pi_\theta(a \mid s) = \theta_{s,a},$ and $\pi_\theta(a_4 \mid s) = 1 - \theta_{s,a_1} - \theta_{s,a_2} - \theta_{s,a_3}$

(this a "direct" param, which is valid inside the simplex)

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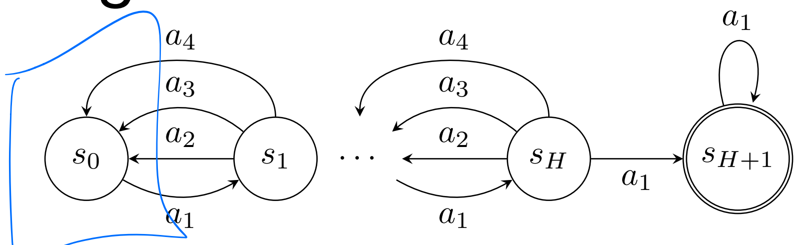

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Theorem: For $0 < \theta < 1$ (componentwise) and $\theta_{s,a_1} < 1/4$ (for all states s).

For all $k \leq O(H/\log(H))$, we have that

$$\|\nabla_\theta^k V^{\pi_\theta}(s_0)\| \leq (1/3)^{H/4}$$

$V^{\pi_\theta}(s_0)$

(where $\|\nabla_\theta^k V^{\pi_\theta}(s_0)\|$ is the operator norm of the tensor $\nabla_\theta^k V^{\pi_\theta}(s_0)$).

Staring state distribution with “coverage”

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- Given our a starting distribution ρ over states, recall our objective is:

$$\max_{\theta \in \Theta} V^{\pi_\theta}(\rho)\,.$$

e.g. $\rho(s) = \mathbb{1}(s = s_0)$

where $\{\pi_\theta \mid \theta \in \Theta \subset \mathbb{R}^d\}$ is some class of parametric policies.

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where $\{\pi_\theta | \theta \in \Theta \subset \mathbb{R}^d\}$ is some class of parametric policies.

- While we are interested in good performance under ρ, it is helpful to optimize under a different measure μ. Specifically, consider optimizing: $V^{\pi_\theta}(\mu)$, i.e.
$$\boxed{\max_{\theta \in \Theta} V^{\pi_\theta}(\mu)\,,}$$
even though our ultimate goal is performance under $V^{\pi_\theta}(\rho)$.

want μ to have "coverage"

“Vanilla” PG for the Softmax

notation (+ overloading)

Today: we will use $d^\pi_{s_0}$ for a state distribution measure.

(it should be clear from context how we use it).

$$d^\pi_{s_0}(s) = (1-\gamma)\sum_{h=0}^{\infty}\gamma^h \mathbb{P}(s_h = s \mid s_0, \pi)$$

$$d^\pi_\mu(s) = \mathbb{E}_{s_0 \sim \mu}\left[d^\pi_{s_0}(s)\right]$$

$$d^\pi_{s_0}(s,a) = (1-\gamma)\sum_{h=0}^{\infty}\gamma^h \mathbb{P}(s_h = s, a_h = a \mid s_0, \pi)$$

suppose π is deterministic

$$A^\pi(s,\pi(s)) = 0$$

Advantage function: $A^\pi(s,a) = Q^\pi(s,a) - V^\pi(s)$

The Softmax Policy Class

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- We have that:
  $$\frac{\partial \log \pi_\theta(a \mid s)}{\partial \theta_{s',a'}} = \mathbf{1}\Big[s = s'\Big]\Big(\mathbf{1}\Big[a = a'\Big] - \pi_\theta(a' \mid s)\Big)$$
  where $\mathbf{1}[\,\cdot\,]$ is the indicator function.

suppose $s = s'$

$$= \begin{cases} 1 - \pi(a|s) & \text{if } a = a' \\ -\pi(a|s) & \text{else} \end{cases}$$

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- Lemma: For the softmax policy class, we have:
  $$\frac{\partial V^{\pi_\theta}(\mu)}{\partial \theta_{s,a}} = \frac{1}{1-\gamma} d_\mu^{\pi_\theta}(s)\pi_\theta(a \mid s) A^{\pi_\theta}(s, a)$$

Remember: The Performance Difference Lemma

For all π, π', s_0:

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\mathbb{E}_{a\sim\pi(\cdot|s)}\left[A^{\pi'}(s,a)\right]$$

$$d^{\pi}_{s_0}(s) = (1-\gamma)\sum_{h=0}^{\infty}\gamma^h \mathbb{P}(s_h = s \mid s_0, \pi)$$

$$V^{*}(\mu) - V^{\pi}(\mu)$$

$$= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{*}_{\mu}}\mathbb{E}_{a\sim\pi^{*}(\cdot|s)}\left[A^{\pi}(s,a)\right]$$

Proof

See book for more concise proof

$$
\begin{aligned}
\frac{\partial V^{\pi_\theta}(\mu)}{\partial \theta_{s,a}} &= = E_{\tau \sim \Pr_\mu^{\pi_\theta}} \left[\sum_{t=0}^{\infty} \gamma^t \nabla_\theta \ln \pi_\theta(a \,|\, s) A^{\pi_\theta}(s,a) \right] \\
&= E_{\tau \sim \Pr_\mu^{\pi_\theta}} \left[\sum_{t=0}^{\infty} \gamma^t \mathbf{1}[s_t = s] \Big(\mathbf{1}[a_t = a] A^{\pi_\theta}(s,a) - \pi_\theta(a \,|\, s) A^{\pi_\theta}(s_t, a_t) \Big) \right] \\
&= E_{\tau \sim \Pr_\mu^{\pi_\theta}} \left[\sum_{t=0}^{\infty} \gamma^t \mathbf{1}[(s_t, a_t) = (s,a)] A^{\pi_\theta}(s,a) \right] + \pi_\theta(a \,|\, s) \sum_{t=0}^{\infty} \gamma^t E_{\tau \sim \Pr_\mu^{\pi_\theta}} \Big[\mathbf{1}[s_t = s] A^{\pi_\theta}(s_t, a_t) \Big] \\
&= \frac{1}{1-\gamma} E_{(s',a') \sim d^{\pi_\theta}} \Big[\mathbf{1}[(s',a') = (s,a)] A^{\pi_\theta}(s,a) \Big] + 0 \\
&= \frac{1}{1-\gamma} d^{\pi_\theta}(s,a) A^{\pi_\theta}(s,a) \,,
\end{aligned}
$$

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- Theorem: Assume the μ is strictly positive i.e. $\mu(s) > 0$ for all states s. For $\eta \leq (1-\gamma)^3/8$, then we have that for all states $s, V^{(t)}(s) \to V^\star(s),$ as $t \to \infty$.

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- Comments:
  - rate could be exponentially slow in S, H.
  - need $\mu > 0$ is necessary.

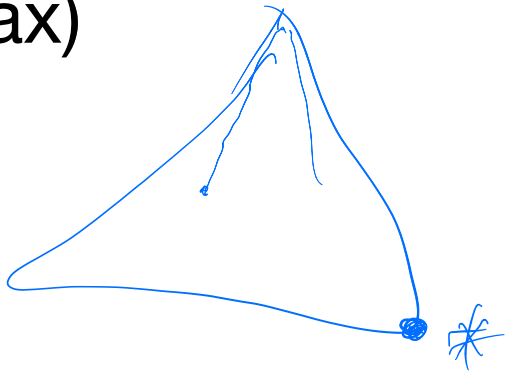
even if

$\mu = \frac{1}{\# \text{ states}}$

PG+Log Barrier Regularization

(for the softmax)

prob. simplex



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- Consider the log barrier λ-regularized objective:
$$L_\lambda(\theta) := V^{\pi_\theta}(\mu) - \lambda\, E_{s \sim \mathrm{Unif}_S}\left[\mathrm{KL}(\mathrm{Unif}_A, \pi_\theta(\cdot \mid s))\right]$$
$$= V^{\pi_\theta}(\mu) + \frac{\lambda}{SA} \sum_{s,a} \log \pi_\theta(a \mid s) + \lambda \log A$$

(aside: $\frac{\text{aside:}}{\text{not entropy}}$)

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- Gradient Ascent:
$\theta^{(t+1)} = \theta^{(t)} + \eta\, \nabla_\theta L_\lambda(\theta^{(t)})$

$E_{s\sim ??}\, H(\pi(\cdot|s))$

$H(p)$

$= \sum_x p(x) \log \frac{1}{p(x)}$

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- Gradient Ascent:
  $\theta^{(t+1)} = \theta^{(t)} + \eta\, \nabla_\theta L_\lambda(\theta^{(t)})$
- Do small gradients imply a globally optimal policy?

Stationarity and Optimality

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- Log barrier regularized objective:

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Stationarity and Optimality

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- Theorem: (Log barrier regularization) Suppose θ is such that:

$\|\nabla_\theta L_\lambda(\theta)\|_2 \le \epsilon_{opt}$ and $\epsilon_{opt} \le \lambda/(2SA)$

then we have for all starting state distributions ρ:

$$V^{\pi_\theta}(\rho) \ge V^\star(\rho) - \frac{2\lambda}{1-\gamma}\left\|\frac{d_\rho^{\pi^\star}}{\mu}\right\|_\infty$$

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$-\frac{2\lambda S}{1-\gamma}$

- where the "distribution mismatch coefficient" is

$$\left\|\frac{d_\rho^{\pi^\star}}{\mu}\right\|_\infty = \max_s\left(\frac{d_\rho^{\pi^\star}(s)}{\mu(s)}\right)$$ (componentwise division notation)

? $\le S$

suppose

$\mu(s) = \frac{1}{S}$

\# states

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- Corollary: (Iteration complexity with log barrier regularization)

  Set $\lambda = \frac{\epsilon(1-\gamma)}{2\left\|\frac{d_\rho^{\pi^\star}}{\mu}\right\|_\infty}$ and $\eta = 1/\beta_\lambda$. Starting from any initial $\theta^{(0)}$,

  poly rate

  then for all starting state distributions ρ, we have

  $$\min_{t<T}\left\{V^\star(\rho) - V^{(t)}(\rho)\right\} \le \epsilon \quad \text{whenever} \quad T \ge c\frac{S^2A^2}{(1-\gamma)^6\,\epsilon^2}\left\|\frac{d_\rho^{\pi^\star}}{\mu}\right\|_\infty^2$$

  (for constant c).

Remember: The Performance Difference Lemma

For all π, π', s_0:

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma} \mathbb{E}_{s \sim d^{\pi}_{s_0}} \mathbb{E}_{a \sim \pi(\cdot|s)} \left[A^{\pi'}(s, a) \right]$$

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- The proof consists of showing that: $\max_a A^{\pi_\theta}(s,a) \le 2\lambda/(\mu(s)S)$ for all states s.

- To see that this is sufficient, observe that by the performance difference lemma:

$$V^\star(\rho) - V^{\pi_\theta}(\rho) = \frac{1}{1-\gamma}\sum_{s,a} d_\rho^{\pi^\star}(s)\pi^\star(a\,|\,s)A^{\pi_\theta}(s,a)$$

$$\le \frac{1}{1-\gamma}\sum_{s} d_\rho^{\pi^\star}(s)\max_{a\in A} A^{\pi_\theta}(s,a)$$

$$\le \frac{1}{1-\gamma}\sum_{s} 2d_\rho^{\pi^\star}(s)\lambda/(\mu(s)S)$$

by

$$\le \frac{2\lambda}{1-\gamma}\max_s\left(\frac{d_\rho^{\pi^\star}(s)}{\mu(s)}\right).$$

which would then complete the proof.

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- Recall $\dfrac{\partial L_\lambda(\theta)}{\partial \theta_{s,a}} = \dfrac{1}{1-\gamma} d_\mu^{\pi_\theta}(s)\pi_\theta(a \mid s)A^{\pi_\theta}(s,a) + \dfrac{\lambda}{S}\left(\dfrac{1}{A} - \pi_\theta(a \mid s)\right)$

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- Solving for $A^{\pi_\theta}(s,a)$ in the first step and using $\|\nabla_\theta L_\lambda(\theta)\|_2 \leq \epsilon_{opt} \leq \lambda/(2SA)$,

solve for A

$$A^{\pi_\theta}(s,a) = \frac{1-\gamma}{d_\mu^{\pi_\theta}(s)}\left(\frac{1}{\pi_\theta(a|s)}\frac{\partial L_\lambda(\theta)}{\partial \theta_{s,a}} + \frac{\lambda}{S}\left(1 - \frac{1}{\pi_\theta(a|s)A}\right)\right)$$

$$\leq \frac{1-\gamma}{d_\mu^{\pi_\theta}(s)}\left(\frac{1}{\pi_\theta(a|s)}\frac{\lambda}{2SA} + \frac{\lambda}{S}\right)$$

$$\frac{\partial L_\lambda}{\partial \theta_{s,a}} \leq \|\nabla L_\lambda\| \leq \varepsilon_{opt} \leq \frac{\lambda}{SA}$$

$$\leq \frac{1}{\mu(s)}\left(\frac{1}{\pi_\theta(a|s)}\frac{\lambda}{2SA} + \frac{\lambda}{S}\right)$$

using that $d_\mu^{\pi_\theta}(s) \geq (1-\gamma)\mu(s)$

coverage.

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- Suppose we could show that $\pi_\theta(a\,|\,s) \geq 1/(2A)$, when $A^{\pi_\theta}(s,a) \geq 0$, then

$$\frac{1}{\mu(s)}\left(\frac{1}{\pi_\theta(a\,|\,s)}\frac{\lambda}{2SA} + \frac{\lambda}{S}\right) \leq \frac{1}{\mu(s)}\left(2A\frac{\lambda}{2SA} + \frac{\lambda}{S}\right) = \frac{2\lambda}{\mu(s)S}$$ and the proof is done!

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$$\epsilon_{opt} \geq \frac{\partial L_\lambda(\theta)}{\partial \theta_{s,a}} = \frac{1}{1-\gamma} d_\mu^{\pi_\theta}(s) \pi_\theta(a \mid s) A^{\pi_\theta}(s, a) + \frac{\lambda}{S}\left(\frac{1}{A} - \pi_\theta(a \mid s)\right)$$

$$\geq 0 + \frac{\lambda}{S}\left(\frac{1}{A} - \pi_\theta(a \mid s)\right) \qquad \text{using } \ A^{\pi_\theta}(s, a) \geq 0$$

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- Rearranging and using our assumption $\epsilon_{opt} \leq \lambda/(2SA)$,

$$\pi_\theta(a \mid s) \geq \frac{1}{A} - \frac{\epsilon_{opt} S}{\lambda} \geq \frac{1}{2A}.$$