Convex Parameterization for Linear Dynamical Systems

Sham Kakade and Wen Sun

CS 6789: Foundations of Reinforcement Learning

Setting: Linear Dynamical System and Convex Cost functions

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Goal: find the best linear controllers: $\{-K_t^\star\}_{t=0}^{H-1} := \arg\min_{\pi \in \Pi} \mathbb{E}_{\pi} \left[\sum_{t=0}^{H-1} c(x_t, u_t) \right]$

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Unfortunately, even with quadratic cost, $\max_{\pi} \mathbb{E}_{\pi}\left[\sum_{h=0}^{H-1} c(x_h, a_h)\right]$ is not convex with respect to $\pi := \{-K_t\}_{t=0}^{H-1}$

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Solution: Over-Parameterization

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What we covered:

1. LQR formulation and DP for LQR (Riccati Equation)

2. You don’t really need to remember the exact quadratic formulation and linear control,
I always derive them from scratch when I use LQR (but it could be time consuming..)

3. Another form of over-parameterized controllers which leads to a convex
parameterization (hence we can do gradient descent).

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$$\left(J^k(\pi) = \sum_{h=0}^{H-1} c^k(x_h, u_h), \text{where } x_{h+1} = Ax_h + Bu_t + w_t, u_t = \pi(x_t) \right)$$