

# **Convex Parameterization for Linear Dynamical Systems**

**Sham Kakade and Wen Sun**

**CS 6789: Foundations of Reinforcement Learning**

## Setting: Linear Dynamical System and Convex Cost functions

$$x_{t+1} = Ax_t + Bu_t + w_t, x_0 \sim \mathcal{D}, w_t \sim \mathcal{N}(0, \sigma^2 I)$$

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$$\frac{1}{2} x^T Q x + \frac{1}{2} u^T R u$$

Convex Function!

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$$\Pi = \{\pi = \{-K_t\}_{t=0}^{H-1} : K_t \in \mathcal{K}, \forall t\}$$

Goal: find the best linear controllers:  $\{-K_t^*\}_{t=0}^{H-1} := \arg \min_{\pi \in \Pi} \mathbb{E}_{\pi} \left[ \sum_{t=0}^{H-1} c(x_t, u_t) \right]$



# A New Convex Parameterization of Controllers

Unfortunately, even with quadratic cost,  $\max_{\pi} \mathbb{E}_{\pi} \left[ \sum_{h=0}^{H-1} c(x_h, a_h) \right]$  is not convex  
with respect to  $\pi := \{-K_t\}_{t=0}^{H-1}$

$\frac{\partial}{\partial \pi} \sum_{h=0}^{H-1} c(x_h, a_h)$

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Solution: Over-Parameterization

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$x_t$

$$w_{t-1} = x_t - Ax_{t-1} - Bu_{t-1}, w_{t-2} = x_{t-1} - Ax_{t-2} - Bu_{t-2}, \dots$$

$$u_t = M_0 x_0 + M_1 w_0 + \dots + M_{t-1} w_{t-1}$$

$\Downarrow$        $\Delta$        $\Delta$        $\Delta$

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$$\begin{aligned} u_t &= -K_t x_t \quad \leftarrow \quad x_t = Ax_{t-1} + Bu_{t-1} + w_{t-1} \\ &= -K_t w_{t-1} - K_t(A - BK_{t-1})x_{t-1} \quad \underbrace{\qquad\qquad\qquad}_{\Delta} \quad \leftarrow \quad u_{t-1} = -K_{t-1} x_{t-1} \\ &\quad \text{---} \\ &\quad \leftarrow \quad x_{t-1} = Ax_{t-2} + Bu_{t-2} + w_{t-2} \\ &\quad \leftarrow \quad u_{t-2} = -K_{t-2} x_{t-2} \end{aligned}$$

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$$= -K_t w_{t-1} - K_t(A - BK_{t-1})x_{t-1}$$

$$= -K_t w_{t-1} - K_t(A - BK_{t-1})(Ax_{t-2} - BK_{t-2}x_{t-2} + \underbrace{w_{t-2}}_{\sim x_{t-1}})$$

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$$= \underbrace{-K_t w_{t-1}}_{:= M_{t-1:t}} - \underbrace{K_t(A - BK_{t-1}) w_{t-2}}_{:= M_{t-2:t}} - \underbrace{K_t(A - BK_{t-1})(A - BK_{t-2}) x_{t-2}}_{:= M_{t-3:t}} \xrightarrow{A}$$

$$x_{t-2} = Ax_{t-3} + Bu_{t-3} + w_{t-3}$$

T

$$-K_{t-3} x_{t-3}$$

$$-K_t (A - BK_{t-1})(A - BK_{t-2})(A - BK_{t-3}) : w_{t-3}$$

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$$= \underbrace{\left[ -K_t \left( \prod_{\tau=1}^t (A - BK_{t-\tau}) \right) \right] x_0}_{M_t} + \sum_{\tau=0}^{t-1} \underbrace{\left[ -K_t \prod_{h=1}^{t-1-\tau} (A - BK_{t-h}) \right]}_{M_{\tau;t}} w_{\tau}$$

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# A New Convex Parameterization of Controllers

[Claim] For any linear controllers  $\pi := \{-K_t\}_{t=0}^{H-1}$ , there exists a parameterization  $\{\{M_t, M_{t-1;t}, \dots M_{0;t}\}\}_{t=0}^{H-1}$ , that generates the same sequence trajectory, given any fixed  $x_0$ , and fixed noise  $w_0, \dots, w_{H-1}$ .

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$$x_0 \longrightarrow u_0 = \underbrace{-K_0 x_0}_0$$

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$$x_0 \longrightarrow u_0 = -K_0 x_0 \longrightarrow \underset{\checkmark}{x_1}$$

$$x_0 \longrightarrow u_0 = M_0 x_0 \longrightarrow \underset{\Delta}{x_1}, \underset{\Delta}{w_0}$$

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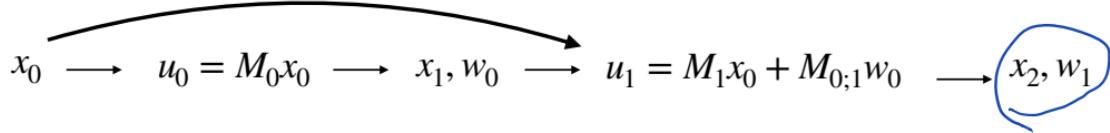
$$x_0 \longrightarrow u_0 = -K_0 x_0 \longrightarrow x_1 \longrightarrow u_1 = -K_1 x_1$$

$$x_0 \xrightarrow{u_0 = M_0 x_0} x_1 \xrightarrow{u_1 = \underbrace{M_1 x_0 + M_{0;1} w_0}_{\text{noise}}}$$

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Δ

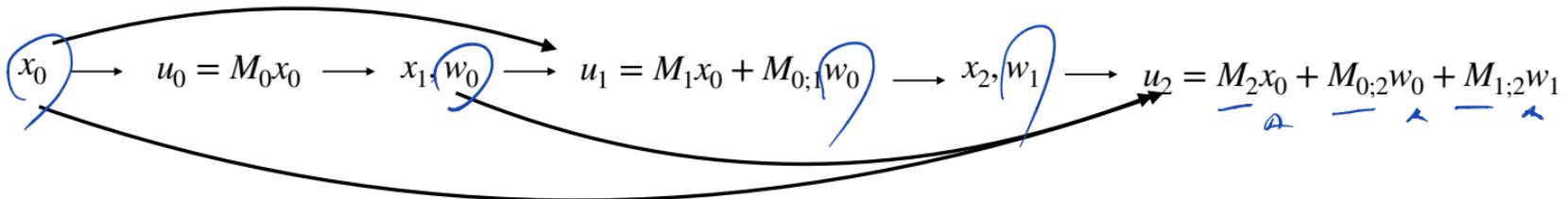
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$$x_1 = Ax_0 + BM_0 x_0 + w_0$$

*linear*

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*$x_2$  is linear w.r.t  $M_s$*

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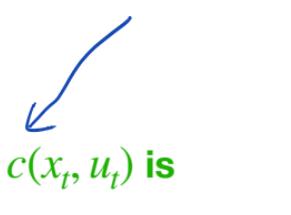
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$$\left\{ \{M_t, M_{0:t}, \dots M_{t-1:t}\} \right\}_{t=0}^{H-1}$$

$$K_t \in \mathbb{R}^{k \times d}$$

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For  $\pi := \{-K_t\}_{t=0}^{H-1}$ , we have  $H \times \underline{d \times k}$  parameters  
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## What we covered:

1. LQR formulation and DP for LQR (Riccati Equation)  

2. You don't really need to remember the exact quadratic formulation and linear control,  
I always derive them from scratch when I use LQR (but it could be time consuming..)
3. Another form of over-parameterized controllers which leads to a convex  
parameterization (hence we can do gradient descent).

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Goal: No-Regret  $\sum_{k=0}^{K-1} \sum_{h=0}^H c^k(x_h^k, a_h^k) - \min_{\{-K_h^\star\}_{h=0}^{H-1}} \sum_{k=0}^{K-1} J^k(\{-K_h^\star\}_{h=0}^{H-1}) = o(K),$

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Goal: No-Regret

$$\sum_{k=0}^{K-1} \sum_{h=0}^H c^k(x_h^k, a_h^k) - \min_{\{-K_h^\star\}_{h=0}^{H-1}} \sum_{k=0}^{K-1} J^k(\{-K_h^\star\}_{h=0}^{H-1}) = o(K),$$

$\frac{\sqrt{K}}{K} \rightarrow 0$  as  $K \rightarrow \infty$

$J^k(\pi) = \sum_{h=0}^{H-1} c^k(x_h, u_h)$ , where  $x_{h+1} = Ax_h + Bu_t + w_t$ ,  $u_t = \pi(x_t)$

$$u \sim N(0, \beta^2 I)$$

Collect  $(x, u, x')$

$$\begin{aligned} (\hat{A}, \hat{B}) = \underset{A, B}{\operatorname{arg\,min}} & \sum \|Ax + Bu - x'\|_2^2 \\ & + \lambda \|A\|_F^2 \\ & + \lambda \|B\|_F^2 \end{aligned}$$

$$\hat{\pi} = \text{RunLQR}(\hat{A}, \hat{B})$$