CS 6789: Foundations of Reinforcement Learning

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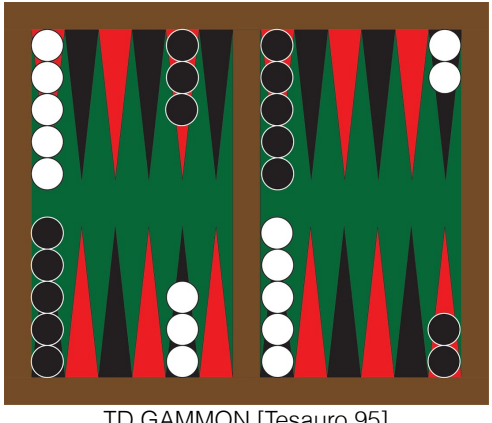
TA: Jonathan Chang

https://wensun.github.io/CS6789.html

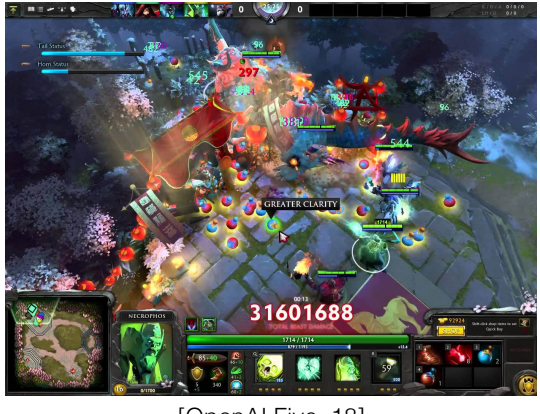
Fall 2020

Progress of RL in Practice



TD GAMMON [Tesauro 95]

[AlphaZero, Silver et.al, 17]



[OpenAI Five, 18]

This course focuses on RL Theory

We care about sample complexity:

Total number of environment interactions
needed to learn a high quality policy (i.e., achieve the task)

Four main themes we will cover in this course:

1. Exploration strategies (not just random)
2. Policy Optimization (gradient descent)
3. Control (LQR and nonlinear control)
4. Imitation Learning (i.e., learning from demonstrations)

Logistics

Four (HW0-HW3) assignments (total 55%) and Course Project (45%)

(HW0 10%, HW1-3 15% each)

HW0 is out today and due in one week

Prerequisites (HW0)

Deep understanding of Machine Learning, Optimization, Statistics

ML: sample complexity analysis for supervised learning (PAC)

Opt: Convex (linear) optimization, e.g., gradient decent for convex functions

Stats: basics of concentration (e.g., Hoeffding's), tricks such as union bound

Prerequisites (HW0)

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Check out HW0 asap!

Course projects (45%)

- Team work: size 3
- Midterm report (5%), Final presentation (15%), and Final report (25%)

- Basics: **survey** of a set of similar RL theory papers. Reproduce analysis and provide a coherent story
- Advanced: **identify** extensions of existing RL papers, **formulate** theory questions, and **provide** proofs

Course Notes:
Reinforcement Learning Theory & Algorithms

- Book website: https://rltheorybook.github.io/

- Many lectures will correspond to chapters in Version 2.
  - V2 (in progress): will use unnormalized rewards.
  - V1: feel free to read ahead.
    Be aware V1 uses normalized values functions
- Please do the assigned readings, especially if you want a mastery of the material.

- Please let us know if you find typos/errors in the book!
  We appreciate it!

Basics of Markov Decision Processes

Supervised Learning

Supervised Learning

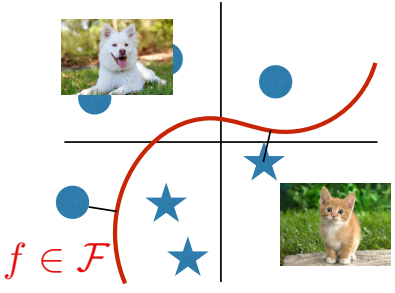
Given i.i.d examples at training:

( ,cat) ( ,cat) ( ,dog)

Supervised Learning

Given i.i.d examples at training:

( ,cat) ( ,cat) ( ,dog)



Supervised Learning

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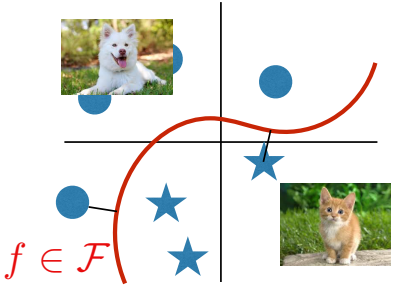
( ,cat) ( ,cat) ( ,dog)



Passive:

Prediction

Data Distribution

Supervised Learning

Given i.i.d examples at training:

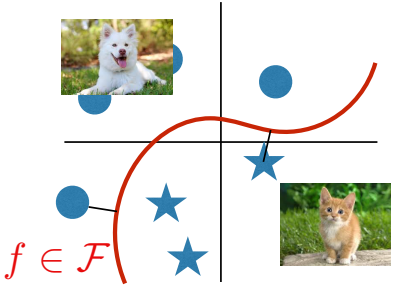
Cersei

,cat

,cat

Tasha

,dog



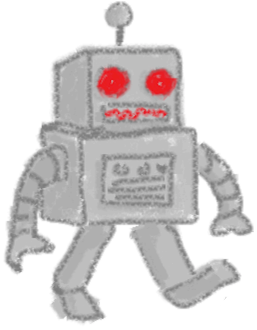
Passive:

Prediction

Data Distribution

Markov Decision Process

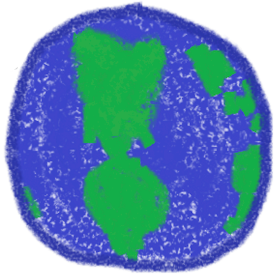
Learning Agent



$$\pi(s) \rightarrow a$$

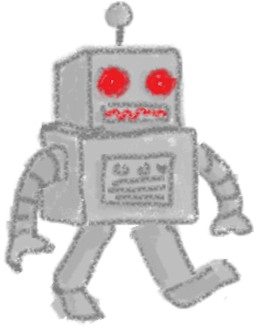
Policy: determine **action** based on **state**

Environmen



Markov Decision Process

Learning Agent



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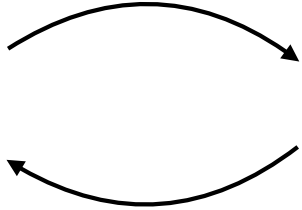
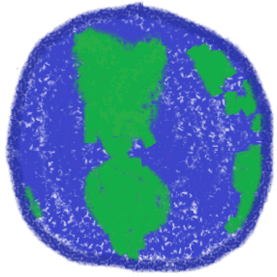


Send **reward** and **next state** from a Markovian transition dynamics

$$r(s,a), s' \sim P(\cdot \mid s,a)$$

Environmen



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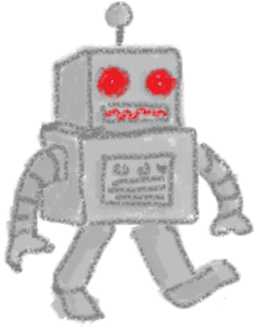

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Multiple Steps

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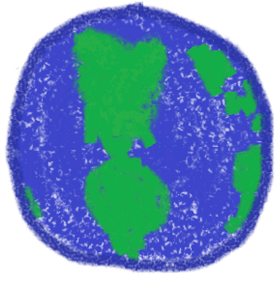

Markov Decision Process

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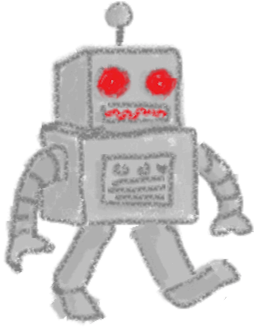

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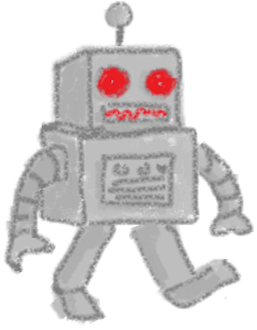

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Supervised Learning					
Reinforcement Learning					

Table content based on slides from Emma Brunskill

	Learn from Experience	Generalize	Interactive	Exploration	Credit assignment
Supervised Learning	✓				
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Infinite horizon Discounted Setting

state space

Action space

$$\mathcal{M} = \{S, A, P, r, \gamma\}$$

$$P : S \times A \mapsto \Delta(S), \quad r : S \times A \to [0,1], \quad \gamma \in [0,1)$$

$s' \sim P(\cdot \mid s, a)$

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Policy $\pi : S \mapsto \Delta(A)$

$\pi(s) \in \Delta(A)$

$a \sim \pi(s)$

$a \sim \pi(\cdot|s)$

$s_0 \quad a_0 \sim \pi(s_0)$

$r(s_0, a_0)$

$s_1 \sim P(\cdot|s_0, a_0)$

$a_1 \sim \pi(\cdot|s_1)$

$\Big\} \; r(s_1, a_1)$

$s_2 \sim P(\cdot|s_1, a_1)$

$\vdots$

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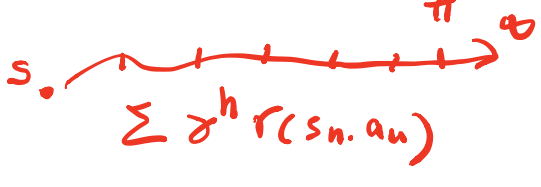
Policy $\pi : S \mapsto \Delta(A)$

Value function $V^{\pi}(s) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \,\middle|\, s_0 = s, a_h \sim \pi(s_h), s_{h+1} \sim P(\cdot \mid s_h, a_h)\right]$



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Bellman Equation:

$$V^{\pi}(s) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \,\middle|\, s_0 = s, a_h \sim \pi(s_h), s_{h+1} \sim P(\cdot \,|\, s_h, a_h)\right]$$

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$$V^{\pi}(s) = \mathbb{E}_{a \sim \pi(s)}\left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^{\pi}(s')\right]$$

$s_0 = s$

$$V^{\pi}(s) = \mathbb{E}\left[r(s_0,a_0) + \gamma r(s_1,a_1) \cdots\right]$$

$$= \mathbb{E}_{a_0 \sim \pi(s_0)}\left[r(s_0,a_0) + \gamma \mathbb{E}\left[r(s_1,a_1) + \gamma r(s_2,a_2) + \cdots\right]\right]$$

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$$\leq V^{\pi}(s_1)$$

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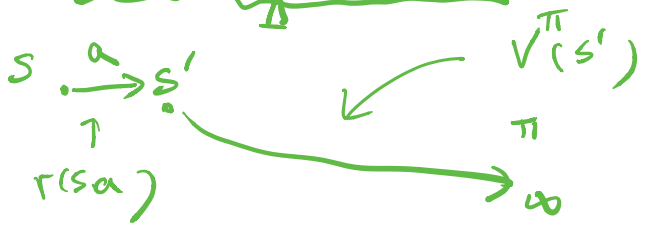

Optimal Policy

For infinite horizon discounted MDP, there exists a deterministic stationary policy

$$\pi^\star : S \mapsto A, \text{ s.t., } V^{\pi^\star}(s) \geq V^{\pi}(s), \forall s, \pi$$

[Puterman 94 chapter 6, also see theorem 1.4 in the RL monograph]

induction

Optimal Policy

For infinite horizon discounted MDP, there exists a deterministic stationary policy $\pi^\star : S \mapsto A$, s.t., $V^{\pi^\star}(s) \geq V^{\pi}(s), \forall s, \pi$

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We denote $V^\star := V^{\pi^\star}, Q^\star := Q^{\pi^\star}$

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Theorem 1: Bellman Optimality

$$V^\star(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^\star(s') \right]$$

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$\Leftrightarrow r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V^*(s')$

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$= Q^*(s,a)$

← Bell-Eqn

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$\max_a \Rightarrow \hat{\pi}(s')$

Proof of Bellman Optimality

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Bellman eq

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$$= r(s, \widehat{\pi}(s)) + \gamma \mathbb{E}_{s' \sim P(s,\widehat{\pi}(s))} \left[r(s', \pi^\star(s')) + \gamma \mathbb{E}_{s'' \sim P(s',\pi^\star(s'))} V^\star(s'') \right]$$

$$\leq r(s, \widehat{\pi}(s)) + \gamma \mathbb{E}_{s' \sim P(s,\widehat{\pi}(s))} \left[r(s', \widehat{\pi}(s')) + \gamma \mathbb{E}_{s'' \sim P(s',\widehat{\pi}(s'))} V^\star(s'') \right]$$

$$\leq r(s, \widehat{\pi}(s)) + \gamma \mathbb{E}_{s' \sim P(s,\widehat{\pi}(s))} \left[r(s', \widehat{\pi}(s')) + \gamma \mathbb{E}_{s'' \sim P(s',\widehat{\pi}(s'))} \left[r(s'', \widehat{\pi}(s'')) + \gamma \mathbb{E}_{s''' \sim P(s'',\widehat{\pi}(s''))} V^\star(s''') \right] \right]$$

$$\leq \mathbb{E}\left[r(s, \widehat{\pi}(s)) + \gamma r(s', \widehat{\pi}(s')) + \ldots \right] = V^{\widehat{\pi}}(s)$$

$V^{\widehat{\pi}}(s) \leq V^\star(s) \leq V^{\widehat{\pi}}(s) \Leftrightarrow V^{\widehat{\pi}}(s) = V^\star(s), \forall s$

Proof of Bellman Optimality

Theorem 1: Bellman Optimality

$$V^\star(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^\star(s') \right]$$

Denote $\widehat{\pi}(s) := \arg\max_a Q^\star(s,a)$, we just proved $V^{\widehat{\pi}}(s) = V^\star(s), \forall s$

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This implies that $\arg\max_a Q^\star(s,a)$ is an optimal policy

$\hat{\pi}$

$V^{\hat{\pi}}(s) = V^*(s), \forall s \in S$

Proof of Bellman Optimality

Theorem 2:

For any $V : S \to \mathbb{R}$, if $V(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \right]$ for all s,

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$\forall s \in S$

$$|V(s) - V^\star(s)| = \left| \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right|$$

Condition

Bellman-opt

Proof of Bellman Optimality

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$$\leq \max_a \left| (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right|$$

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$$\leq \max_a \left| (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right|$$

$$\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left| V(s') - V^\star(s') \right|$$

← Recursion

Proof of Bellman Optimality

Theorem 2:

For any $V : S \to \mathbb{R}$, if $V(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \right]$ for all s,

then $V(s) = V^\star(s), \forall s$

$$
\begin{aligned}
|V(s) - V^\star(s)| &= \left| \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right| \\
&\leq \max_a \left| (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right| \\
&\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left| V(s') - V^\star(s') \right| \\
&\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left(\max_{a'} \gamma \mathbb{E}_{s'' \sim P(s',a')} \left| V(s'') - V^\star(s'') \right| \right)
\end{aligned}
$$

Recursion

Proof of Bellman Optimality

Theorem 2:

For any $V : S \to \mathbb{R}$, if $V(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \right]$ for all s,

then $V(s) = V^\star(s), \forall s$

$$
\begin{aligned}
|V(s) - V^\star(s)| &= \left| \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - \max_a (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right| \\
&\leq \max_a \left| (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V(s')) - (r(s,a) + \gamma \mathbb{E}_{s \sim P(s,a)} V^\star(s')) \right| \\
&\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left| V(s') - V^\star(s') \right| \\
&\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left(\max_{a'} \gamma \mathbb{E}_{s'' \sim P(s',a')} \left| V(s'') - V^\star(s'') \right| \right) \\
&\leq \max_{a_1, a_2, \ldots a_{k-1}} \gamma^k \mathbb{E}_{s_k} |V(s_k) - V^\star(s_k)|
\end{aligned}
$$

$0 = \forall s \in S \quad V = V^*$

$k \to \infty \quad \gamma^k \to 0$

Finite Horizon Setting

$$\mathscr{M} = \{S, A, P, r, \mu_0, H\}$$

$$P : S \times A \mapsto \Delta(S), \quad r : S \times A \to [0,1], \quad H \in \mathbb{N}^+, \quad \mu_0 \in \Delta(S)$$

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Given $\pi : S \mapsto \Delta(A)$

$$s_0 \sim \mu_0, a_0 \sim \pi(s_0), r(s_0, a_0), s_1 \sim P(\cdot \mid s_0, a_0), \ldots, s_{H-1} \sim P(\cdot \mid s_{H-2}, a_{H-2}), a_{H-1} \sim \pi(s_{H-1}), r(s_{H-1}, a_{H-1})$$

$a_0 \cdots a_{H-1}$

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Objective function: $J(\pi) = \mathbb{E}\left[\sum_{h=0}^{H-1} r(s_h, a_h)\right]$

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Time-dependent value/Q function:

$$V_h^\pi(s) = \mathbb{E}\left[\sum_{t=h}^{H-1} r(s_t, a_t) \,|\, s_h = s, a_t \sim \pi(s_t)\right], \quad Q_h^\pi(s, a) = \mathbb{E}\left[\sum_{t=h}^{H-1} r(s_t, a_t) \,|\, s_h = s, a_h = a, a_t \sim \pi(s_t)\right]$$

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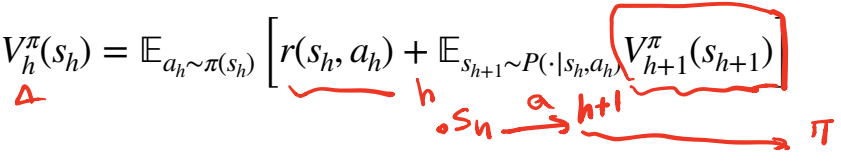
$$V_h^\pi(s_h) = \mathbb{E}_{a_h \sim \pi(s_h)}\left[r(s_h, a_h) + \mathbb{E}_{s_{h+1} \sim P(\cdot \mid s_h, a_h)} V_{h+1}^\pi(s_{h+1})\right]$$



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Time-dependent optimal policy $\pi^\star = \{\pi_0^\star, \ldots, \pi_{H-1}^\star\}$:

$\pi_h^\star : S \to \Delta(A)$

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Time-dependent optimal policy $\pi^\star = \{\pi_0^\star, \ldots, \pi_{H-1}^\star\}$:

$$Q_{H-1}^\star(s,a) = r(s,a), \pi_{H-1}^\star(s) = \arg\max_a Q_{H-1}^\star(s,a), V_{H-1}^\star(s) = \max_a Q_{H-1}^\star(s,a)$$

$= r(s,a)$

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$$Q_h^\star(s,a) = r(s,a) + \mathbb{E}_{s' \sim P(\cdot|s,a)} V_{h+1}^\star(s'), \pi_h^\star(s) = \arg\max_a Q_h^\star(s,a)$$

State (action) Occupancy

$\mathbb{P}_h(s; s_0, \pi)$: probability of π visiting s at time step $h \in \mathbb{N}$, starting at s_0

$$d^{\pi}_{s_0}(s) = (1-\gamma)\sum_{h=0}^{\infty}\gamma^h \mathbb{P}_h(s; s_0, \pi)$$

$$V^{\pi}(s_0) = \frac{1}{1-\gamma}\sum_{s,a} d^{\pi}_{s_0}(s)\pi(a\,|\,s)r(s,a)$$

$\sum_{h=0}^{\infty}\gamma^h = \frac{1}{1-\gamma}$

Valid distribution

$s_0, a_0, s_1, a_1, \ldots s_h$

$\Pr(s_0, a_0, s_1, a_1 \ldots s_h) = \pi(a_0|s_0)P(s_1|s_0,a_0)\cdots\pi(a_{h-1}|s_{h-1})P(s_h|s_{h-1},a_{h-1})$

$\mathbb{P}_h(s; s_0, \pi) = \sum_{a_0, s_1 \ldots a_{h-1}, s_{h-1}} \Pr(s_0, a_0 \ldots s_h = s)$

State (action) Occupancy

$\mathbb{P}_h(s, a; s_0, \pi)$: probability of π visiting (s, a) at time step $h \in \mathbb{N}$, starting at s_0

$$d^{\pi}_{s_0}(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h(s, a; s_0, \pi)$$

$$V^{\pi}(s_0) = \frac{1}{1 - \gamma} \sum_{s,a} d^{\pi}_{s_0}(s, a) r(s, a)$$

$$= \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \,\middle|\, s_0 = s, \pi \right]$$

π^*, Given P, r

Δ

Linear progrg