

Linear Bandits

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1 Recap

2 Linear Bandits

- Setting
- LinUCB
- An Optimal Regret Bound

3 Analysis

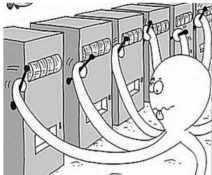
- Regret Analysis
- Confidence Analysis

- (distribution free) Agnostic learning is not possible in RL:
we showed that to get $O(\log |\Pi|)$ sample complexity we need either:
 - $\text{poly}(|\mathcal{S}|)$ samples OR
 - $\exp(H)$ samples.in order to learn the best policy in some policy class.
- upshot: we need stronger assumptions for RL analysis.

Multi-Armed-Bandits: High-level picture

Setting

- Set of alternatives (arms)
- Each arm has a reward distribution
- Learner adaptively selects arms
- **Challenge:** Distributions not known



Images from:

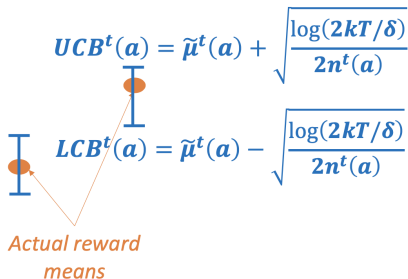
<https://towardsdatascience.com/beyond-a-b-testing-multi-armed-bandit-experiments-1493f209f804>
<https://www.aquasagtechnologies.com/wp-content/uploads/2017/04/Online-advertising.jpg>

Upper Confidence Bound (UCB)

Pick arm with highest **Upper Confidence Bound**

By Hoeffding and union bound, with probability $\geq 1 - \delta$, it holds $\forall a \in [k], t \in [T]$:

$$\mu(a) \in [LCB^t(a), UCB^t(a)]$$



Claim : In the event that all confidence intervals hold, the regret is at most $\sum_t (UCB^t(a^t) - LCB^t(a^t)) + \delta \cdot T$

Proof: $Reg^t = \mu(a^*) - \mu(a^t)$

$$\leq UCB^t(a^*) - LCB^t(a^t)$$
$$\leq UCB^t(a^t) - LCB^t(a^t)$$

Upper Confidence Bound (UCB)

$$UCB^t(a) = \tilde{\mu}^t(a) + \sqrt{\frac{\log(2kT/\delta)}{2n^t(a)}} \quad LCB^t(a) = \tilde{\mu}^t(a) - \sqrt{\frac{\log(2kT/\delta)}{2n^t(a)}}$$

Claim : In the event that all confidence intervals hold, the regret is at most $\sum_t (UCB^t(a^t) - LCB^t(a^t)) + \delta \cdot T$

Regret bound by confidence sum

$$\begin{aligned} \sum_t (UCB^t(a^t) - LCB^t(a^t)) &\leq 2 \cdot \sum_t \sqrt{\frac{\log\left(\frac{2kT}{\delta}\right)}{2n^t(a^t)}} = \sum_a \sum_{j=1}^{N(a)} \sqrt{\frac{\log\left(\frac{2kT}{\delta}\right)}{2 \cdot j}} \\ &\leq \sum_a \sum_{j=1}^{\frac{T}{k}} \sqrt{\frac{\log\left(\frac{2kT}{\delta}\right)}{2 \cdot j}} \leq k \cdot \sqrt{\log\left(\frac{2kT}{\delta}\right) \cdot \frac{T}{k}} = \mathcal{O}\left(\sqrt{T \cdot k \cdot \log\left(\frac{kT}{\delta}\right)}\right) \end{aligned}$$

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Handling Large Actions Spaces

- On each round, we must choose a decision $x_t \in D \subset R^d$.
- Obtain a reward $r_t \in [-1, 1]$, where

$$\mathbb{E}[r_t | x_t = x] = \mu^* \cdot x \in [-1, 1],$$

- so the the conditional expectation of r_t is linear)
- Also, we have the *noise sequence*,

$$\eta_t = r_t - \mu^* \cdot x_t$$

is i.i.d noise.

model due to Abe & Long '99

Our Objective

If x_0, \dots, x_{T-1} are our decisions, then our **cumulative regret** is

$$R_T = \mu^* \cdot x^* - \sum_{t=0}^{T-1} \mu^* \cdot x_t$$

where $x^* \in D$ is an optimal decision for μ^* , i.e.

$$x^* \in \operatorname{argmax}_{x \in D} \mu^* \cdot x$$

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The “Confidence Ball”

After t rounds, define our uncertainty region BALL_t : with center, $\hat{\mu}_t$, and shape, Σ_t , using the λ -regularized least squares solution:

$$\begin{aligned}\hat{\mu}_t &= \arg \min_{\mu} \sum_{\tau=0}^{t-1} \|\mu \cdot \mathbf{x}_{\tau} - r_{\tau}\|_2^2 + \lambda \|\mu\|_2^2 \\ &= \Sigma_t^{-1} \sum_{\tau=0}^{t-1} r_{\tau} \mathbf{x}_{\tau}, \\ \Sigma_t &= \lambda I + \sum_{\tau=0}^{t-1} \mathbf{x}_{\tau} \mathbf{x}_{\tau}^{\top}, \text{ with } \Sigma_0 = \lambda I.\end{aligned}$$

Define the uncertainty region:

$$\text{BALL}_t = \left\{ (\hat{\mu}_t - \mu^*)^{\top} \Sigma_t^{-1} (\hat{\mu}_t - \mu^*) \leq \beta_t \right\},$$

where β_t is a parameter of the algorithm.

LinUCB (the algo)

- 1 Input: λ, β_t
- 2 For $t = 0, 1, \dots$
 - 1 Execute

$$X_t = \operatorname{argmax}_{X \in D} \max_{\mu \in \text{BALL}_t} \mu \cdot X$$

and observe the reward r_t .

- 2 Update BALL_{t+1} .

LinUCB Regret Bound

Sublinear regret: $R_T \leq O^*(d\sqrt{T})$

poly dependence on d , no dependence on the cardinality $|D|$.

Theorem

Suppose: bounded noise $|\eta_t| \leq \sigma$, that $\|\mu^*\| \leq W$, and that $\|x\| \leq B$ for all $x \in D$. Set $\lambda = \sigma^2/W^2$ and

$$\beta_t := \sigma^2 \left(2 + 4d \log \left(1 + \frac{TB^2W^2}{d} \right) + 8 \log(4/\delta) \right).$$

With probability greater than $1 - \delta$, that for all $t \geq 0$,

$$R_T \leq c\sigma\sqrt{T} \left(d \log \left(1 + \frac{TB^2W^2}{d\sigma^2} \right) + \log(4/\delta) \right)$$

where c is an absolute constant.

due to Dani, Hayes, K. '09

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In establishing the upper bounds there are two main propositions from which the upper bounds follow. The first is in showing that the confidence region is valid.

Proposition

(Confidence) Let $\delta > 0$. We have that

$$\Pr(\forall t, \mu^* \in \text{BALL}_t) \geq 1 - \delta.$$

Sum of Squares Regret Bound

Assuming the confidence event holds, the following controls on the growth of the regret.

Proposition

(Sum of Squares Regret Bound) Define:

$$\text{regret}_t = \mu^* \cdot x^* - \mu^* \cdot x_t$$

Suppose $\|x\| \leq B$ for $x \in D$. Suppose β_t is increasing and larger than 1. Suppose $\mu^ \in \text{BALL}_t$ for all t , then*

$$\sum_{t=0}^{T-1} \text{regret}_t^2 \leq 4\beta_T d \log \left(1 + \frac{TB^2}{d\lambda} \right)$$

Completing the Proof

Proof:[Proof of Theorem 1] With the two previous Propositions, along with the Cauchy-Schwarz inequality, we have, with probability at least $1 - \delta$,

$$R_T = \sum_{t=0}^{T-1} \text{regret}_t \leq \sqrt{T \sum_{t=0}^{T-1} \text{regret}_t^2} \leq \sqrt{4T\beta_T d \log \left(1 + \frac{TB^2}{d\lambda} \right)}.$$

The remainder of the proof follows from using our chosen value of β_T and algebraic manipulations. ■

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“Width” of Confidence Ball

Lemma

Let $x \in D$. If $\mu \in \text{BALL}_t$ and $x \in D$. Then

$$|(\mu - \hat{\mu}_t)^\top x| \leq \sqrt{\beta_t x^\top \Sigma_t^{-1} x}$$

Proof: By Cauchy-Schwarz, we have:

$$\begin{aligned} |(\mu - \hat{\mu}_t)^\top x| &= |(\mu - \hat{\mu}_t)^\top \Sigma_t^{1/2} \Sigma_t^{-1/2} x| = |(\Sigma_t^{1/2} (\mu - \hat{\mu}_t))^\top \Sigma_t^{-1/2} x| \\ &\leq \|\Sigma_t^{1/2} (\mu - \hat{\mu}_t)\| \|\Sigma_t^{-1/2} x\| = \|\Sigma_t^{1/2} (\mu - \hat{\mu}_t)\| \sqrt{x^\top \Sigma_t^{-1} x} \leq \sqrt{\beta_t x^\top \Sigma_t^{-1} x} \end{aligned}$$

where the last inequality holds since $\mu \in \text{BALL}_t$. ■

Instantaneous Regret Lemma

Define

$$w_t := \sqrt{x_t^\top \Sigma_t^{-1} x_t}$$

which is the “normalized width” at time t in the direction of our decision.

Lemma

Fix $t \leq T$. If $\mu^* \in \text{BALL}_t$, then

$$\text{regret}_t \leq 2 \min(\sqrt{\beta_t} w_t, 1) \leq 2\sqrt{\beta_T} \min(w_t, 1)$$

Proof: Let $\tilde{\mu} \in \text{BALL}_t$ denote the vector which minimizes the dot product $\tilde{\mu}^\top x_t$. By choice of x_t , we have

$$\tilde{\mu}^\top x_t = \max_{\mu \in \text{BALL}_t} \max_{x \in D} \mu^\top x \geq (\mu^*)^\top x^*,$$

where the inequality used the hypothesis $\mu^* \in \text{BALL}_t$. Hence,

$$\begin{aligned} \text{regret}_t &= (\mu^*)^\top x^* - (\mu^*)^\top x_t \leq (\tilde{\mu} - \mu^*)^\top x_t \\ &= (\tilde{\mu} - \hat{\mu}_t)^\top x_t + (\hat{\mu}_t - \mu^*)^\top x_t \leq 2\sqrt{\beta_t} w_t \end{aligned}$$

Geometric Argument: Part 1

The next two lemmas give us 'geometric' potential function argument, where we can bound the sum of widths independently of the choices made by the algorithm.

Lemma

We have:

$$\det \Sigma_T = \det \Sigma_0 \prod_{t=0}^{T-1} (1 + w_t^2).$$

Proof: By the definition of Σ_{t+1} , we have

$$\begin{aligned} \det \Sigma_{t+1} &= \det(\Sigma_t + x_t x_t^\top) = \det(\Sigma_t^{1/2} (I + \Sigma_t^{-1/2} x_t x_t^\top \Sigma_t^{-1/2}) \Sigma_t^{1/2}) \\ &= \det(\Sigma_t) \det(I + \Sigma_t^{-1/2} x_t (\Sigma_t^{-1/2} x_t)^\top) = \det(\Sigma_t) \det(I + v_t v_t^\top), \end{aligned}$$

where $v_t := \Sigma_t^{-1/2} x_t$. Now observe that $v_t^\top v_t = w_t^2$ and ... ■

Geometric Argument: Part 2

Lemma

For any sequence x_0, \dots, x_{T-1} such that, for $t < T$, $\|x_t\|_2 \leq B$, we have:

$$\log \left(\det \Sigma_{T-1} / \det \Sigma_0 \right) = \log \det \left(I + \frac{1}{\lambda} \sum_{t=0}^{T-1} x_t x_t^\top \right) \leq d \log \left(1 + \frac{TB^2}{d\lambda} \right).$$

Proof: Denote the eigenvalues of $\sum_{t=0}^{T-1} x_t x_t^\top$ as $\sigma_1, \dots, \sigma_d$, and note:

$$\sum_{i=1}^d \sigma_i = \text{Trace} \left(\sum_{t=0}^{T-1} x_t x_t^\top \right) = \sum_{t=0}^{T-1} \|x_t\|^2 \leq TB^2.$$

Using the AM-GM inequality,

$$\begin{aligned} \log \det \left(I + \frac{1}{\lambda} \sum_{t=0}^{T-1} x_t x_t^\top \right) &= \log \left(\prod_{i=1}^d (1 + \sigma_i / \lambda) \right) \\ &= d \log \left(\prod_{i=1}^d (1 + \sigma_i / \lambda) \right)^{1/d} \leq d \log \left(\frac{1}{d} \sum_{i=1}^d (1 + \sigma_i / \lambda) \right) \leq d \log \left(1 + \frac{TB^2}{d\lambda} \right) \end{aligned}$$

Proving “sum of squares regret” Proposition

Proof:[Proof of Proposition 3] Assume $\mu^* \in \text{BALL}_t$ for all t . We have:

$$\begin{aligned} \sum_{t=0}^{T-1} \text{regret}_t^2 &\leq \sum_{t=0}^{T-1} 4\beta_t \min(w_t^2, 1) \leq 4\beta_T \sum_{t=0}^{T-1} \min(w_t^2, 1) \\ &\leq 4\beta_T \sum_{t=0}^{T-1} \ln(1 + w_t^2) \leq 4\beta_T \log \left(\det \Sigma_{T-1} / \det \Sigma_0 \right) \\ &= 4\beta_T d \log \left(1 + \frac{TB^2}{d\lambda} \right) \end{aligned}$$

where the first inequality follow from by Lemma 5; the second from that β_t is an increasing function of t ; the third uses that for $0 \leq y \leq 1$, $\ln(1 + y) \geq y/2$; the final two inequalities follow by Lemmas 6 and 7. ■

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Confidence [Proof of Proposition 2]

Proof: Since $r_\tau = \mathbf{x}_\tau \cdot \mu^* + \eta_\tau$, we have:

$$\begin{aligned}\hat{\mu}_t - \mu^* &= \Sigma_t^{-1} \sum_{\tau=0}^{t-1} r_\tau \mathbf{x}_\tau - \mu^* = \Sigma_t^{-1} \sum_{\tau=0}^{t-1} \mathbf{x}_\tau (\mathbf{x}_\tau \cdot \mu^* + \eta_\tau) - \mu^* \\ &= \Sigma_t^{-1} \left(\sum_{\tau=0}^{t-1} \mathbf{x}_\tau (\mathbf{x}_\tau)^\top \right) \mu^* - \mu^* + \Sigma_t^{-1} \sum_{\tau=0}^{t-1} \eta_\tau \mathbf{x}_\tau \\ &= \lambda \Sigma_t^{-1} \mu^* + \Sigma_t^{-1} \sum_{\tau=0}^{t-1} \eta_\tau \mathbf{x}_\tau\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}\sqrt{(\hat{\mu}_t - \mu^*)^\top \Sigma_t (\hat{\mu}_t - \mu^*)} &\leq \left\| \lambda \Sigma_t^{-1/2} \mu^* \right\| + \left\| \Sigma_t^{-1/2} \sum_{\tau=0}^{t-1} \eta_\tau \mathbf{x}_\tau \right\| \\ &\leq \sqrt{\lambda} \|\mu^*\| + ??.\end{aligned}$$

How can we bound “??” To be continued...

Self-Normalizing Sum

Lemma (Self-Normalized Bound for Vector-Valued Martingales)

(Abassi et. al '11) Suppose $\{\varepsilon_i\}_{i=1}^{\infty}$ are mean zero random variables (can be generalized to martingales), and ε_i is bounded by σ . Let $\{X_i\}_{i=1}^{\infty}$ be a stochastic process. Define $\Sigma_t = \Sigma_0 + \sum_{i=1}^t X_i X_i^{\top}$. With probability at least $1 - \delta$, we have for all $t \geq 1$:

$$\left\| \sum_{i=1}^t X_i \varepsilon_i \right\|_{\Sigma_t^{-1}}^2 \leq \sigma^2 \log \left(\frac{\det(\Sigma_t) \det(\Sigma_0)^{-1}}{\delta^2} \right).$$

Continued... [Proof of Proposition 2]

Proof:

$$\begin{aligned}(\widehat{\mu}_t - \mu^*)^\top \Sigma_t (\widehat{\mu}_t - \mu^*) &\leq \left\| \lambda \Sigma_t^{-1/2} \mu^* \right\|^2 + \left\| \Sigma_t^{-1/2} \sum_{\tau=0}^{t-1} \eta_\tau X_\tau \right\|^2 \\ &\leq \sqrt{\lambda} \|\mu^*\| + \sqrt{2\sigma^2 \log(\det(\Sigma_t) \det(\Sigma^0)^{-1} / \delta_t)}.\end{aligned}$$

We seek to lower bound $\Pr(\forall t, \mu^* \in \text{BALL}_t)$. Assign failure probability $\delta_t = (3/\pi^2)/t^2$ for the t -th event, which gives us:

$$\begin{aligned}1 - \Pr(\forall t, \mu^* \in \text{BALL}_t) &= \Pr(\exists t, \mu^* \notin \text{BALL}_t) \leq \sum_{t=1}^{\infty} \Pr(\mu^* \notin \text{BALL}_t) \\ &< \sum_{t=1}^{\infty} (1/t^2)(3/\pi^2) = 1/2.\end{aligned}$$

This along with Lemma 7 completes the proof. ■