Exploration in Linear MDPs

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$$\text{Regret} = \mathbb{E}\left[\sum_{n=1}^{N} \theta^\star \cdot x^\star - \sum_{n=1}^{N} \theta^\star x_n\right]$$

Important Lemma:

Lemma [Self Normalized Bound for Vector-Valued Martingales] Suppose $\{\epsilon_n\}_{n=1}^{\infty}$ are mean zero random variables with $|\epsilon_n| \leq \alpha$, for all n; Let $\{x_i \in \mathbb{R}^d\}_{n=1}^{\infty}$ be some stochastic random process; Define $\Lambda^n = \lambda I + \sum_{i=1}^{n} x_i x_i^\top$, then with probability at least $1 - \delta$, for all $n \geq 1$:

$$\left\| \sum_{i=1}^{n} x_i \epsilon_i \right\|_{(\Lambda^n)^{-1}}^2 \leq 2\sigma^2 \ln\left(\frac{\det(\Lambda^n)^{1/2} \det(\lambda I)^{-1/2}}{\delta} \right)$$

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$$2\sigma^2 \ln\left(det(\Lambda^n)^{1/2} det(\lambda I)^{-1/2} / \delta \right) \leq \sigma^2 \left(d \ln(1 + n/\lambda) + 2 \ln(1/\delta) \right)$$

Notations and Useful Inequalities

For real-value matrix A:

$$\|A\|_F^2 = \sum_{i,j} A_{i,j}^2 \qquad \|A\|_2 = \sup_{x:\|x\|_2 \leq 1} \|Ax\|_2$$

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$$\phi_h^i := \phi(s_h^i, a_h^i)$$

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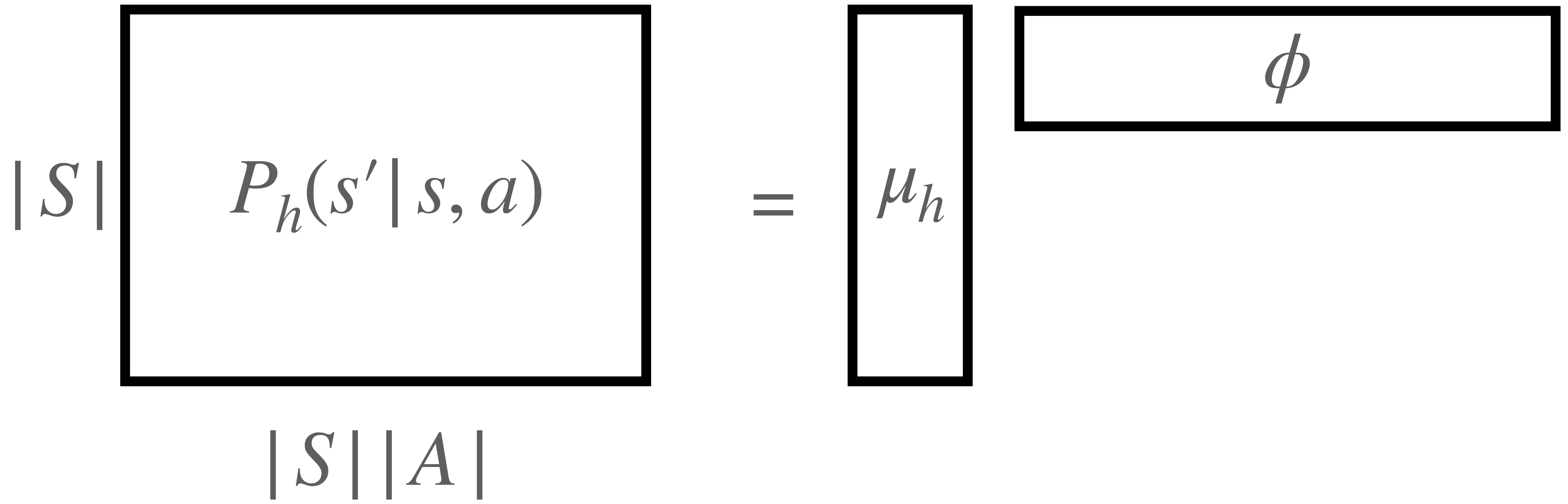

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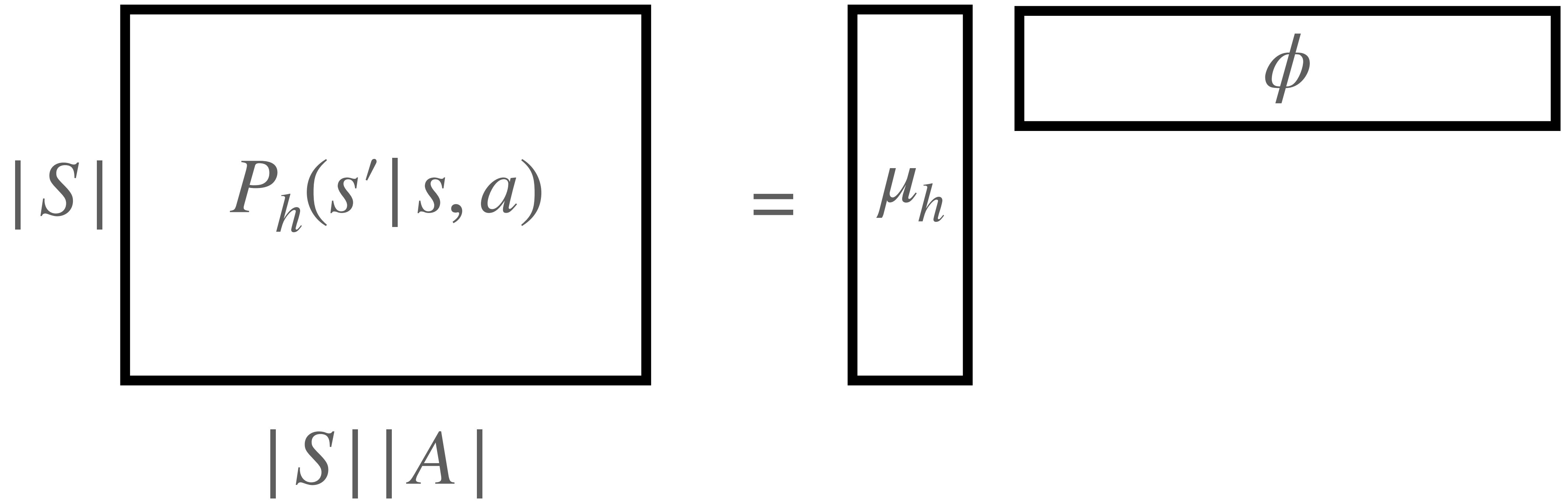

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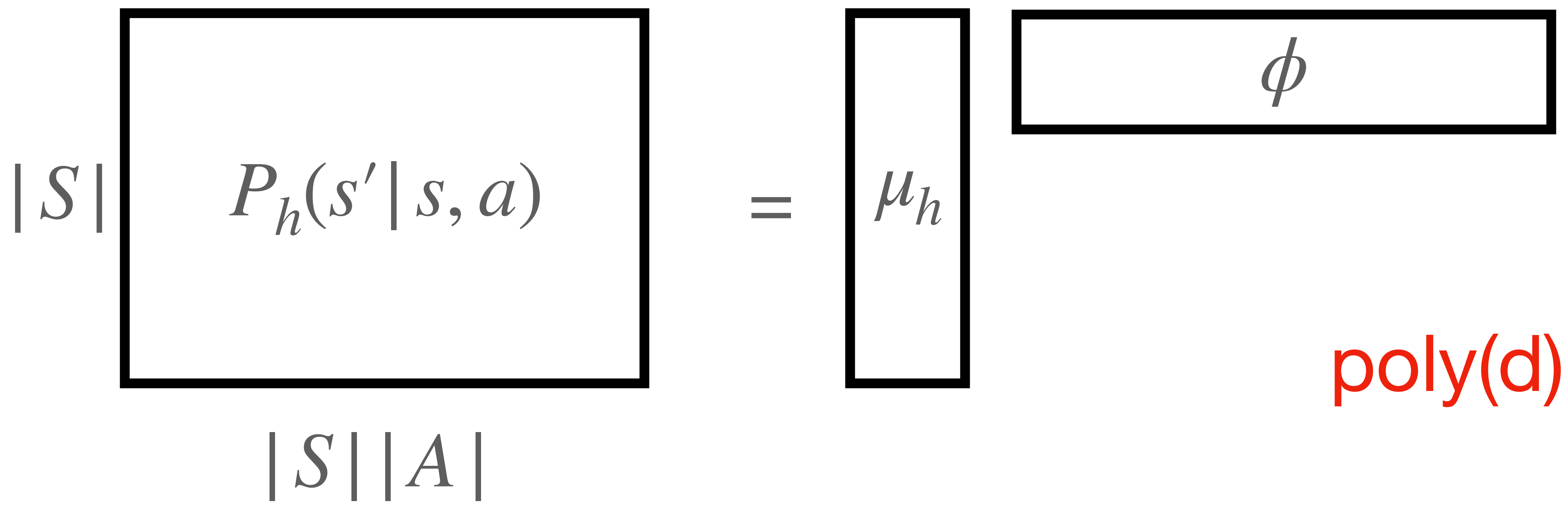

poly(d) rather than poly(SA)

Linear MDP Definition

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Feature map ϕ is known to the learner!

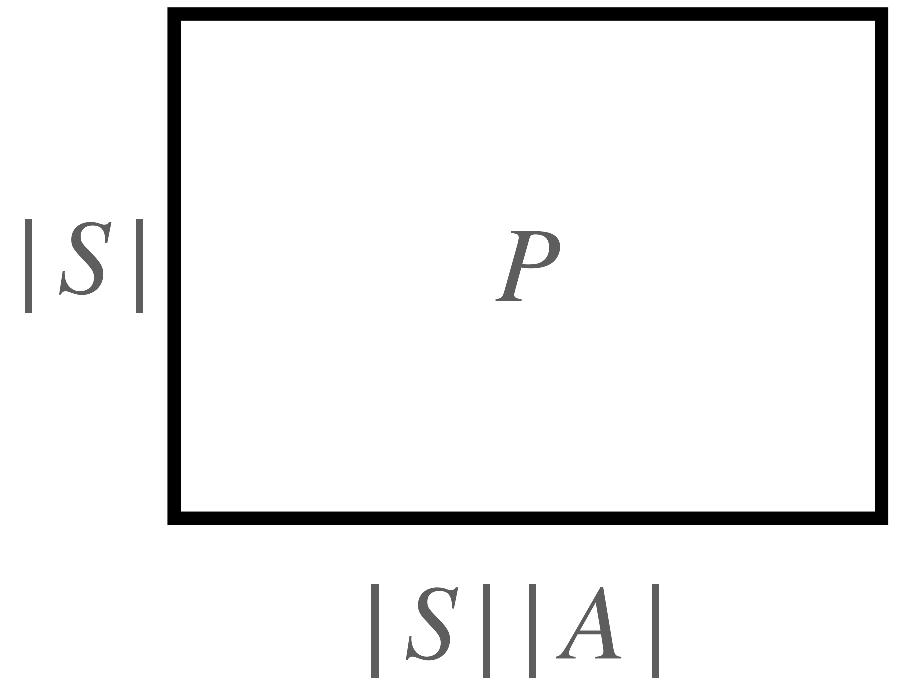
(We assume reward is known, i.e., $\theta^\star$ is known)

Linear MDP Example

It generalizes tabular MDPs: $\phi(s, a)$ one-hot vector

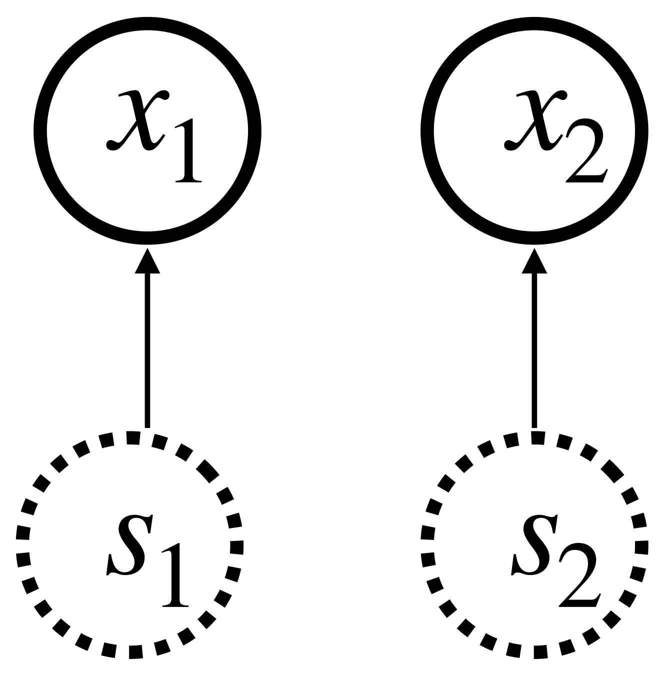
$$P(\cdot \mid s, a) = P\phi(s, a)$$

where $P \in \mathbb{R}^{|S| \times |SA|}$ is the transition matrix



Low-Rank Example

Can encode latent variables: block-MDPs



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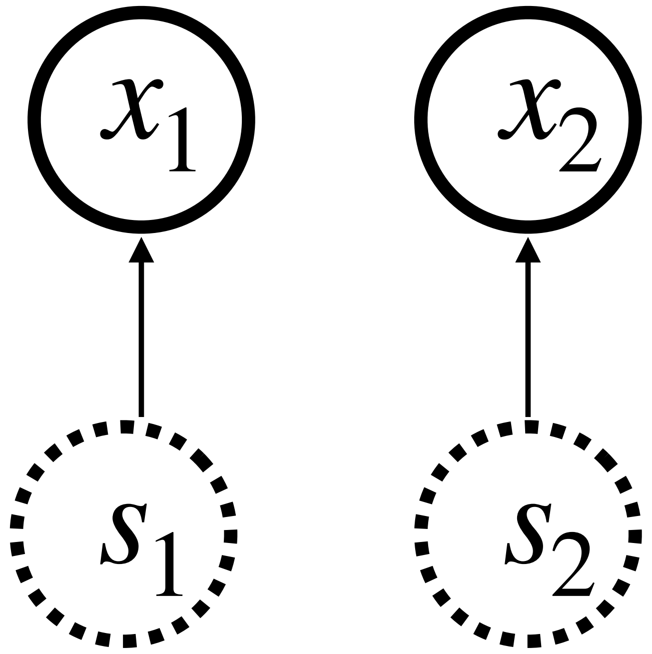

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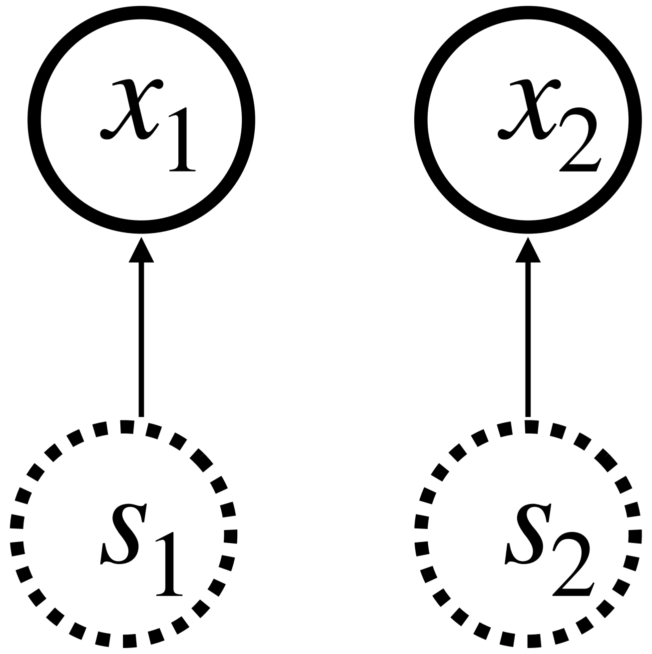

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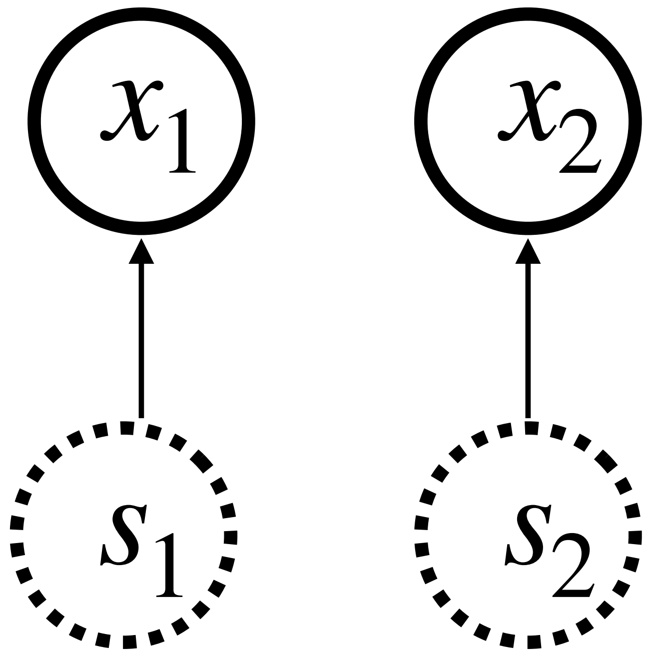

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Each state s has an emission distribution $\mu_s \in \Delta(X)$, also μ_s and $\mu_{s'}$ have **disjoint support** for any $s \neq s'$ (i.e, latent state is decodable)

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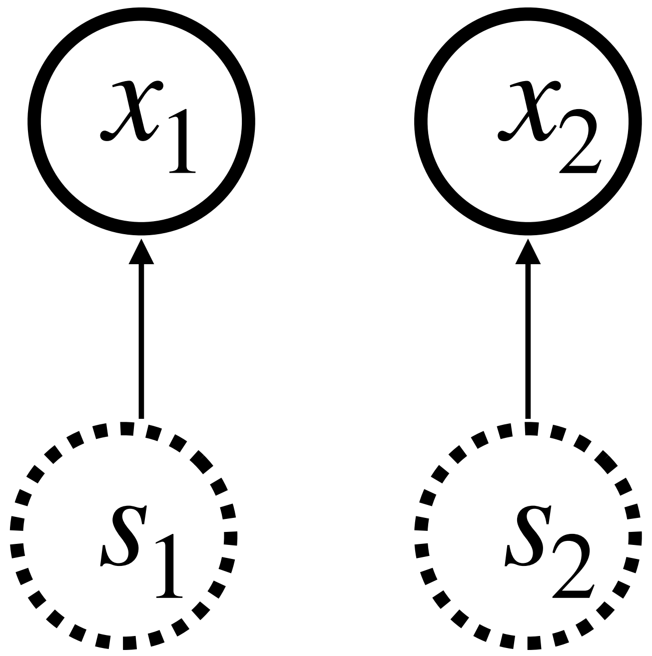

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$$P(x' | x, a) = \sum_{s' \in \{s_1, s_2, s_3\}} T(s' | \omega(x), a) \mu_{s'}(x') = [\mu_{s_1}(x'), \mu_{s_2}(x'), \mu_{s_3}(x')] \begin{bmatrix} T(s_1 | \omega(x), a) \\ T(s_2 | \omega(x), a) \\ T(s_3 | \omega(x), a) \end{bmatrix}$$

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"Topic modelling"

We have d topics, $v_1, \ldots, v_d$

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We study Linear MDPs here.
Learning in Low-rank MDP is much harder!

Planning in Linear MDP: Value Iteration

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Indeed we can show that $Q_h^\pi(\cdot, \cdot)$

Is linear with respect to ϕ as well, for any π, h

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3. Plan: $\pi^{n+1} = \text{Value-Iter}\left(\{\widehat{P}^n\}_h, \{r_h + b_h^n\}\right)$

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Norm bounds:

$$\sup_{s,a} \|\phi(s, a)\|_2 \leq 1, \ \|\theta_h^\star\|_2 \leq W, \quad \|v^\top \mu_h^\star\|_2 \leq \sqrt{d}, \forall v \text{ s.t. } \|v\|_\infty \leq 1$$

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Denote $\epsilon_{s,a} = \delta(s') - P(\cdot \mid s, a)$, we have $\mathbb{E}_{s'}[\epsilon_{s,a}] = 0$, and $\|\epsilon_{s,a}\|_1 \leq 2$

1. Model Learning in Linear MDPs

$$\mathcal{D}_h^n = \left\{ s_h^i, a_h^i, s_{h+1}^i \right\}_{i=1}^{n-1} \qquad \Lambda_h^n = \sum_{i=1}^{n-1} \phi(s_h^i, a_h^i)\phi(s_h^i, a_h^i)^\top + \lambda I$$

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Ridge Linear Regression:

$$\min_{\mu} \sum_{i=1}^{n-1} \|\mu\phi(s_h^i, a_h^i) - \delta(s_{h+1}^i)\|_2^2 + \lambda\|\mu\|_F^2$$

1. Model Learning in Linear MDPs

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1. Model Learning in Linear MDPs

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$$\widehat{P}_h^n(\cdot \mid s, a) = \widehat{\mu}_h^n \phi(s, a)$$

1. Model Learning in Linear MDPs

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$$\widehat{P}_h^n(\cdot \mid s, a) = \widehat{\mu}_h^n\phi(s, a)$$

Can we bound the ℓ_1 error on distributions, i.e., $\|\widehat{P}^n(\cdot \mid s, a) - P(\cdot \mid s, a)\|_1$?

1. Model Learning in Linear MDPs

Ridge Linear Regression: $$\min_{\mu} \sum_{i=1}^{n-1} \|\mu\phi(s_h^i, a_h^i) - \delta(s_{h+1}^i)\|_2^2 + \lambda\|\mu\|_F^2$$

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As in tabular-UCBVI and Generative Model, we care **average model error:**

1. Model Learning in Linear MDPs

Ridge Linear Regression: $\min_{\mu} \sum_{i=1}^{n-1} \|\mu\phi(s_h^i, a_h^i) - \delta(s_{h+1}^i)\|_2^2 + \lambda\|\mu\|_F^2$

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Can we bound the ℓ_1 error on distributions, i.e., $\|\widehat{P}^n(\cdot \mid s, a) - P(\cdot \mid s, a)\|_1$?

As in tabular-UCBVI and Generative Model, we care **average model error:**

Consider a fixed function $V : S \mapsto [0,H]$, we can bound:

$$\left| \left(\widehat{P}_h^n(\cdot \mid s, a) - P_h(\cdot \mid s, a) \right) \cdot V \right|$$

1. Model Learning in Linear MDPs

Ridge Linear Regression: $$\min_{\mu} \sum_{i=1}^{n-1} \|\mu\phi(s_h^i, a_h^i) - \delta(s_{h+1}^i)\|_2^2 + \|\mu\|_F^2$$

$$\widehat{\mu}^n = \sum_{i=1}^{n-1} \delta(s_{h+1}^i)\phi(s_h^i, a_h^i)^\top (\Lambda_h^n)^{-1}, \quad \Lambda_h^n = \sum_{i=1}^{n-1} \phi(s_h^i, a_h^i)\phi(s_h^i, a_h^i)^\top + \lambda I$$

$$\widehat{P}_h^n(\cdot \mid s, a) = \widehat{\mu}_h^n \phi(s, a)$$

Lemma [Model Average Error under a fixed V]:

Consider a fixed $V : S \to [0,H]$. With probability at least $1 - \delta$, for any s, a, h, n, we have:

$$\left| \left(\widehat{P}_h^n(\cdot \mid s, a) - P_h(\cdot \mid s, a) \right) \cdot V \right| \leq \|\phi(s, a)^\top\|_{(\Lambda_h^n)^{-1}} \times \left(2H\sqrt{\ln \frac{H \det(\Lambda_h^n)^{1/2} \det(\lambda I)^{-1/2}}{\delta}} + H\sqrt{\lambda d} \right)$$

1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

1. Model Learning in Linear MDPs

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1. Model Learning in Linear MDPs

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$$\widehat{\mu}_h^n = \sum_{i=1}^{n-1} \left(P(\cdot \mid s_h^i, a_h^i) + \epsilon_h^i\right) \phi(s_h^i, a_h^i)^\top (\Lambda_h^n)^{-1}$$

1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

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1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda\mu_h^\star\left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1}\epsilon_h^i\phi(s_h^i, a_h^i)^\top\left(\Lambda_h^n\right)^{-1}$$

$$\widehat{\mu}_h^n = \sum_{i=1}^{n-1}\delta(s_{h+1}^i)\phi(s_h^i, a_h^i)^\top(\Lambda_h^n)^{-1}, \quad \Lambda_h^n = \sum_{i=1}^{n-1}\phi(s_h^i, a_h^i)\phi(s_h^i, a_h^i)^\top + \lambda I$$

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$$= \mu_h^\star\left(\sum_{i=1}^{n-1}\phi(s_h^i, a_h^i)\phi(s_h^i, a_h^i)^\top\right)(\Lambda_h^n)^{-1} + \sum_{i=1}^{n-1}\epsilon_h^i\phi(s_h^i, a_h^i)^\top(\Lambda_h^n)^{-1}$$

1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

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$$= \mu_h^\star \left(\sum_{i=1}^{n-1} \phi(s_h^i, a_h^i) \phi(s_h^i, a_h^i)^\top \right) (\Lambda_h^n)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top (\Lambda_h^n)^{-1}$$

$$= \mu_h^\star - \lambda \mu_h^\star (\Lambda_h^n)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top (\Lambda_h^n)^{-1}$$

1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

$$\left((\widehat{\mu}_h^n - \mu_h^\star)\phi(s,a)\right)^\top V$$

1. Model Learning in Linear MDPs

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1. Model Learning in Linear MDPs

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$$\left| \lambda \phi(s,a)^\top \left(\Lambda_h^n\right)^{-1} \left(\mu_h^\star\right)^\top V \right| \leq \lambda \| \phi(s,a)^\top (\Lambda_h^n)^{-1/2} \|_2 \| (\Lambda_h^n)^{-1/2} (\mu_h^\star)^\top V \|_2$$

1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

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$$\left| \lambda \phi(s,a)^\top \left(\Lambda_h^n\right)^{-1} \left(\mu_h^\star\right)^\top V \right| \leq \lambda \|\phi(s,a)^\top (\Lambda_h^n)^{-1/2}\|_2 \|(\Lambda_h^n)^{-1/2} (\mu_h^\star)^\top V\|_2$$

$$= \lambda \|\phi(s,a)^\top\|_{(\Lambda_h^n)^{-1}} \|(\Lambda_h^n)^{-1/2}(\mu_h^\star)^\top V\|_2$$

1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

$$\left((\widehat{\mu}_h^n - \mu_h^\star)\phi(s,a)\right)^\top V$$

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$$\leq \lambda \|\phi(s,a)^\top\|_{(\Lambda_h^n)^{-1}} \|(\Lambda_h^n)^{-1/2}\|_2 \|(\mu_h^\star)^\top V\|_2$$

1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

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$$\left| \lambda \phi(s,a)^\top \left(\Lambda_h^n\right)^{-1} \left(\mu_h^\star\right)^\top V \right| \leq \lambda \|\phi(s,a)^\top (\Lambda_h^n)^{-1/2}\|_2 \|(\Lambda_h^n)^{-1/2} (\mu_h^\star)^\top V\|_2$$

$$= \lambda \|\phi(s,a)^\top\|_{(\Lambda_h^n)^{-1}} \|(\Lambda_h^n)^{-1/2} (\mu_h^\star)^\top V\|_2$$

$$\leq \lambda \|\phi(s,a)^\top\|_{(\Lambda_h^n)^{-1}} \|(\Lambda_h^n)^{-1/2}\|_2 \|(\mu_h^\star)^\top V\|_2 \ \leq \lambda \|\phi(s,a)^\top\|_{(\Lambda_h^n)^{-1}} \frac{H\sqrt{d}}{\sqrt{\lambda}}$$

1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

$$\left| \left((\widehat{\mu}_h^n - \mu_h^\star) \phi(s, a) \right)^\top V \right| \leq H \sqrt{\lambda d} \| \phi(s, a)^\top \|_{(\Lambda_h^n)^{-1}} + \left| \sum_{i=1}^{n-1} \phi(s, a)^\top (\Lambda_h^n)^{-1} \phi(s_h^i, a_h^i) (\epsilon_h^i)^\top V \right|$$

1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

$$\left| \left((\widehat{\mu}_h^n - \mu_h^\star)\phi(s,a) \right)^\top V \right| \leq H\sqrt{\lambda d} \|\phi(s,a)^\top\|_{(\Lambda_h^n)^{-1}} + \left| \sum_{i=1}^{n-1} \phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right|$$

$$\left| \sum_{i=1}^{n-1} \phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right| \leq \|\phi(s,a)^\top (\Lambda_h^n)^{-1/2}\|_2 \left\| \sum_{i=1}^{n-1} (\Lambda_h^n)^{-1/2} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right\|_2$$

1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

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$$\left| \sum_{i=1}^{n-1} \phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right| \leq \|\phi(s,a)^\top (\Lambda_h^n)^{-1/2}\|_2 \left\| \sum_{i=1}^{n-1} (\Lambda_h^n)^{-1/2} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right\|_2$$

$$= \|\phi(s,a)^\top\|_{(\Lambda_h^n)^{-1}} \left\| \sum_{i=1}^{n-1} \phi(s_h^i, a_h^i)\left((\epsilon_h^i)^\top V\right) \right\|_{(\Lambda_h^n)^{-1}}$$

1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

$$\left| \left((\widehat{\mu}_h^n - \mu_h^\star)\phi(s,a) \right)^\top V \right| \leq H\sqrt{\lambda d} \|\phi(s,a)^\top\|_{(\Lambda_h^n)^{-1}} + \left| \sum_{i=1}^{n-1} \phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right|$$

$$\left| \sum_{i=1}^{n-1} \phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right| \leq \|\phi(s,a)^\top (\Lambda_h^n)^{-1/2}\|_2 \left\| \sum_{i=1}^{n-1} (\Lambda_h^n)^{-1/2} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right\|_2$$

$$= \|\phi(s,a)^\top\|_{(\Lambda_h^n)^{-1}} \left\| \sum_{i=1}^{n-1} \phi(s_h^i, a_h^i)\left((\epsilon_h^i)^\top V\right) \right\|_{(\Lambda_h^n)^{-1}}$$

$$\mathbb{E}\left[(\epsilon_h^i)^\top V \,|\, \mathscr{H}_{i,h}\right] = 0, \quad |(\epsilon_h^i)^\top V| \leq \|\epsilon_h^i\|_1 \|V\|_\infty \leq 2H$$

1. Model Learning in Linear MDPs

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

$$\left| \left((\widehat{\mu}_h^n - \mu_h^\star)\phi(s,a)\right)^\top V \right| \leq H\sqrt{\lambda d} \|\phi(s,a)^\top\|_{(\Lambda_h^n)^{-1}} + \left| \sum_{i=1}^{n-1} \phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right|$$

$$\left| \sum_{i=1}^{n-1} \phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right| \leq \|\phi(s,a)^\top (\Lambda_h^n)^{-1/2}\|_2 \left\| \sum_{i=1}^{n-1} (\Lambda_h^n)^{-1/2} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right\|_2$$

$$= \|\phi(s,a)^\top\|_{(\Lambda_h^n)^{-1}} \left\| \sum_{i=1}^{n-1} \phi(s_h^i, a_h^i)\left((\epsilon_h^i)^\top V\right) \right\|_{(\Lambda_h^n)^{-1}} \qquad \leq \|\phi(s,a)^\top\|_{(\Lambda_h^n)^{-1}} \times 2H \sqrt{\ln \frac{\det(\Lambda_h^n)^{1/2} \det(\lambda I)^{-1/2}}{\delta}}$$

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1. Model Learning in Linear MDPs

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Lemma [Model Average Error under a fixed V]:

Consider a fixed $V : S \to [0,H]$. With probability at least $1-\delta,$ for all s, a, n, h, we have:

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$$b_h^n(s,a) = \beta \sqrt{\phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s,a)}, \ \beta = \widetilde{O}(dH)$$

Detour: Covering Number and Covering Dimension

Consider the ball $\Theta = \{\theta : \theta \in \mathbb{R}^d, \|\theta\|_2 \leq R\}$. Denote ϵ-Net as a subset $\mathcal{N}_\epsilon \subseteq \Theta$, such that $\forall \theta \in \Theta$: $\exists \theta' \in \mathcal{N}_\epsilon$, s.t. $\|\theta' - \theta\|_2 \leq \epsilon$.

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Then $(\epsilon/L)-$Net on Θ gives us an $\epsilon-$Net on $\mathcal{F}$ with $d(f_{\theta_1}, f_{\theta_2}) := \|f_{\theta_1} - f_{\theta_2}\|_\infty$

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$$\sqrt{\lambda}\epsilon/3\text{−net at } \{\beta : \beta \in [0,B]\}, \quad |\mathcal{N}_{\sqrt{\lambda}\epsilon/3,\beta}| \leq 1 + 6B/(\sqrt{\lambda}\epsilon)$$

$$\epsilon^2/(9B^2)\text{−net at } \{A \in \mathbb{R}^{d\times d} : \|A\|_F \leq \sqrt{d}/\lambda\}, \quad |\mathcal{N}_{\epsilon^2/(9B^2),\Lambda}| \leq (1 + 6B\sqrt{d}/(\lambda\epsilon))^{2d}$$

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$$\leq d\ln(1 + 6L/\epsilon) + \ln(1 + 6B/(\sqrt{\lambda}\epsilon)) + d^2\ln(1 + 18B^2\sqrt{d}/(\lambda\epsilon^2))$$

$$\left| f_\theta(s) - f_{\hat\theta}(s) \right| \le \|w - \hat w\|_2 + |\beta - \hat\beta|/\sqrt{\lambda} + B\sqrt{\|\Lambda^{-1} - \hat\Lambda^{-1}\|_F}$$

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$$\begin{aligned}
\ln|\mathcal{N}_\epsilon| &\le \ln|\mathcal{N}_{\epsilon/3,w}| + \ln|\mathcal{N}_{\sqrt{\lambda}\epsilon/3,\beta}| + \ln|\mathcal{N}_{\epsilon^2/(9B^2),\Lambda}| \\
&\le d\ln(1 + 6L/\epsilon) + \ln(1 + 6B/(\sqrt{\lambda}\epsilon)) + d^2\ln(1 + 18B^2\sqrt{d}/(\lambda\epsilon^2)) \\
&\le d\ln(1 + 6L/\epsilon) + 2d^2\ln(1 + 18B^2\sqrt{d}/(\lambda\epsilon^2)) \\
&\le \widetilde{O}(d^2)
\end{aligned}$$

Summary for Today

Lemma [Model Average Error under a fixed V]:

Consider a fixed $V : S \to [0,H]$. With probability at least $1-\delta$, for all s, a, n, h, we have:

$$\left| \left(\widehat{P}_h^n(\cdot \mid s,a) - P_h(\cdot \mid s,a) \right) \cdot V \right| \leq \|\phi(s,a)^\top\|_{(\Lambda_h^n)^{-1}} \times \left(2H\sqrt{\ln \frac{H \det(\Lambda_h^n)^{1/2} \det(\lambda I)^{-1/2}}{\delta}} + H\sqrt{\lambda d} \right)$$

Summary for Today

Define the function $f_{w,\beta,\Lambda} : S \to [0,H]$, $f_{w,\beta,\Lambda}(s) := \min \left\{ \max_a \left(w^\top \phi(s,a) + \beta\sqrt{\phi(s,a)^\top \Lambda^{-1} \phi(s,a)} \right), H \right\}$

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Denote $\mathscr{F} = \{ f_{w,\beta,\Lambda} : \|w\|_2 \leq L, \beta \in [0,H], \sigma_{\min}(\Lambda) \geq \lambda \}$, under ℓ_∞ we have

$$\ln |\mathcal{N}_\epsilon| \leq d \ln(1 + 6L/\epsilon) + 2d^2 \ln(1 + 18B^2\sqrt{d}/(\lambda\epsilon^2))$$

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Q: can we build uniform concentration for **all** $V \in \mathcal{F}$?

i.e., $\left| (\widehat{P}_h^n(\cdot \mid s,a) - P_h(\cdot \mid s,a)) \cdot V \right|, \forall V \in \mathcal{F}$