Statistical Limits of Generalization Part II: Linear Realizability

Sham Kakade and Wen Sun

CS 6789: Foundations of Reinforcement Learning

Part-2: Linear Realizability

What if we impose linearity assumptions?

Let's look at the most natural assumptions.

RL with Linearly Realizable Q^*-Function Approximation

(Does there exist a sample efficient algo?)

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- (A2: Large Suboptimality Gap): for all $a \neq \pi^\star(s)$,

$$V_h^\star(s) - Q_h^\star(s, a) \geq \text{constant}$$

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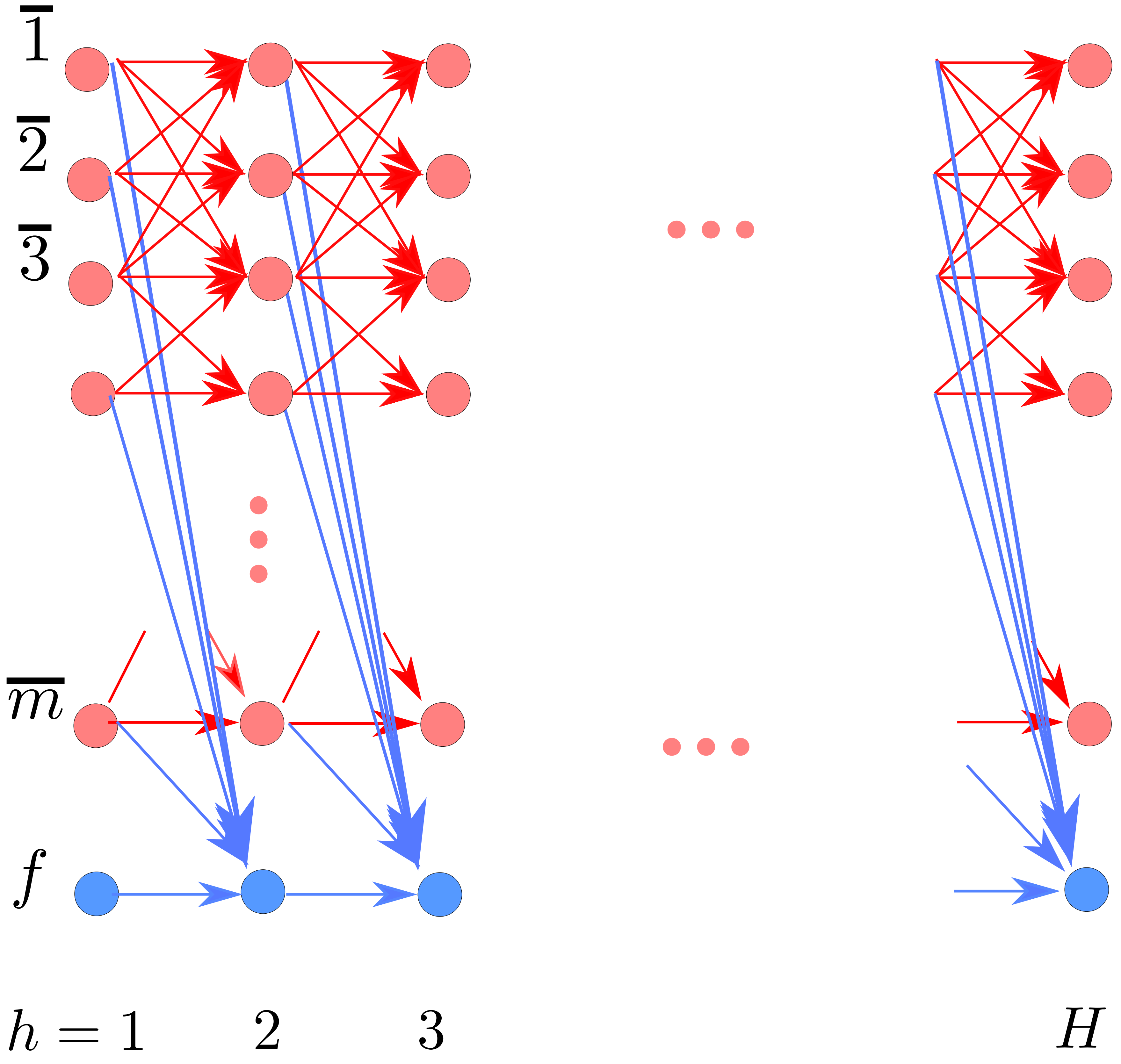
Comments: An exponential separation between online RL vs simulation access.

[Du, K., Wang, Yang '20]: A1+A2+simulator access (input: any s, a; output: $s' \sim P(\cdot \mid s, a), r(s, a)$)

$\Longrightarrow$ there is sample efficient approach to find an ϵ-opt policy.

Construction Sketch: a Hard MDP Family

(A ``leaking complete graph'')



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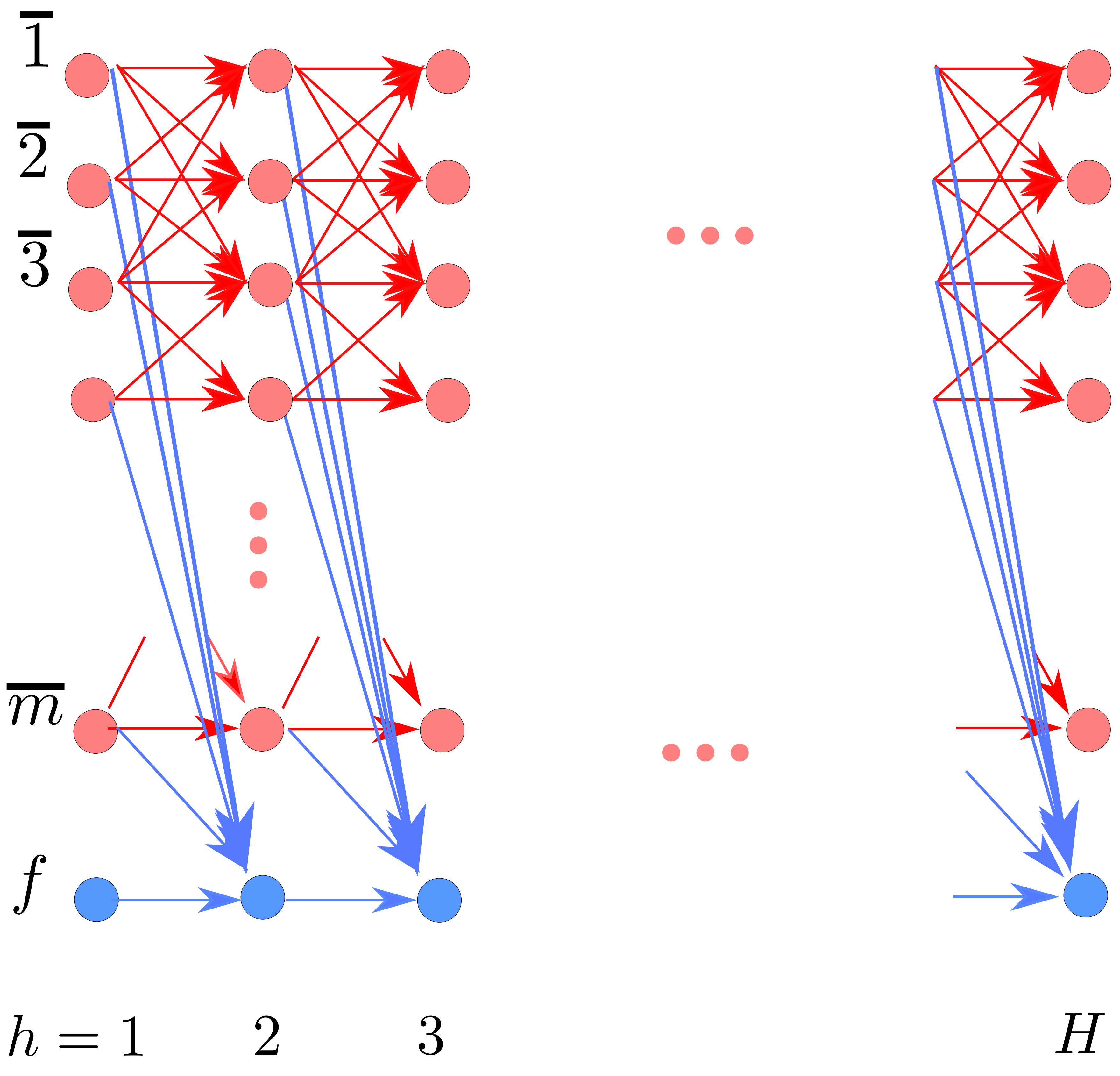

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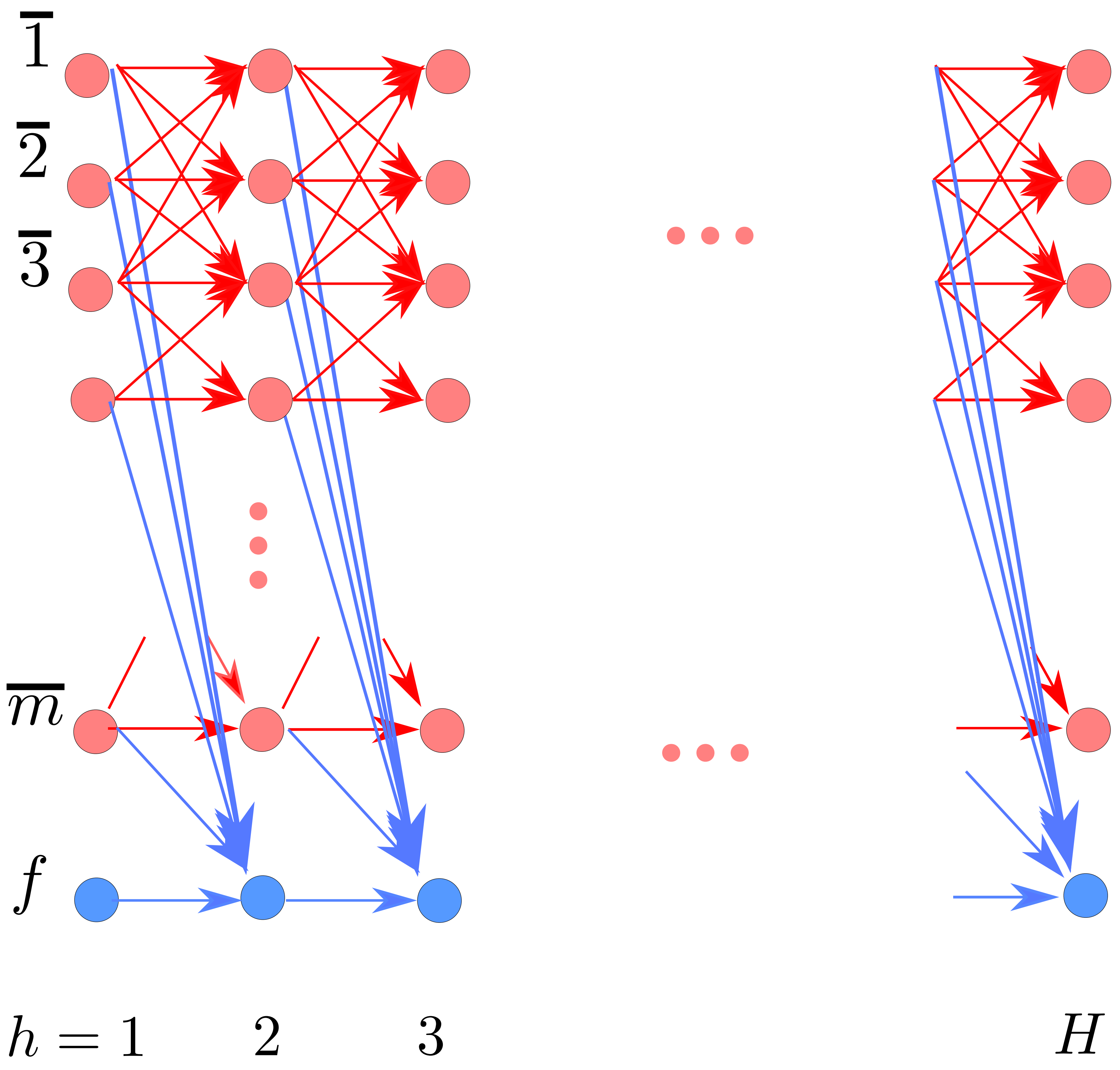

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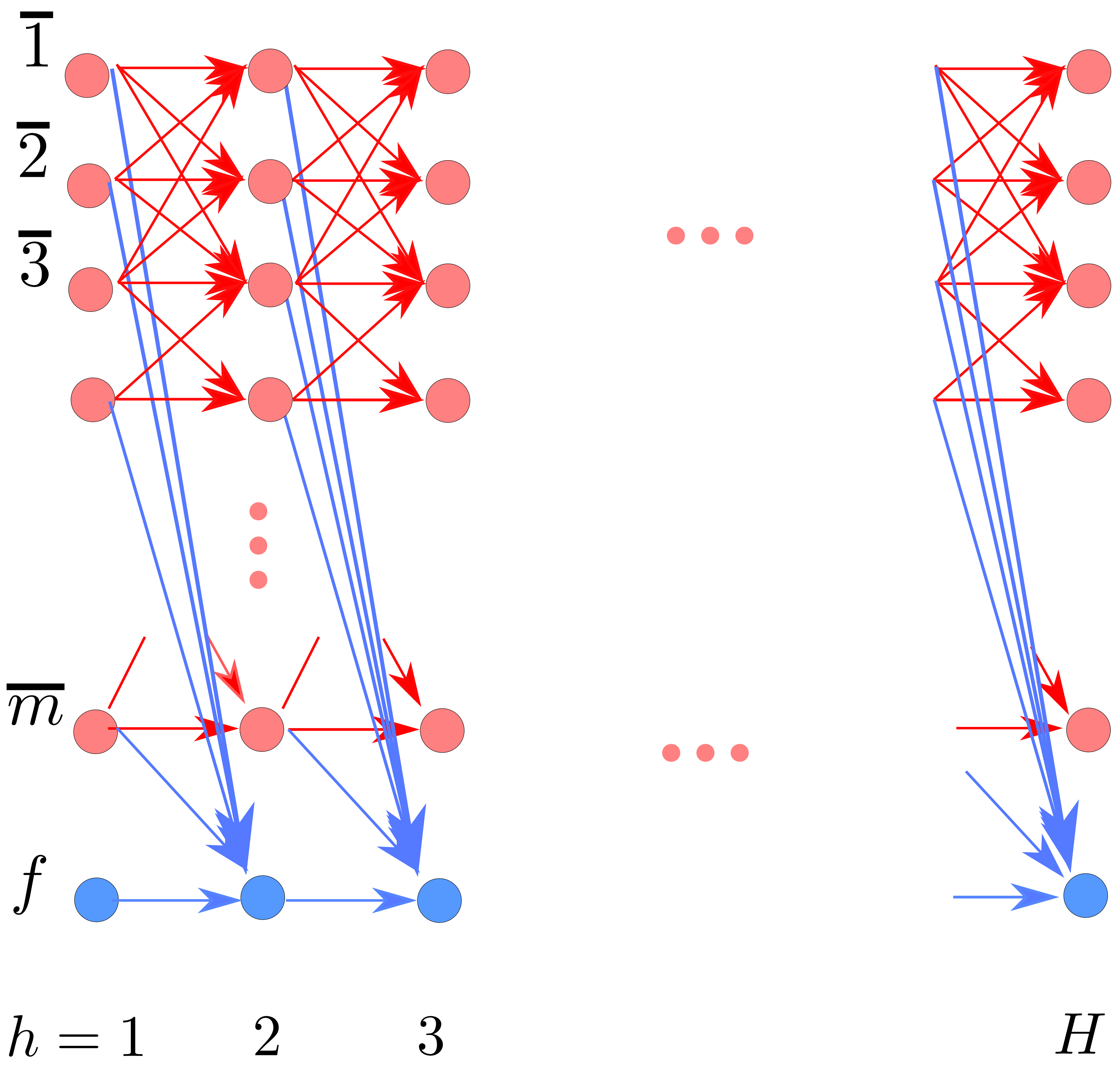

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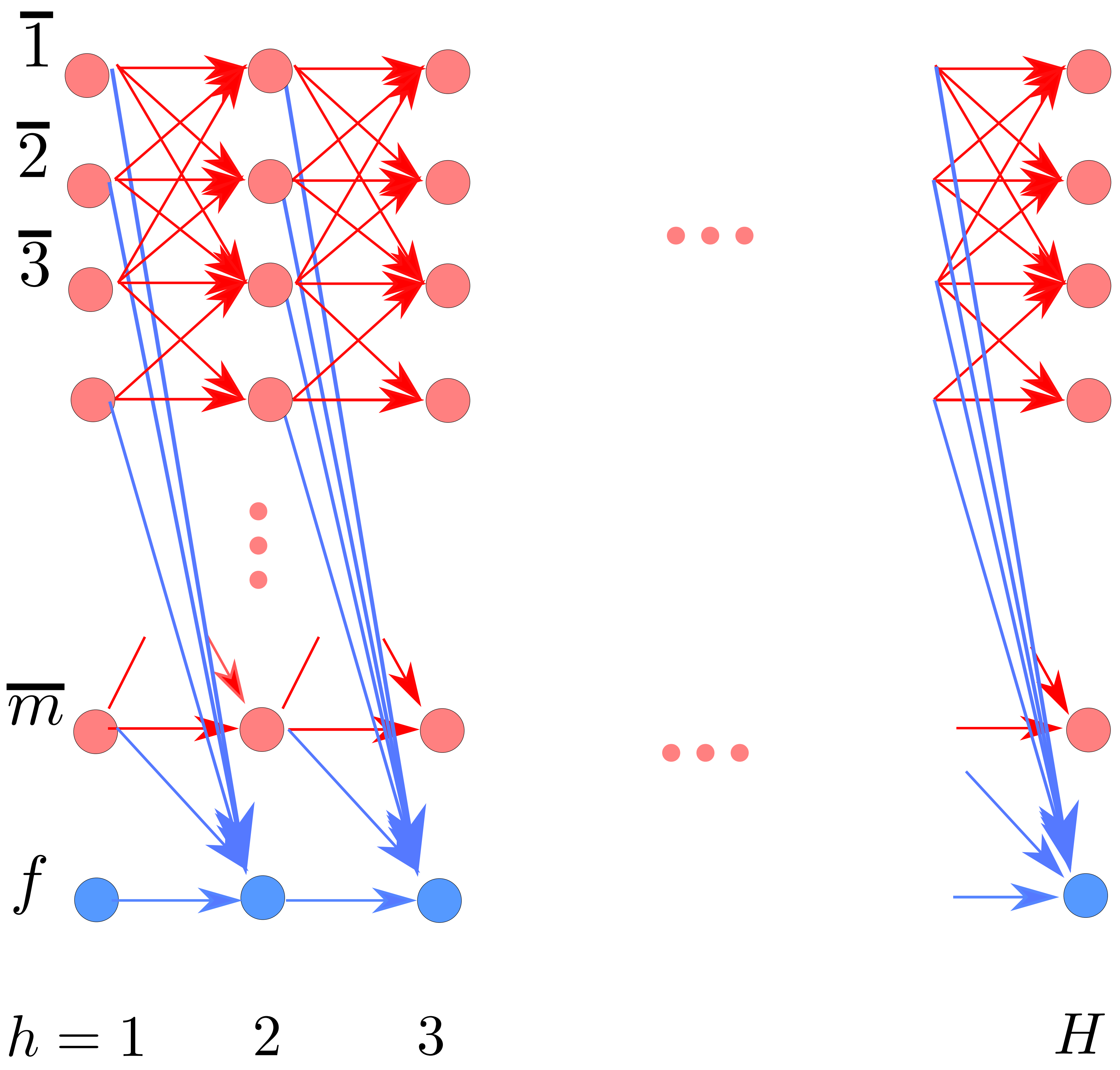

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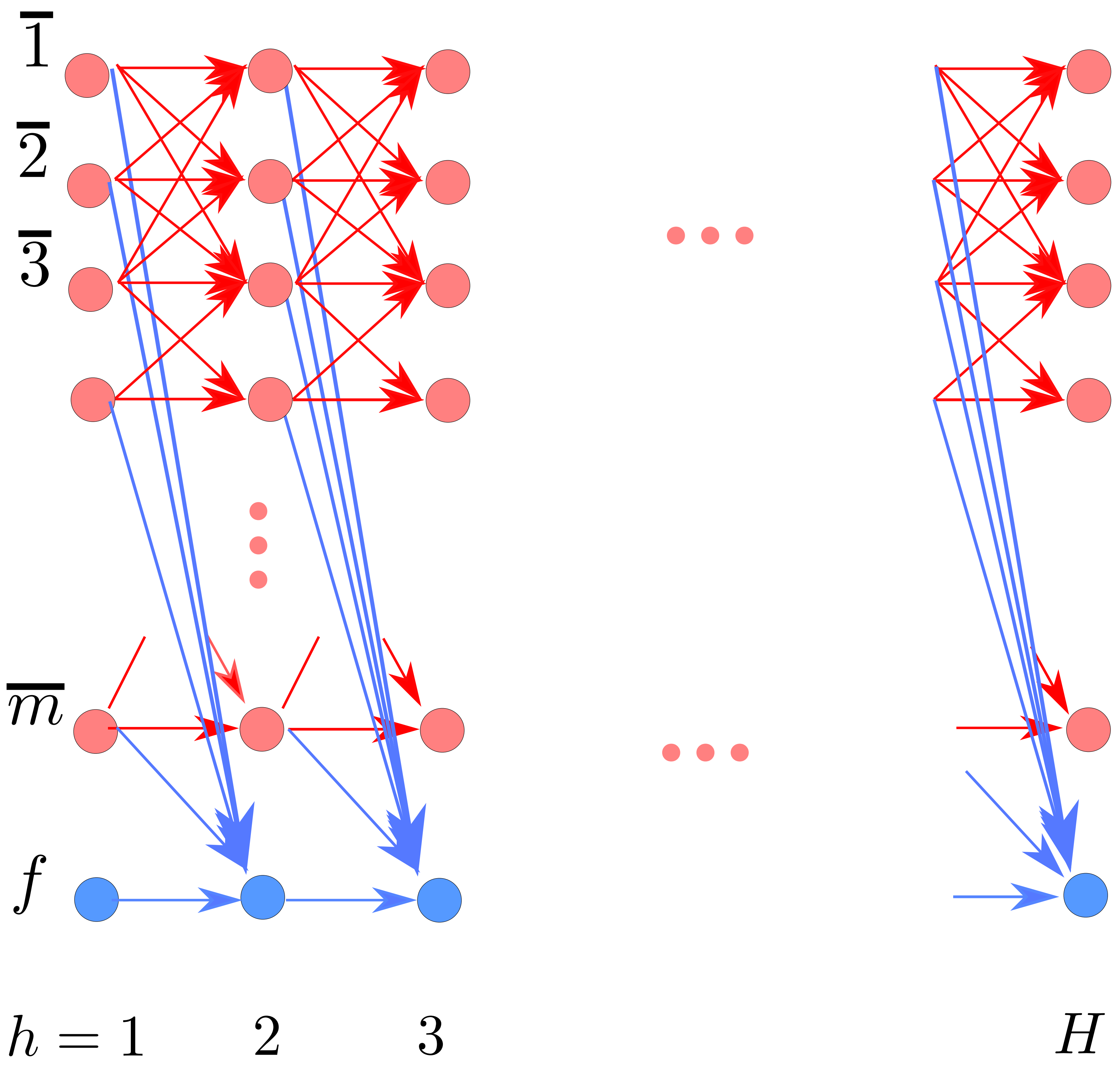

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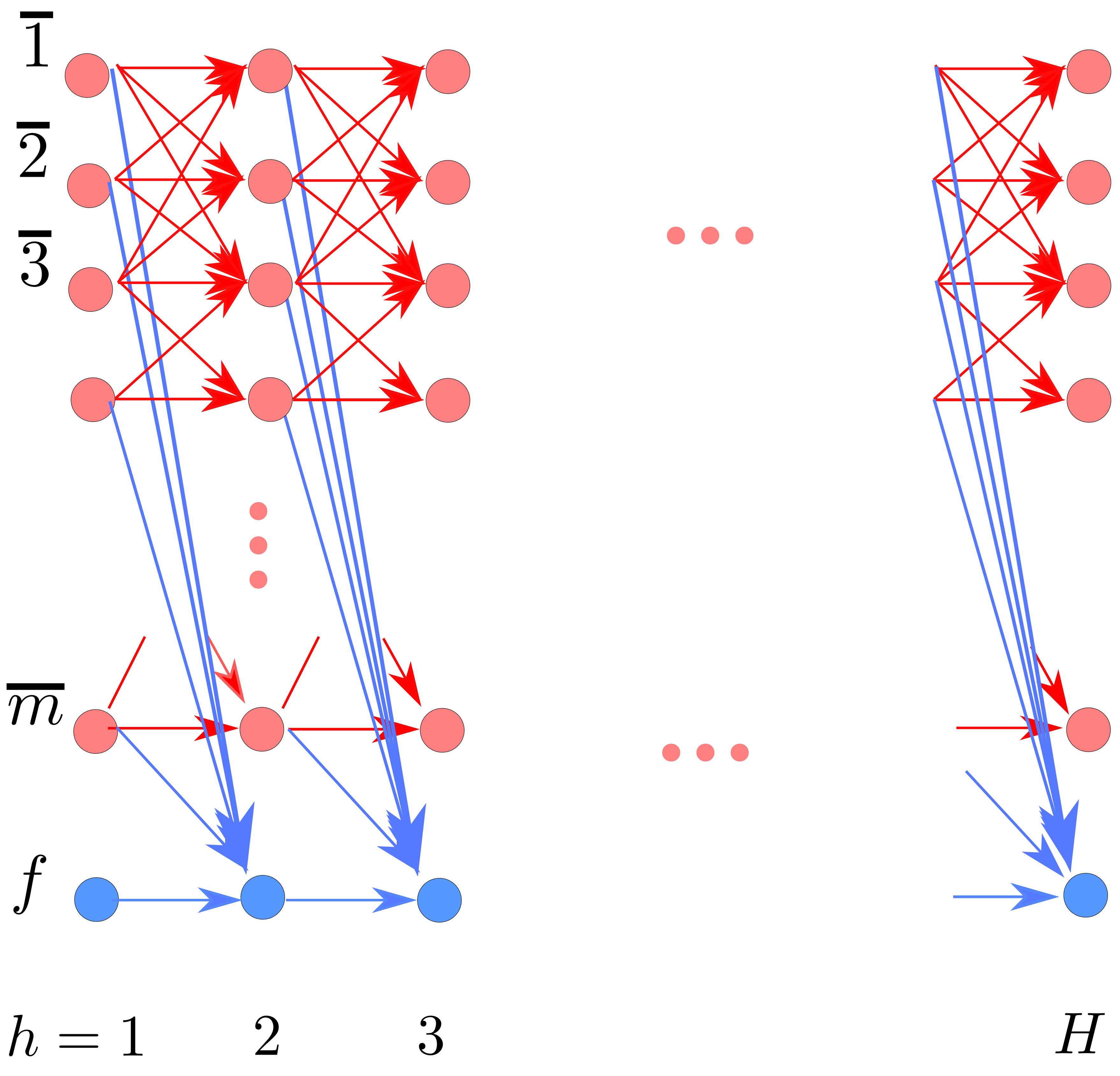

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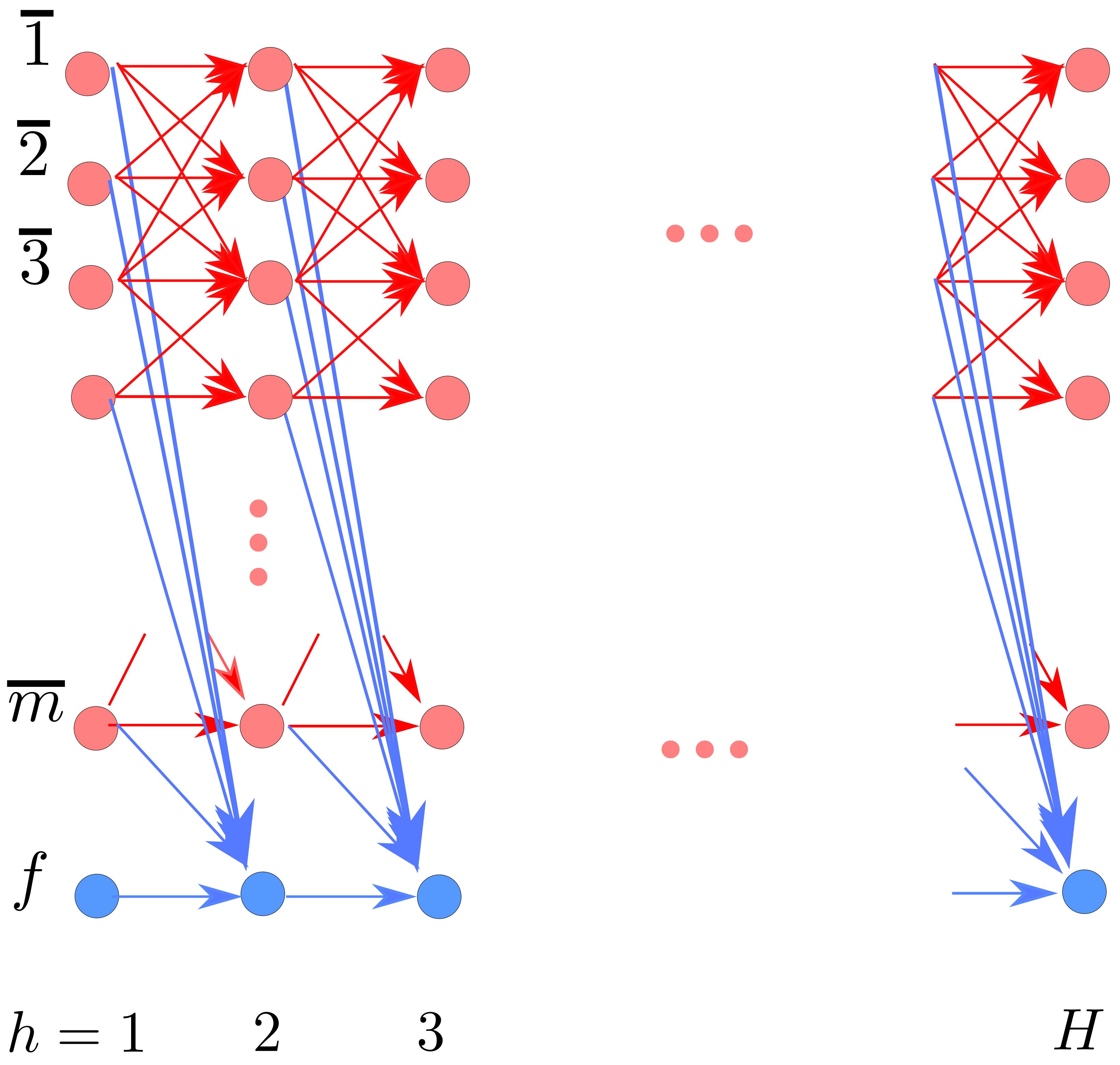
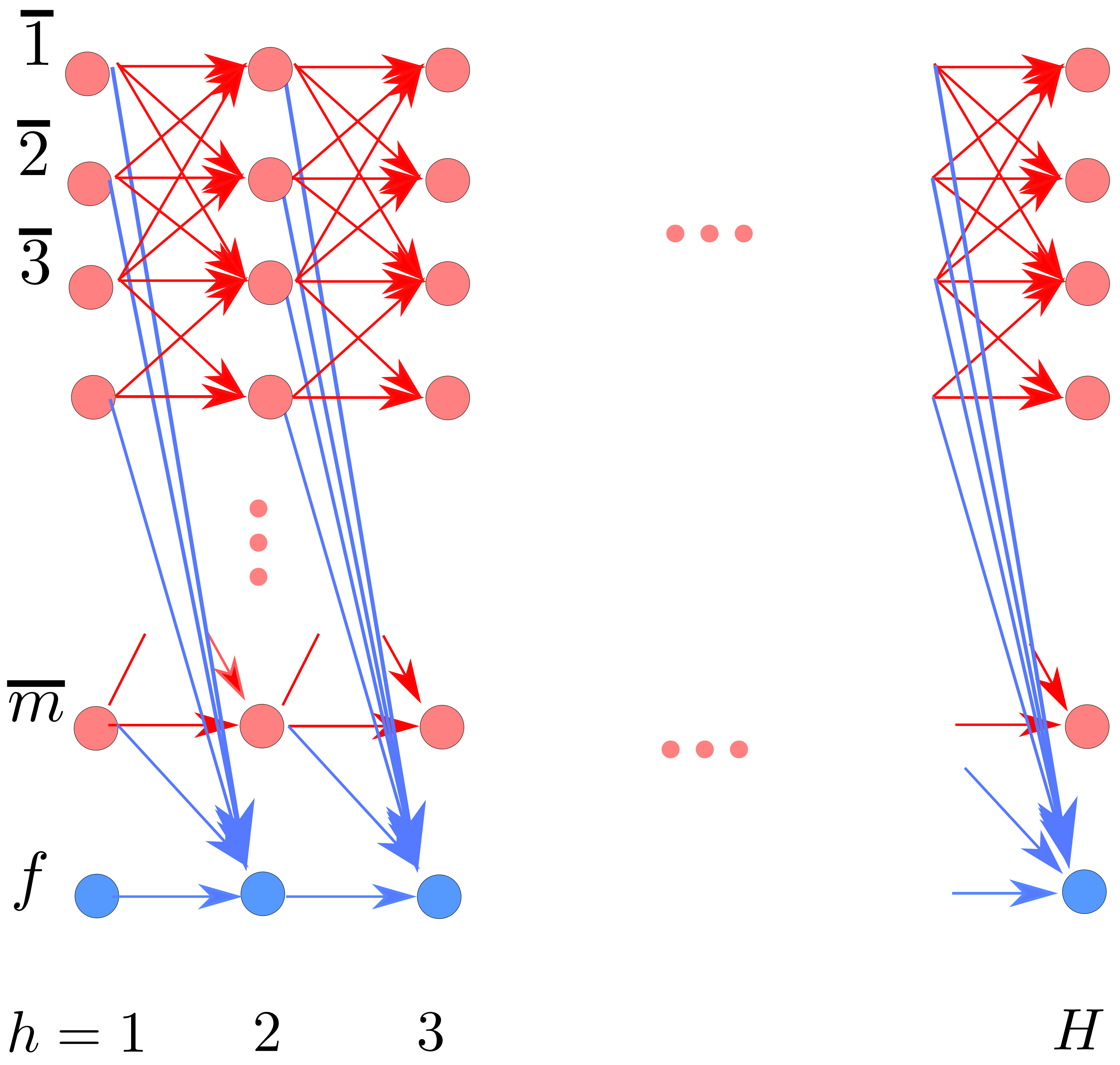
Lemma: For any $\gamma > 0$, there exist $m = \lfloor \exp(\frac{1}{8}\gamma^2 d) \rfloor$ unit vectors $\{v_1, \cdots, v_m\}$ in R^d s.t. $\forall i, j \in [m]$ and $i \neq j$, $|\langle v_i, v_j \rangle| \leq \gamma$.

We will set $\gamma = 1/4$.

(proof: Johnson-Lindenstrauss)

The construction, continued





The construction, continued

- Transitions: $s_0 \sim$ Uniform($[m]$).
  $\Pr[f \mid \overline{a_1}, a^*] = 1,$

$$\Pr[\,\cdot \mid \overline{a_1}, a_2] = \begin{cases} \overline{a_2} : \left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \\ f : 1 - \left\langle v(a_1), v(a_2) \right\rangle - 2\gamma \end{cases}, (a_2 \neq a^*, a_2 \neq a_1)$$

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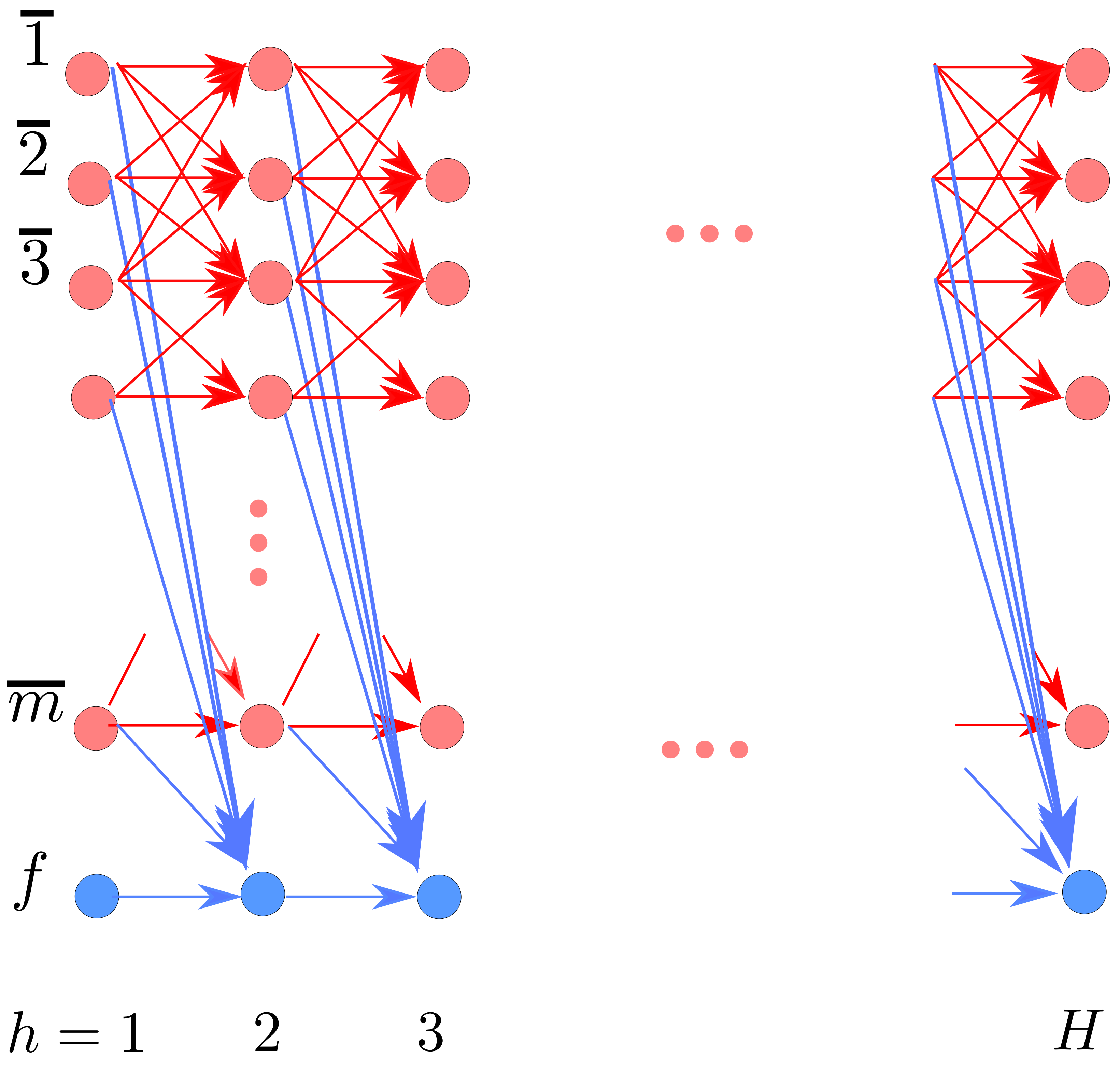

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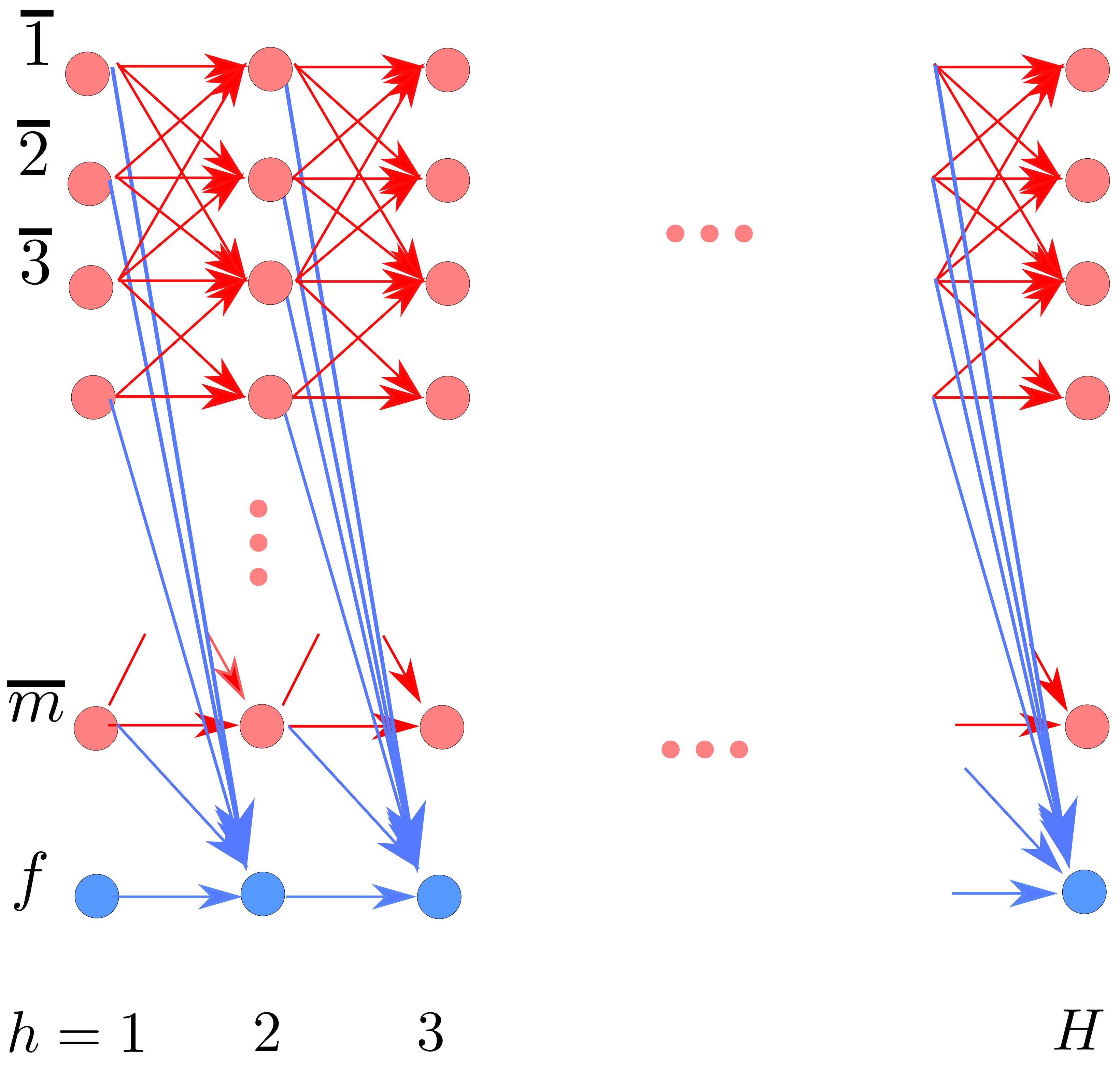

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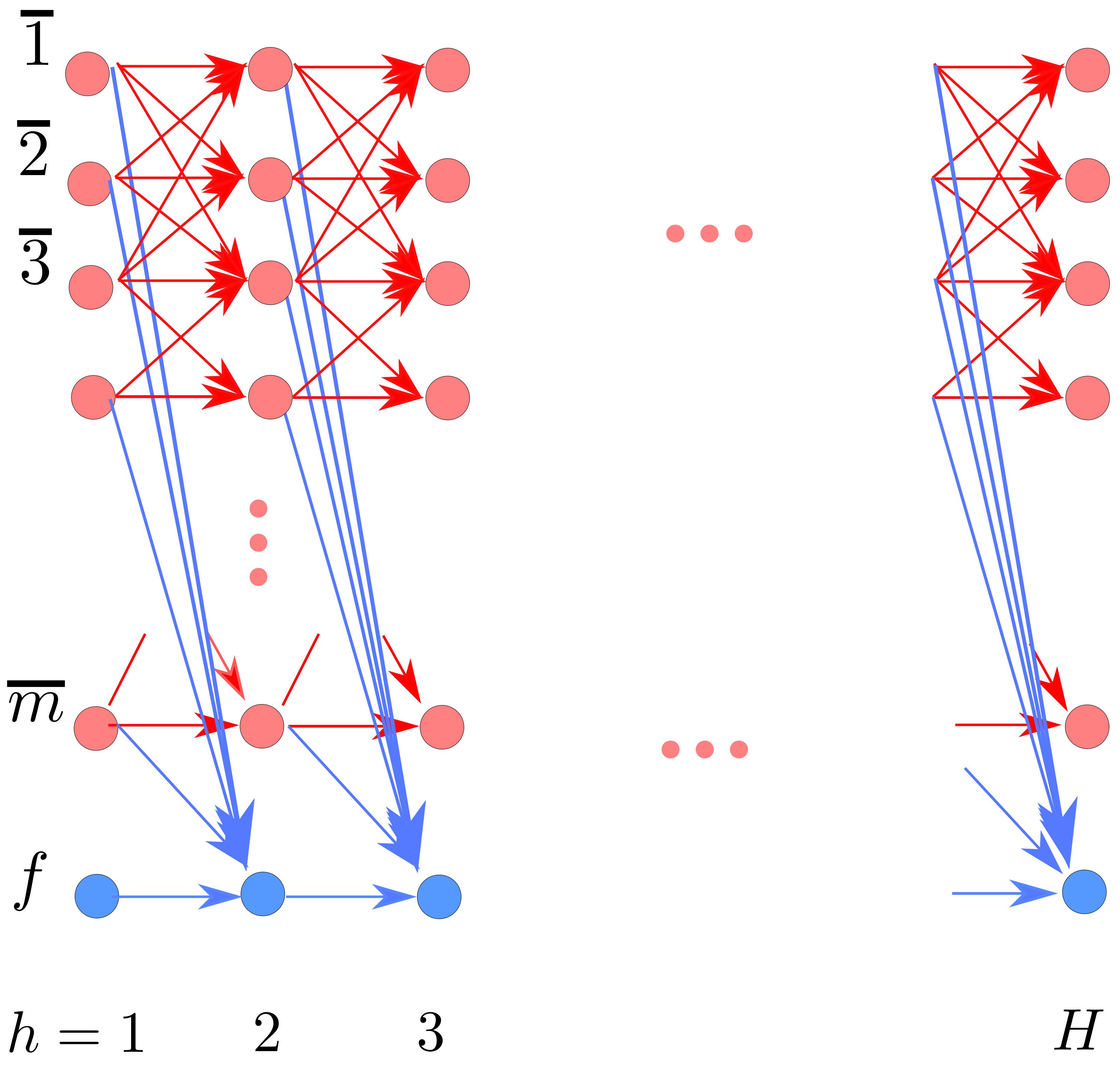

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- The transition probabilities are indeed valid, because $0 < \gamma \leq \left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \leq 3\gamma < 1$.

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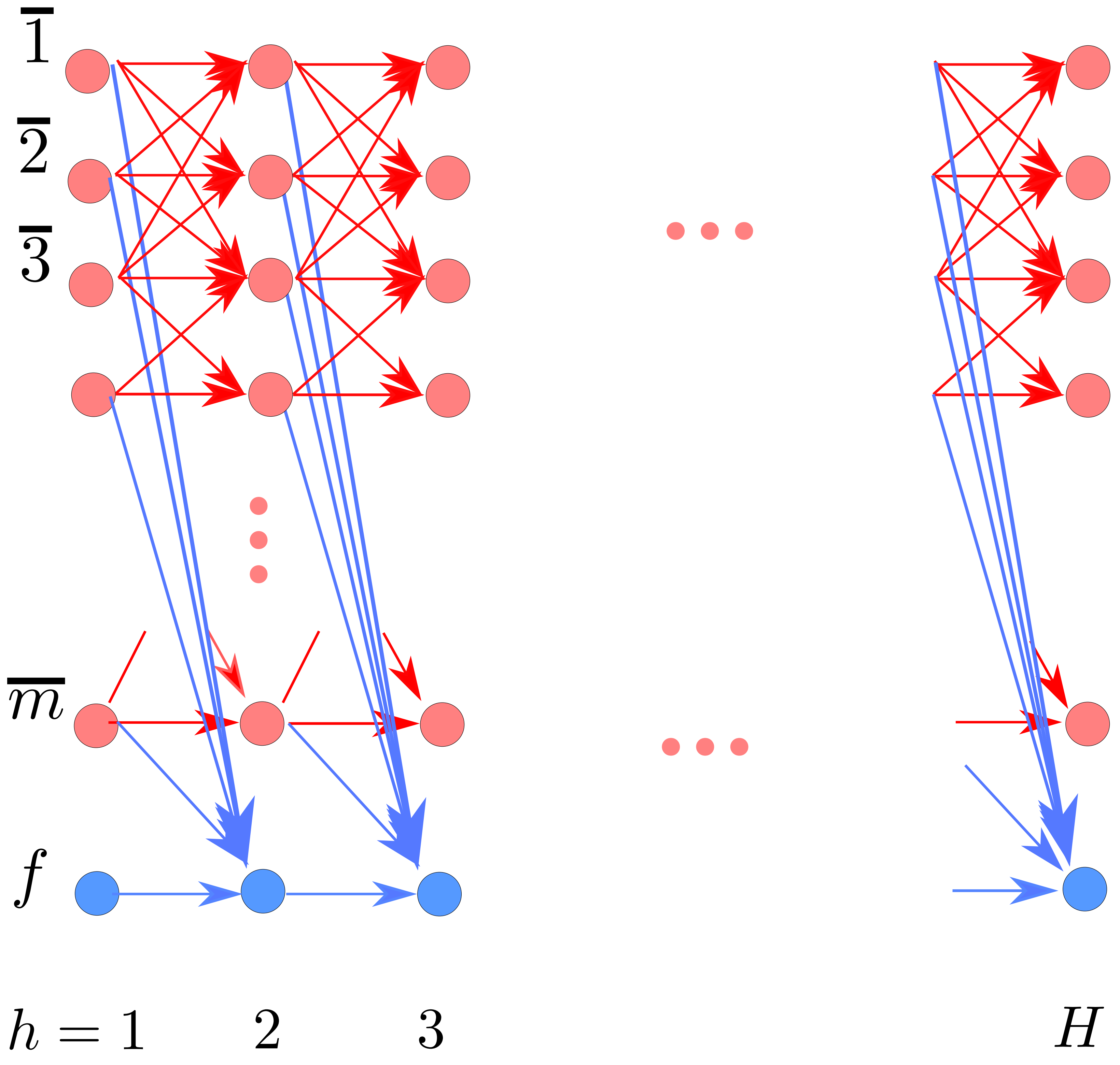

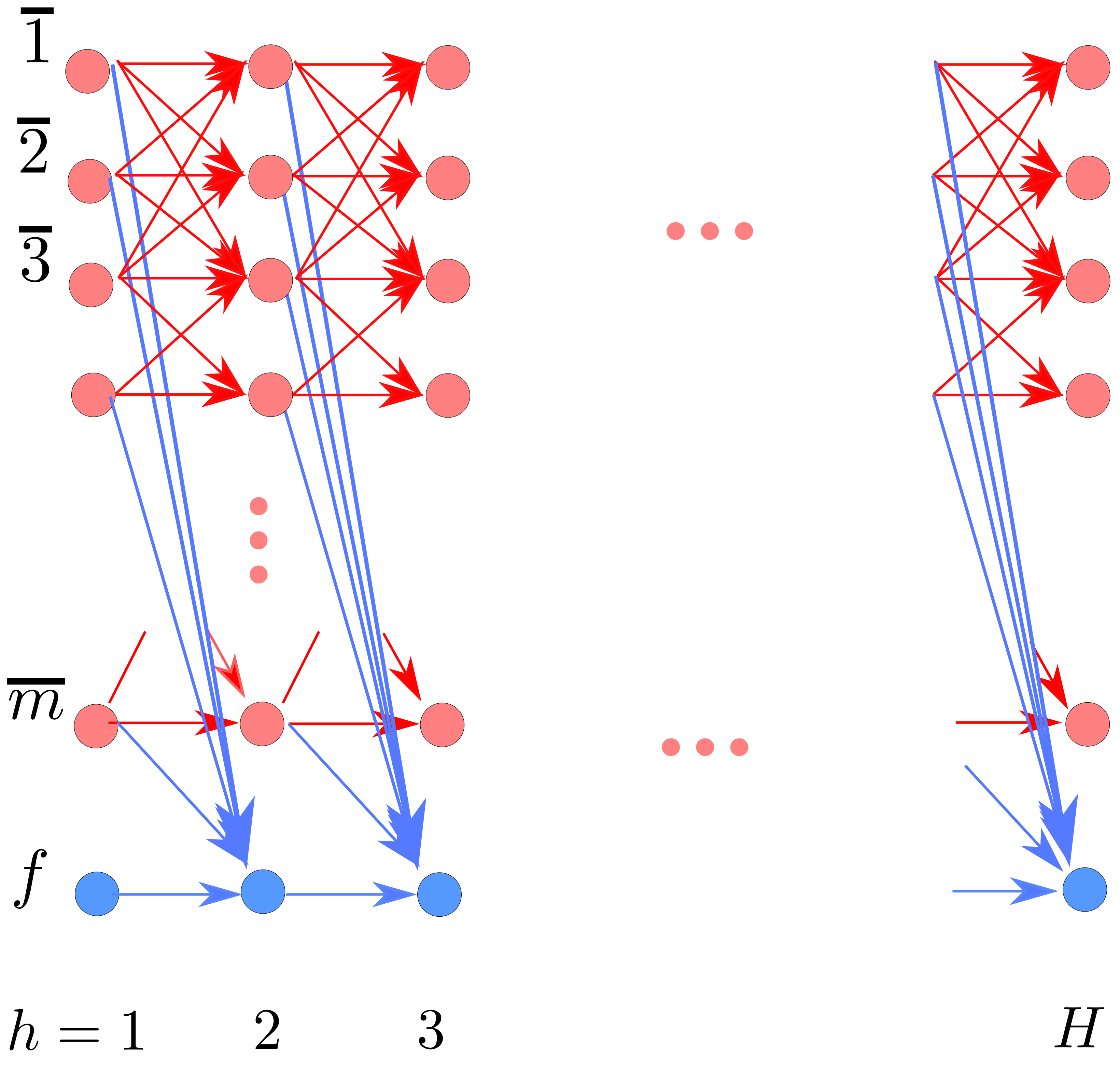

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- Features: of dimension d defined as:

$$\phi(\overline{a_1}, a_2) := \left(\left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \right) \cdot v(a_2), \quad \forall a_1 \neq a_2$$

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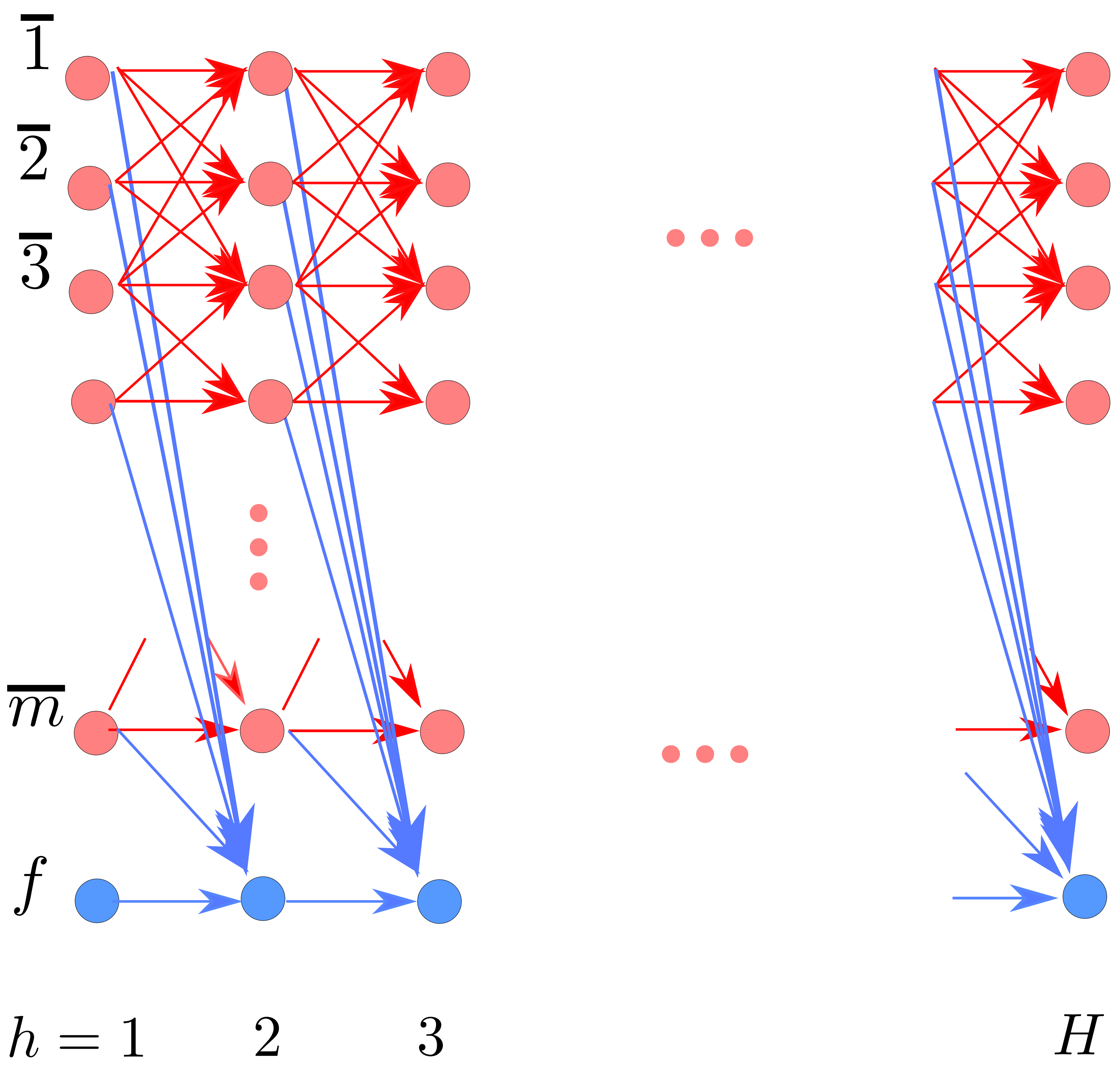

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- Rewards:

for $1 \leq h < H$,

$$R_h(\overline{a_1}, a^*) := \left\langle v(a_1), v(a^*) \right\rangle + 2\gamma,$$

$$R_h(\overline{a_1}, a_2) := -2\gamma \left[\left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \right], \; a_2 \neq a^*, a_2 \neq a_1$$

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for $h = H$,

$$r_H(s, a) := \langle \phi(s, a), v(a^*) \rangle$$

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Lemma: For all (s, a), we have $Q_h^*(s, a) = \langle \phi(s, a), v(a^*) \rangle$ and the "gap" is $\geq \gamma/4$.

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- Proving the large gap: for $a_2 \neq a^*$

$$V_h^*(\overline{a_1}) - Q_h^*(\overline{a_1}, a_2) = Q_h^\pi(\overline{a_1}, a^*) - Q_h^\pi(\overline{a_1}, a_2) > \gamma - 3\gamma^2 \geq \frac{1}{4}\gamma\,.$$

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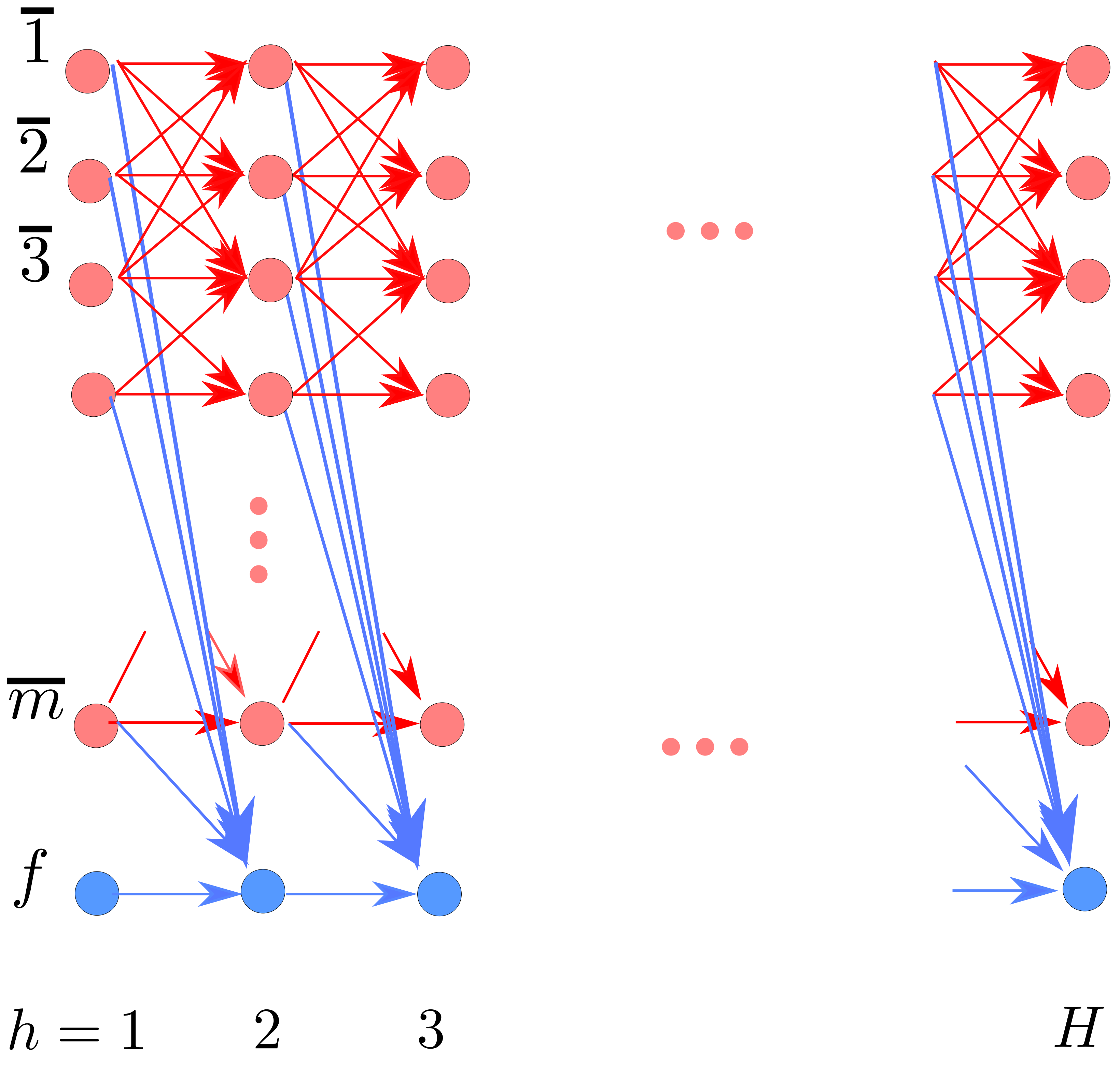

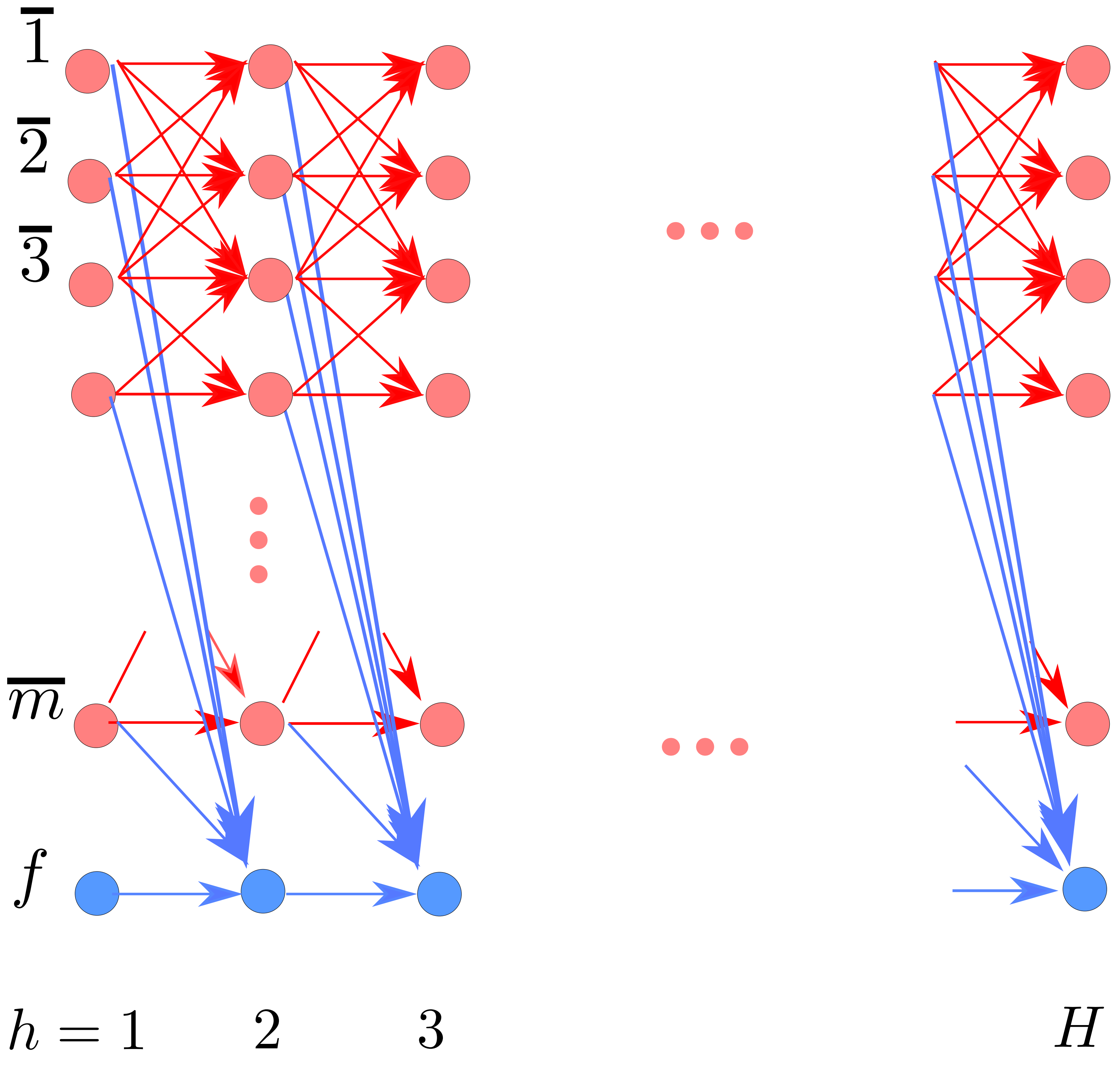

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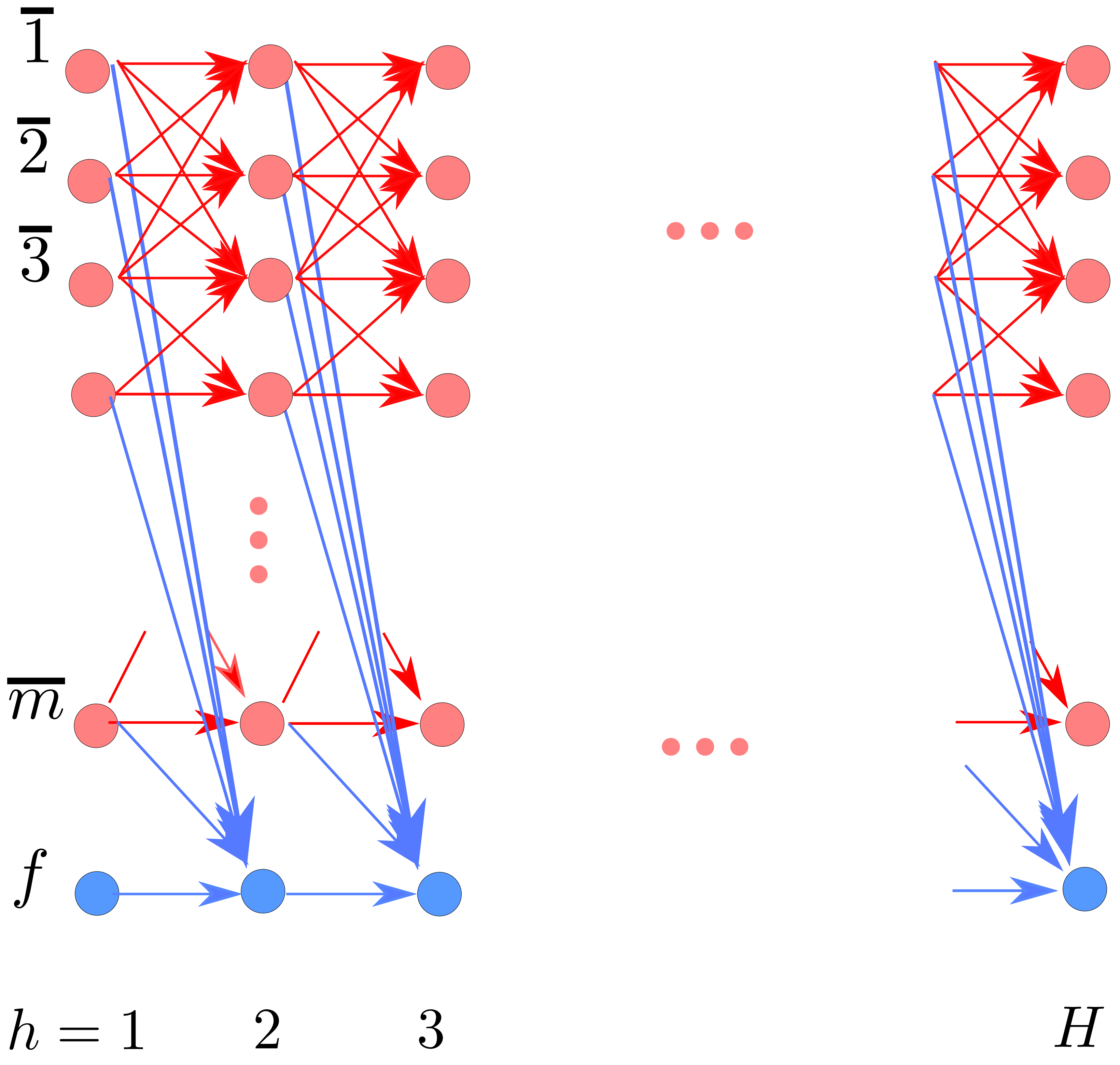

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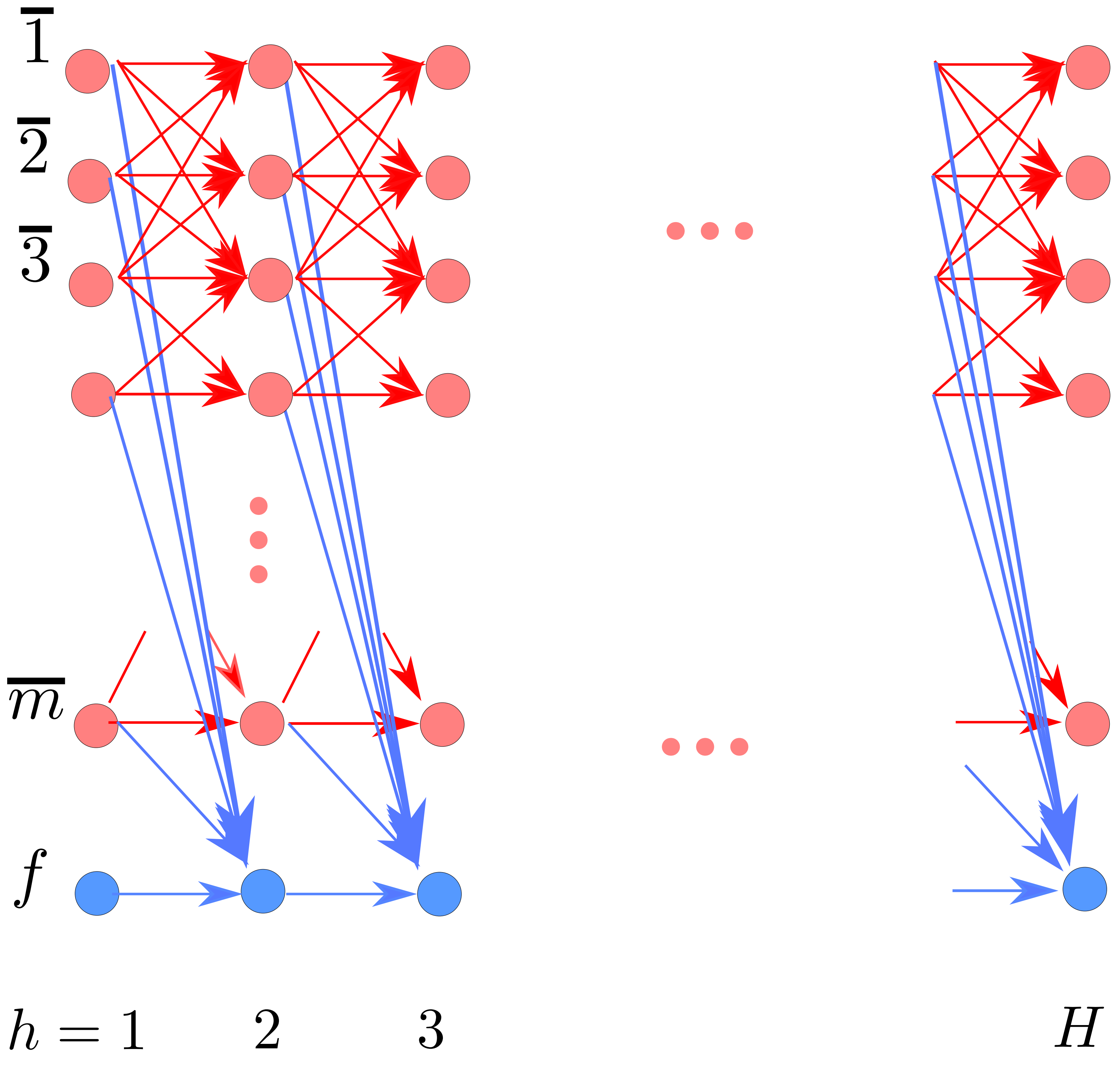

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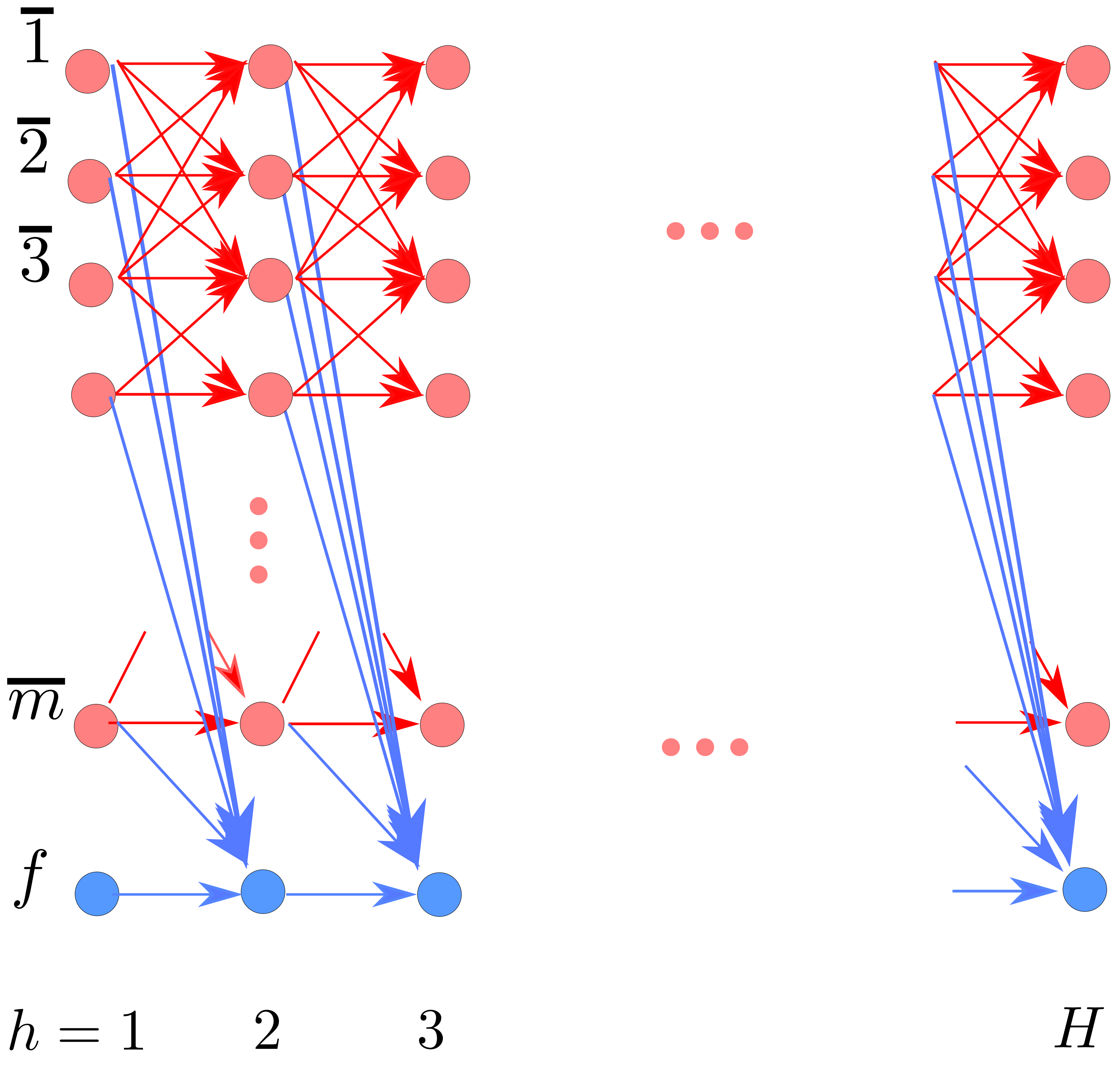

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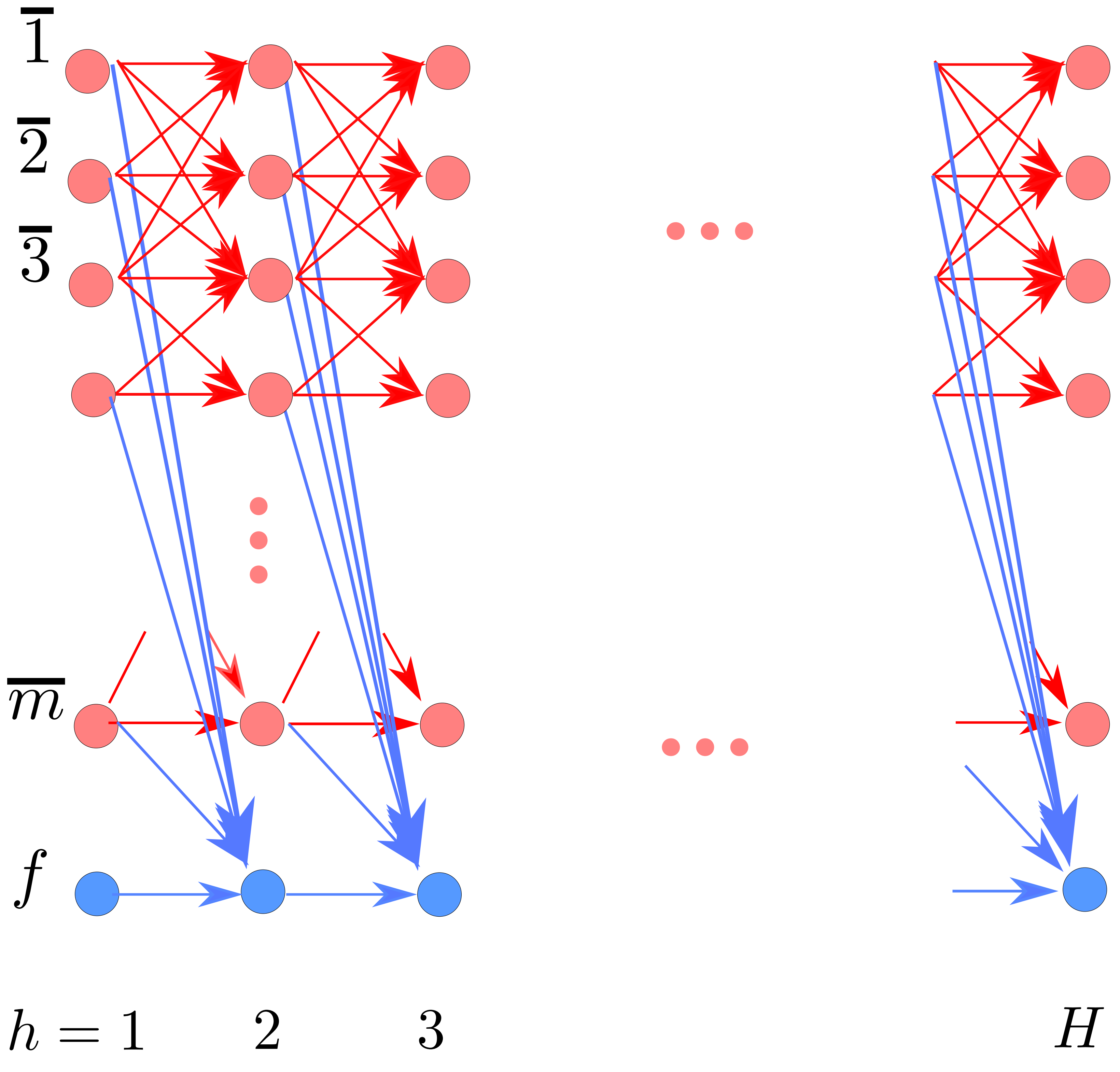
- Rewards: two cases which leak info about $a^\star$
  (1) if we take a^* at any h, then reward leaks info about a^* (but there $m = O(2^d)$ actions)
  (2) also, if we terminate at $s_H \neq f$, then the reward r_H leaks info about on a^*
  - But there is always at least $1/4$ chance of moving to f
  - So need at least $O((4/3)^H)$ trajectories to hit $s_H \neq f$



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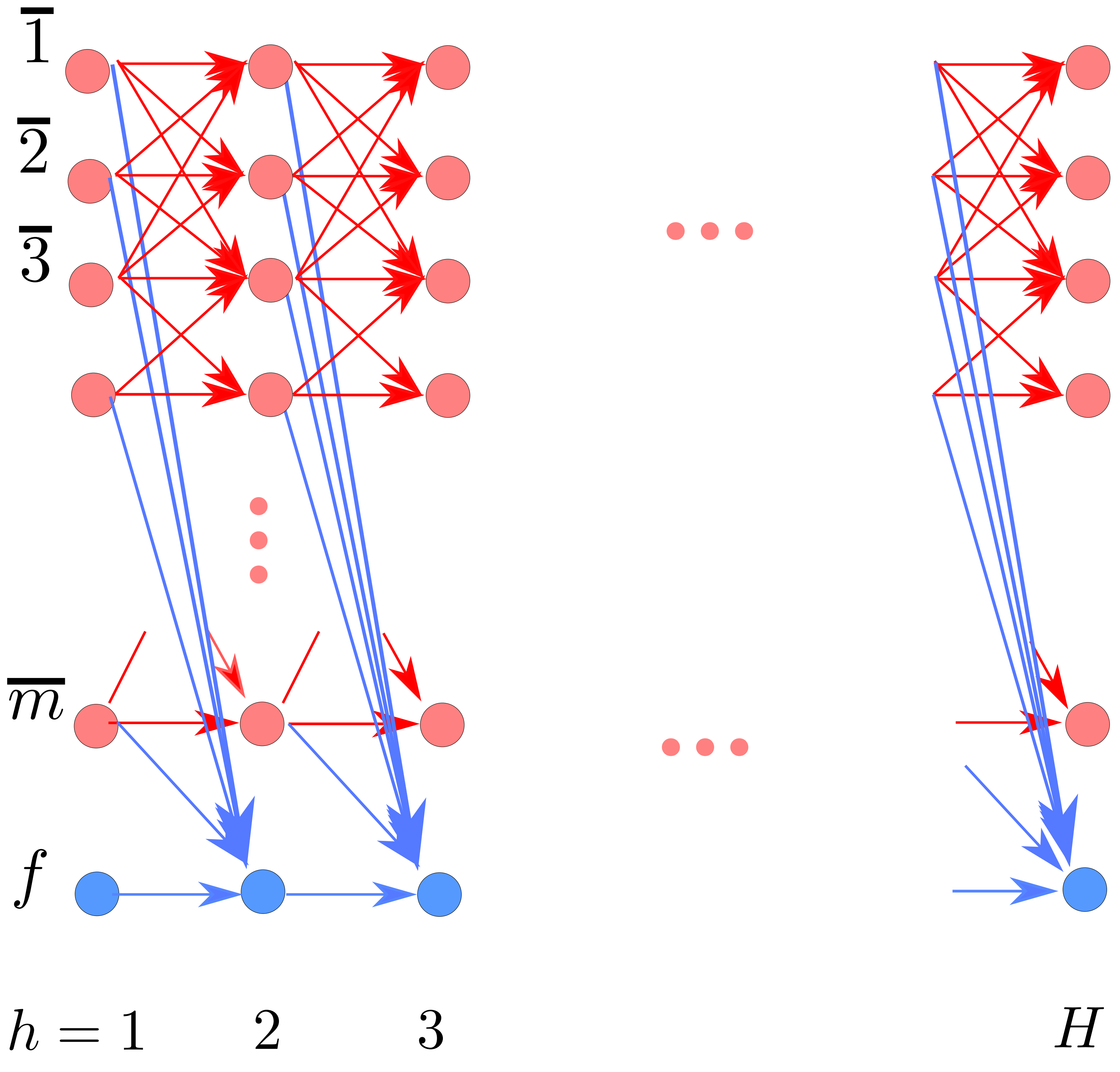

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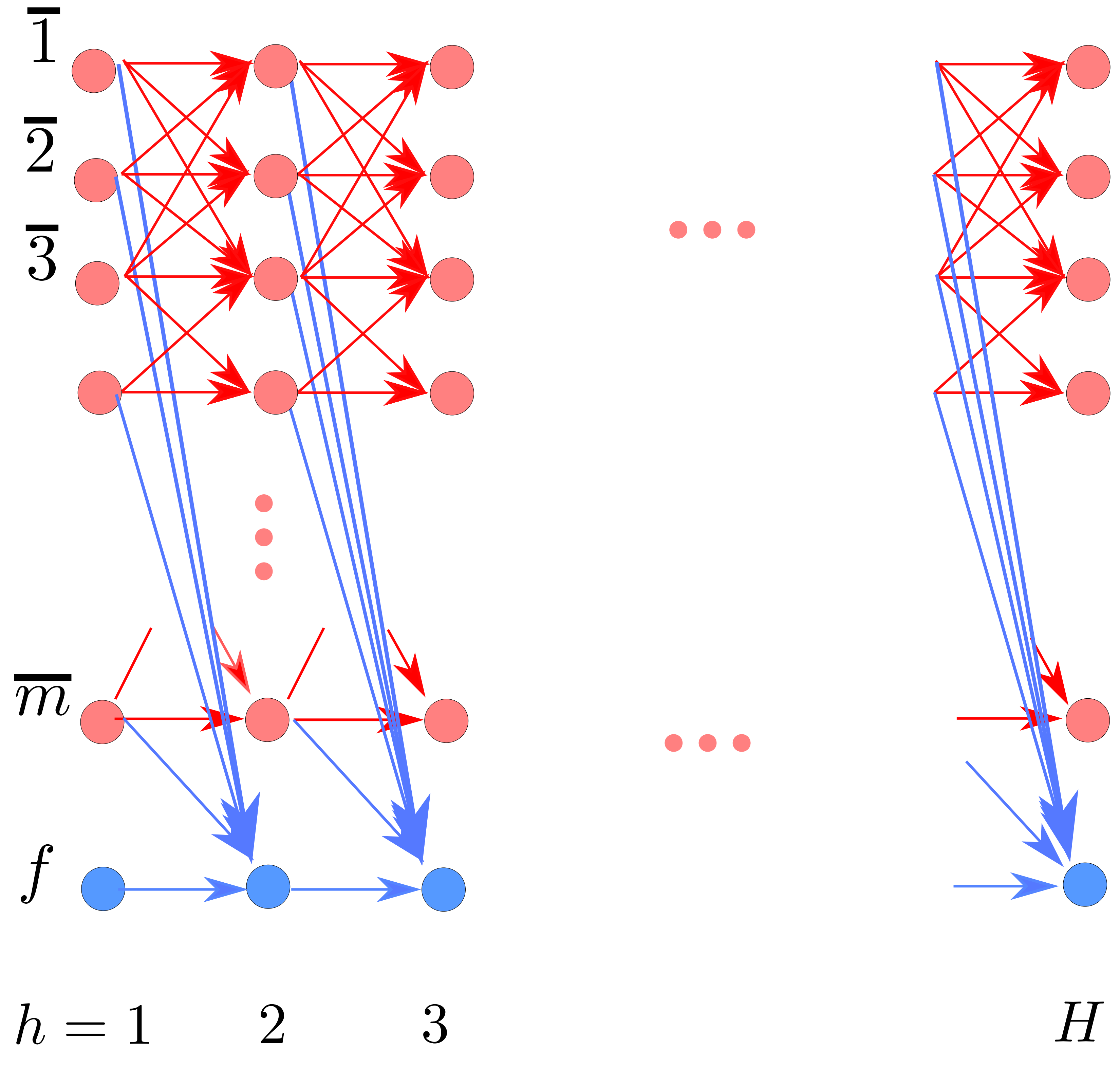

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Open Problem: Can we prove a lower bound with $A = 2$ actions?

<u>Part-3:</u> Discussion

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+ we have seen negative results.

How do we obtain positive results?

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  - agnostic learning is not possible in RL
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  - Imitation learning and behavior cloning: models where the agent has input from, effectively, a teacher.