Statistical Limits of Generalization
Part II: Linear Realizability

Sham Kakade and Wen Sun

CS 6789: Foundations of Reinforcement Learning

Part-2: Linear Realizability

What if we impose linearity assumptions?

Let’s look at the most natural assumptions.

RL with Linearly Realizable Q*-Function Approximation

(Does there exist a sample efficient algo?)

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$d << |S|, |A|$

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- (A1: Linearly Realizable Q*): Assume for all $s, a, h \in [H]$, there exists $w_1^\star, \ldots w_H^\star \in R^d$ s.t.

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- (A2: Large Suboptimality Gap): for all $a \neq \pi^\star(s)$,

$$V_h^\star(s) - Q_h^\star(s, a) \geq \text{constant}$$

$$\shortparallel$$

$$Q_h^\star(s, \pi(s))$$

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  holds only with episodic setting
- [Wang, Wang, K. '21]:
  Let's make the problem even easier, where we also assume A2 (large gap)
  The lower bound holds even with **both** A1 and A2.

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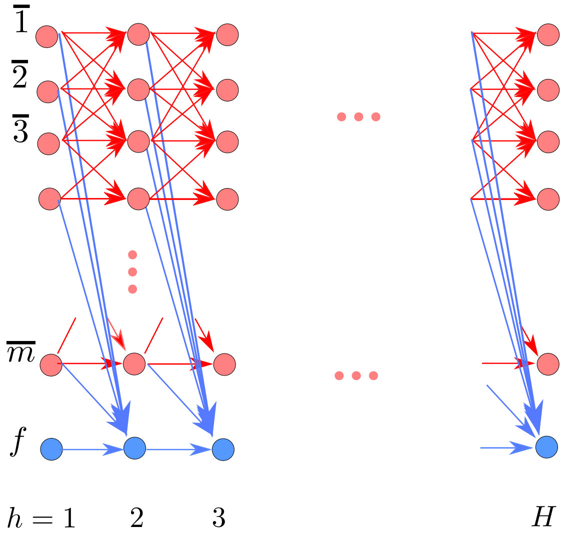
Comments: An exponential separation between online RL vs simulation access.

[Du, K., Wang, Yang ’20]: A1+A2+simulator access (input: any s, a; output: $s' \sim P(\cdot \mid s, a), r(s, a)$)

$\Longrightarrow$ there is sample efficient approach to find an ϵ-opt policy.



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(A ``leaking complete graph'')

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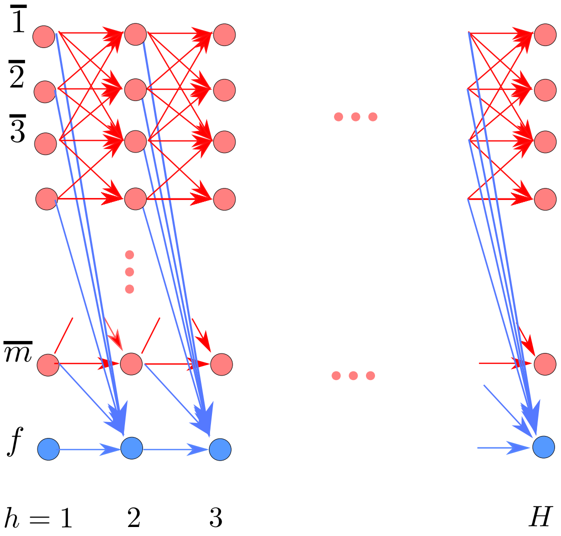

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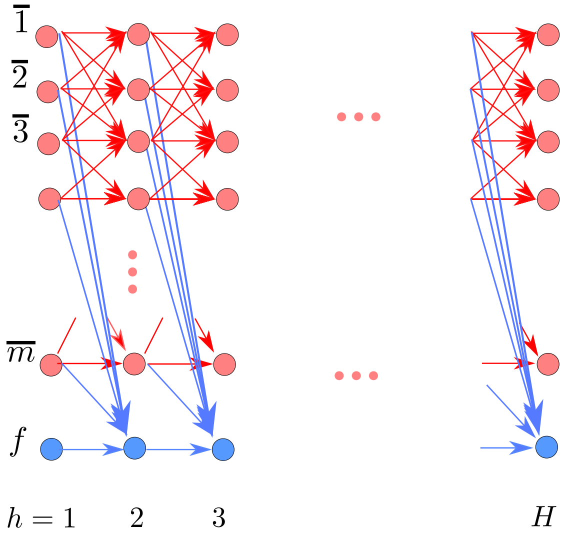

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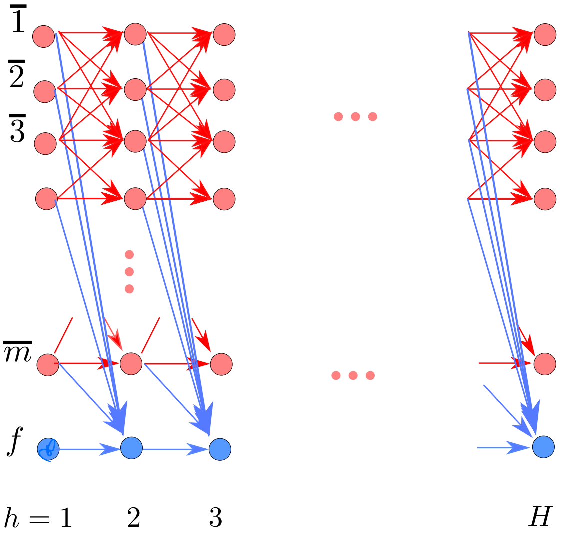

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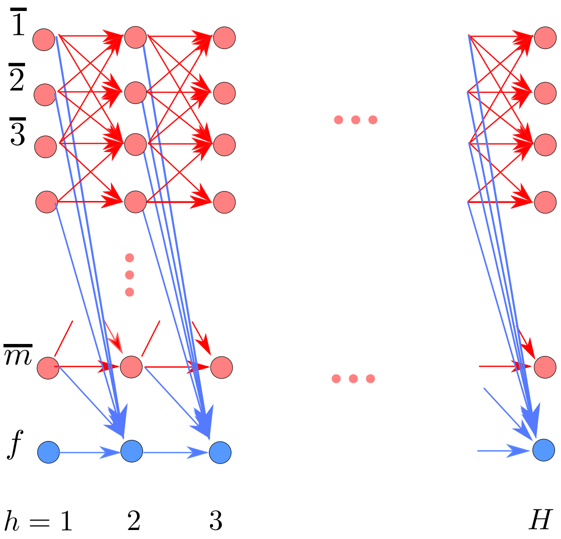

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  at f, the feasible action set is $[m-1]$.

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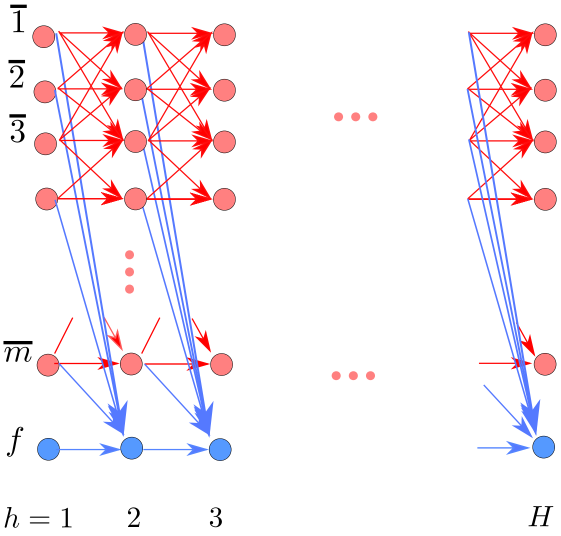

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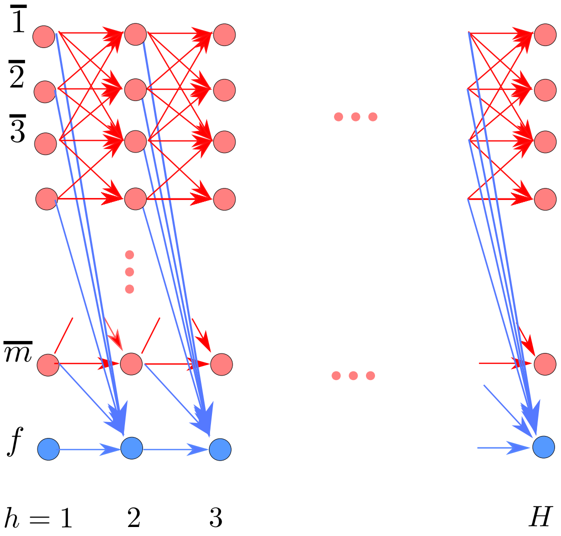

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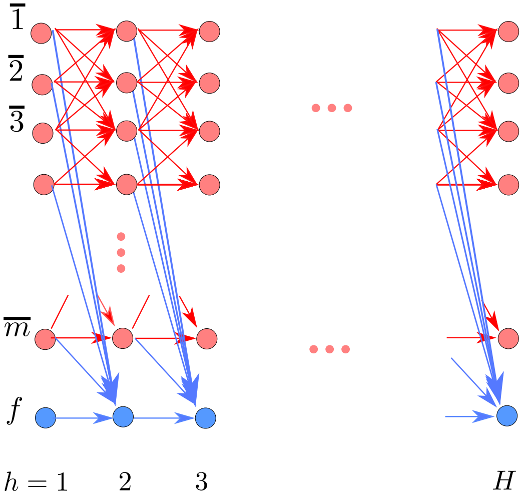
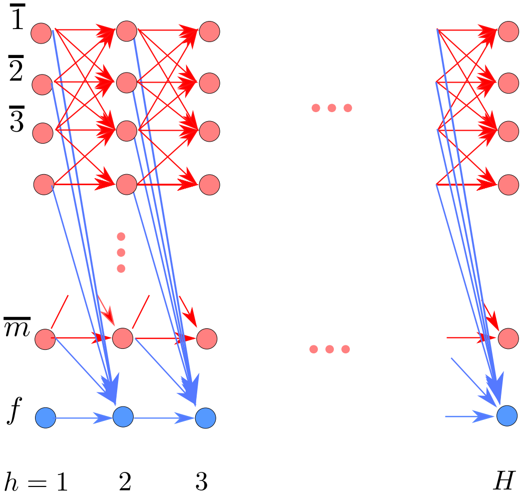
Lemma: For any $\gamma > 0$, there exist $m = \lfloor \exp(\frac{1}{8}\gamma^2 d) \rfloor$ unit vectors $\{v_1, \cdots, v_m\}$ in R^d s.t. $\forall i, j \in [m]$ and $i \neq j$, $|\langle v_i, v_j \rangle| \leq \gamma$.

We will set $\gamma = 1/4$.

(proof: Johnson-Lindenstrauss)

The construction, continued





The construction, continued

- Transitions: $s_0 \sim \text{Uniform}([m])$.

$\Pr[f \mid \overline{a_1}, a^*] = 1$,

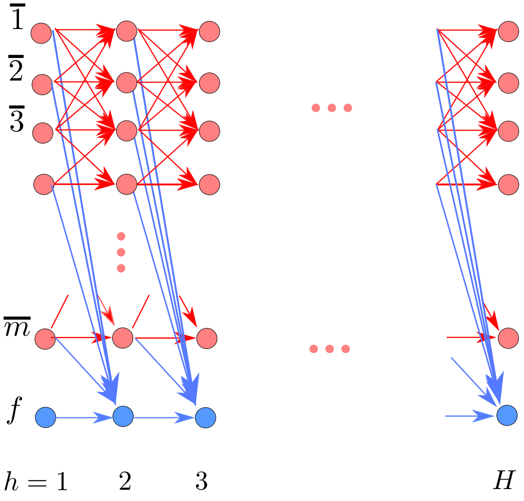
$< \gamma$

$v(3), v(5)$

$$\Pr[\,\cdot \mid \overline{a_1}, a_2] = \begin{cases} \overline{a_2} : \left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \\ f : 1 - \left\langle v(a_1), v(a_2) \right\rangle - 2\gamma \end{cases}, (a_2 \neq a^*, a_2 \neq a_1)$$

$$\Pr[f \mid f, \cdot\,] = 1.$$

$\gamma \le \Pr(\cdot \mid \overline{a_1}, a_2) \le 3\gamma$



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- After taking action a_2, the next state is either $\overline{a_2}$ or f.
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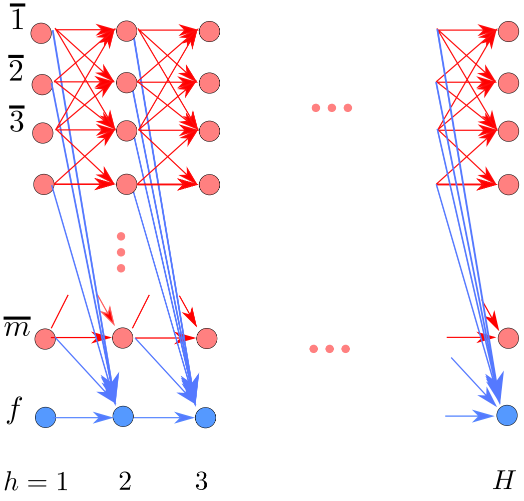

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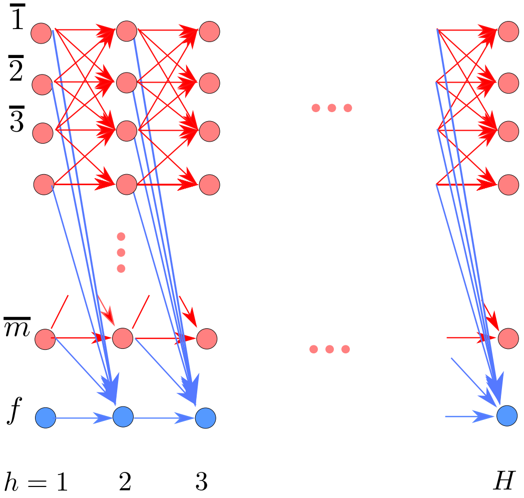

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- It is possible to visit any other state (except for $\overline{a^*}$); however, there is at least $1 - 3\gamma = 1/4$ probability of going to the terminal state f.
- The transition probabilities are indeed valid, because
  $$0 < \gamma \leq \left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \leq 3\gamma < 1.$$

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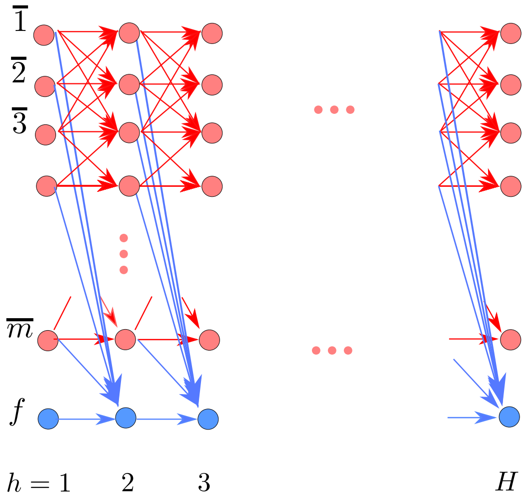
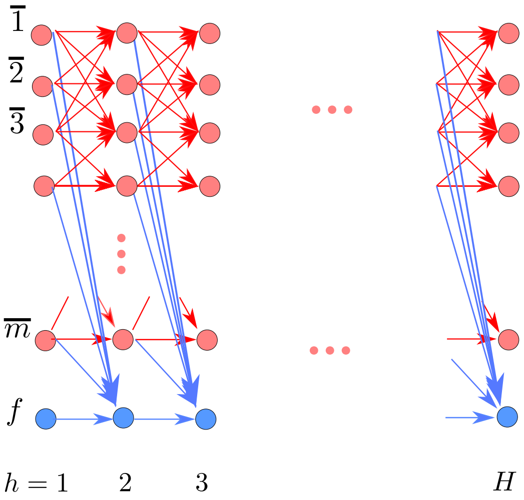




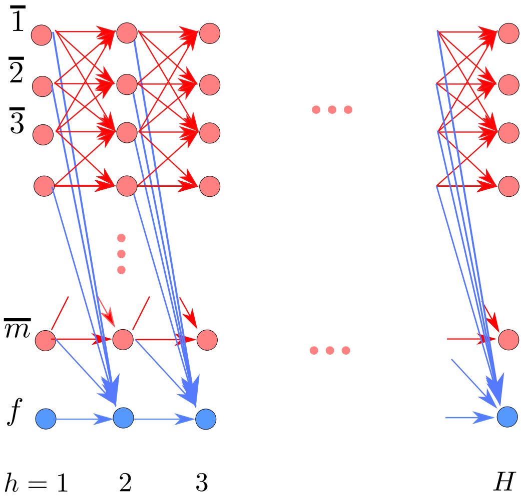
The construction, continued

- Features: of dimension d defined as:

$$\phi(\overline{a_1}, a_2) := \left(\left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \right) \cdot v(a_2), \quad \forall a_1 \neq a_2$$

$$\phi(f, \cdot\,) := \mathbf{0}$$

note: the feature map does not depend of a^*.



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- Rewards:

$\gamma \leq R_h(\overline{a_1}, a^*) \leq 3\gamma$

for $1 \leq h < H$,

$$R_h(\overline{a_1}, a^*) := \left\langle v(a_1), v(a^*) \right\rangle + 2\gamma,$$

$$R_h(\overline{a_1}, a_2) := -2\gamma \left[\left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \right], \; a_2 \neq a^*, a_2 \neq a_1$$

$$R_h(f, \cdot) := 0.$$

for $h = H$,

$$r_H(s, a) := \langle \phi(s, a), v(a^*) \rangle$$

$R_h \leq -2\gamma^2$

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Lemma: For all (s, a), we have $Q_h^*(s, a) = \langle \phi(s, a), v(a^*) \rangle$ and the "gap" is $\geq \gamma/4$.

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  By induction, we can show:

$$Q_h^\pi(\overline{a_1}, a_2) = \left(\left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \right) \cdot \left\langle v(a_2), v(a^*) \right\rangle,$$

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- Proving optimality: for $a_2 \neq a^*, a_1$

$$Q_h^\pi(\overline{a_1}, a_2) \leq 3\gamma^2, \quad Q_h^\pi(\overline{a_1}, a^*) = \left\langle v(a_1), v(a^*) \right\rangle + 2\gamma \geq \gamma > 3\gamma^2$$

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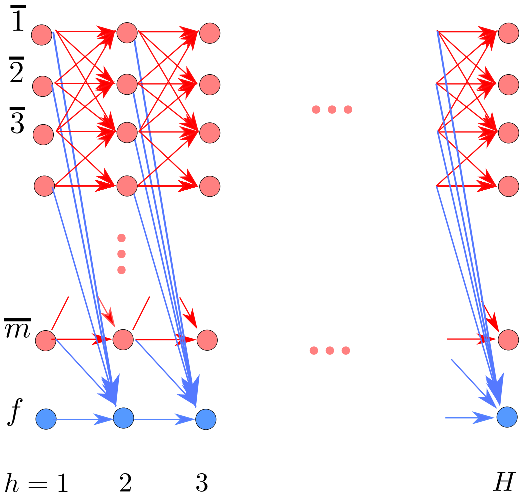
- Proving the large gap: for $a_2 \neq a^*$

$$V_h^*(\overline{a_1}) - Q_h^*(\overline{a_1}, a_2) = Q_h^\pi(\overline{a_1}, a^*) - Q_h^\pi(\overline{a_1}, a_2) > \gamma - 3\gamma^2 \geq \frac{1}{4}\gamma\,.$$



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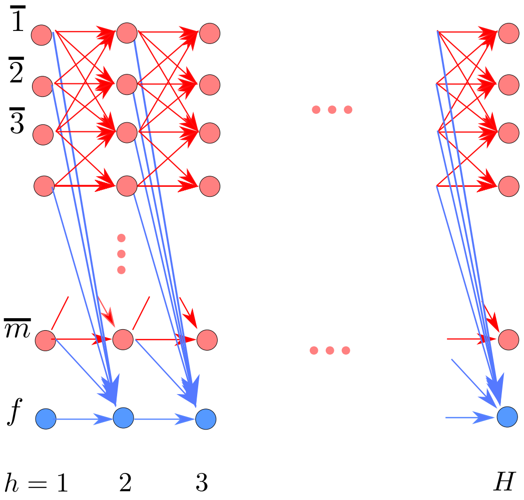

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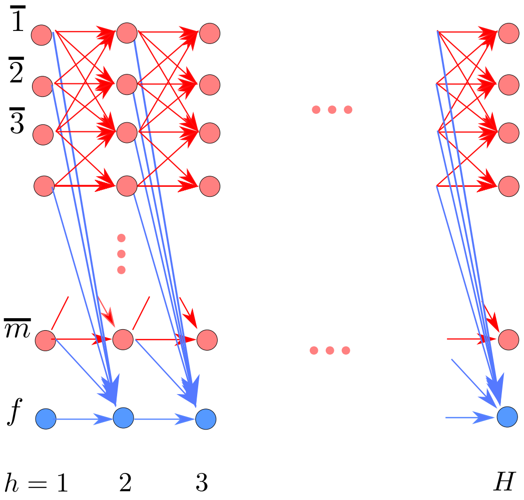

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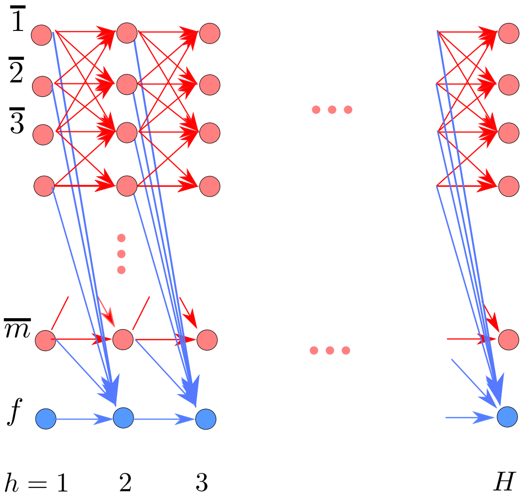

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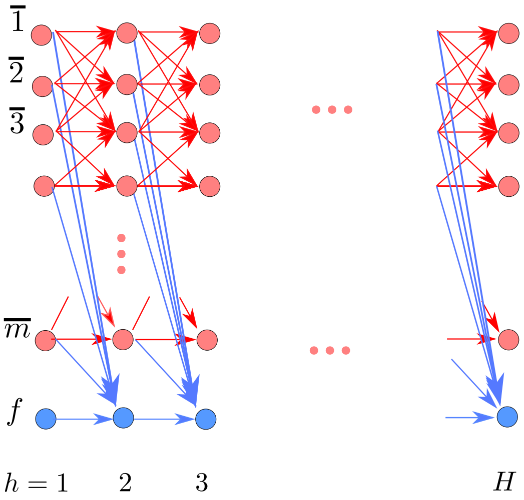

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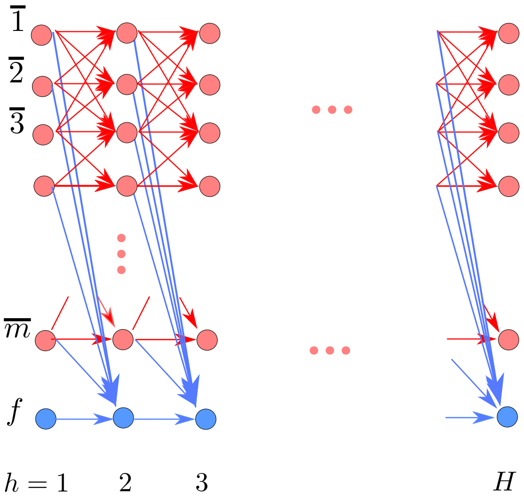
- Features: The construction of ϕ does not depend on $a^\star$.
- Transitions: if we take $a*$, only then does the dynamics leak info about $a*$ (but there $O(2^d)$ actions)
- Rewards: two cases which leak info about $a^\star$

  (1) if we take $a*$ at any h, then reward leaks info about $a*$ (but there $m = O(2^d)$ actions)

  (2) also, if we terminate at $s_H \neq f$, then the reward r_H leaks info about on $a*$

  - But there is always at least $1/4$ chance of moving to f
  - So need at least $O((4/3)^H)$ trajectories to hit $s_H \neq f$



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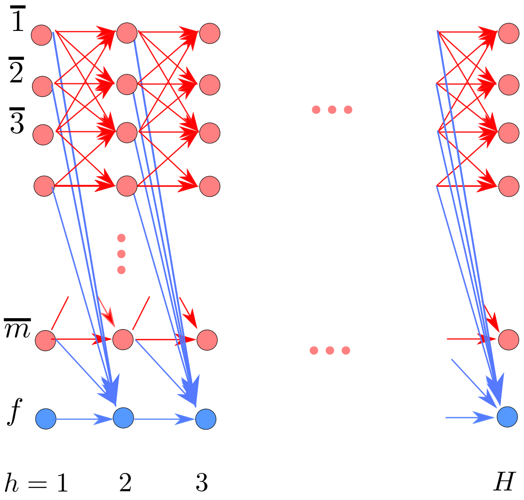

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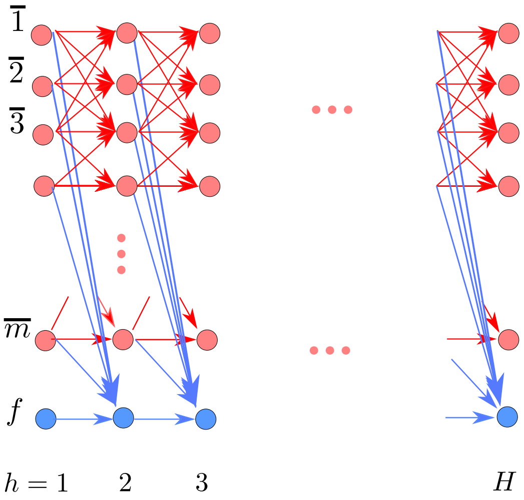

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Open Problem: Can we prove a lower bound with $A = 2$ actions?

Part-3: Discussion

RL is different from SL.

+ we have seen negative results.

How do we obtain positive results?

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  - Imitation learning and behavior cloning: models where the agent has input from, effectively, a teacher.