Policy Gradient: REINFORCE, Variance Reduction, Convergence

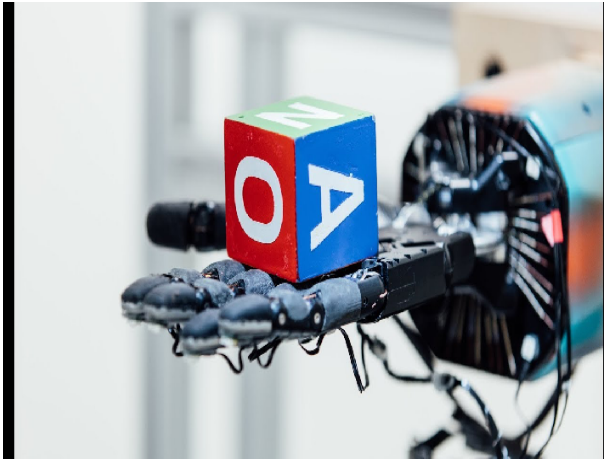
Sham Kakade and Wen Sun

CS 6789: Foundations of Reinforcement Learning

Policy Optimization

[AlphaZero, Silver et.al, 17]

[OpenAI Five, 18]



[OpenAI,19]

Recap: Infinite Horizon Discounted MDPs

$$\mathcal{M} = \{P, r, \gamma, \rho, S, A\}$$

where $s_0 \sim \rho$

Objective: $J(\pi) := \mathbb{E}_\pi \left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \middle| s_0 \sim \rho, s_{h+1} \sim P_{s_h, a_h}, a_h \sim \pi(\cdot \mid s_h) \right]$

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Discounted visitation $d^\pi(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h^\pi(s, a)$

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Discounted visitation $d^\pi(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h^\pi(s, a)$

Advantage function: $A^\pi(s, a) = Q^\pi(s, a) - V^\pi(s)$

Today: Policy Gradient Derivation

e.g., Reinforce, Natural Policy Gradient, TRPO, PPO:

(Williams 92, Kakade 02, Schulman et al 15, 17)

$$\pi_\theta(a \mid s) = \pi(a \mid s; \theta)$$

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Main question for today's lecture:
how to compute the gradient?

Outline for today

1. Two formulations of Policy Gradient

2. Variance Reduction

3. Convergence of SGD

Policy Gradient: Examples of Policy Parameterization (discrete actions)

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Neural network $f_\theta : S \times A \mapsto \mathbb{R}$

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$P_{\theta_0} := \rho$

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$\max_{\theta}$

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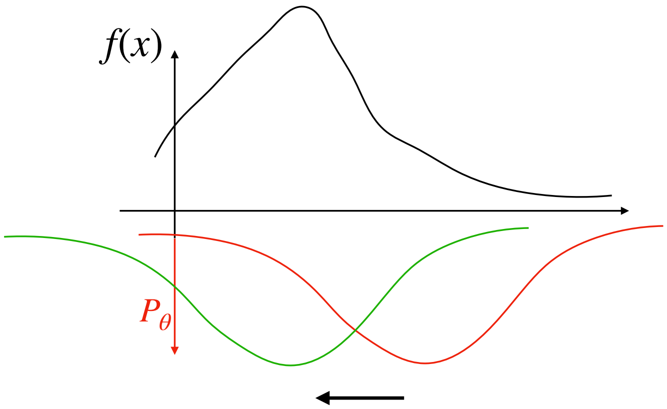

1. Derivation of Policy Gradient: REINFORCE

$$\tau = \{s_0, a_0, s_1, a_1, \dots\}$$

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Recall definition of value function $V^{\pi_\theta}(s)$

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$$= \sum_{a_0} \pi_\theta(a_0|s_0)\, Q^{\pi_\theta}(s_0, a_0)$$

$$\nabla_\theta \sum_{a_0} \pi_\theta(a_0|s_0)\, Q^{\pi_\theta}(s_0,a_0) = \sum_{a_0} \nabla_\theta \pi_\theta(a_0|s_0)\, Q^{\pi_\theta}(s_0,a_0)$$

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$$= \mathbb{E}_{s_0 \sim \rho}\left[\nabla_\theta \mathbb{E}_{a_0 \sim \pi_\theta(s_0)} Q^{\pi_\theta}(s_0, a_0)\right]$$

$$= \mathbb{E}_{s_0 \sim \rho}\left[\sum_{a_0 \in A} \pi_\theta(a_0 \mid s_0)\left[\frac{\nabla_\theta \pi_\theta(a_0 \mid s_0)}{\pi_\theta(a_0 \mid s_0)}\right] \cdot Q^{\pi_\theta}(s_0, a_0) + \gamma \sum_{a_0} \pi_\theta(a_0 \mid s_0) \mathbb{E}_{s_1 \sim P_{s_0, a_0}} \nabla_\theta V^{\pi_\theta}(s_1)\right]$$

$$= \mathbb{E}_{s_0 \sim \rho}\left[\mathbb{E}_{a_0 \sim \pi_\theta(a_0|s_0)} \nabla_\theta \ln \pi_\theta(a_0 \mid s_0) \cdot Q^{\pi_\theta}(s_0, a_0)\right] + \gamma \mathbb{E}_{s_1 \sim \mathbb{P}_1^{\pi_\theta}} \nabla_\theta V^{\pi_\theta}(s_1)$$

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$$= \sum_{h=0}^{\infty} \mathbb{E}_{s_h, a_h \sim \mathbb{P}_h^{\pi_\theta}} \nabla_\theta \ln \pi_\theta(a_h \mid s_h) Q^{\pi_\theta}(s_h, a_h)$$

2. Derivation of Policy Gradient w/ Q^π

Recall definition of value function $V^{\pi_\theta}(s)$

$$\nabla_\theta J(\pi_\theta) = \nabla_\theta \mathbb{E}_{s_0 \sim \rho}\left[V^{\pi_\theta}(s_0)\right]$$

$$= \mathbb{E}_{s_0 \sim \rho}\left[\nabla_\theta \mathbb{E}_{a_0 \sim \pi_\theta(s_0)} Q^{\pi_\theta}(s_0, a_0)\right]$$

$$= \mathbb{E}_{s_0 \sim \rho}\left[\sum_{a_0 \in A} \pi_\theta(a_0 \mid s_0)\left[\frac{\nabla_\theta \pi_\theta(a_0 \mid s_0)}{\pi_\theta(a_0 \mid s_0)}\right] \cdot Q^{\pi_\theta}(s_0, a_0) + \gamma \sum_{a_0} \pi_\theta(a_0 \mid s_0) \mathbb{E}_{s_1 \sim P_{s_0, a_0}} \nabla_\theta V^{\pi_\theta}(s_1)\right]$$

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$$= \sum_{h=0}^{\infty} \mathbb{E}_{s_h, a_h \sim \mathbb{P}_h^{\pi_\theta}} \nabla_\theta \ln \pi_\theta(a_h \mid s_h) Q^{\pi_\theta}(s_h, a_h) = \frac{1}{1-\gamma} \mathbb{E}_{s, a \sim d^{\pi_\theta}} \nabla_\theta \ln \pi_\theta(a \mid s) Q^{\pi_\theta}(s, a)$$

Derivation of unbiased Stochastic Policy Gradient

$$\nabla_\theta J(\theta) := \frac{1}{1-\gamma}\mathbb{E}_{s,a\sim d^{\pi_\theta}}\left[\nabla_\theta \ln \pi_\theta(a \mid s) Q^{\pi_\theta}(s,a)\right]$$

PI:

$\arg\max_a Q^{\pi_\theta}(s,a)$

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Draw $h \propto \gamma^h$, **roll-in** π_θ to generate $s_h, a_h \sim \mathbb{P}_h^{\pi_\theta}$

$$d^{\pi_\theta} = \underline{(1-\gamma)} \left[\mathbb{P}_0^{\pi} + \gamma \mathbb{P}_1^{\pi} + \gamma^2 \mathbb{P}_2^{\pi} + \cdots \right]$$

$$(1-\gamma), \quad (1-\gamma)\gamma, \quad (1-\gamma)\gamma^2, \quad \ldots$$

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Roll-out π_θ from (s_h, a_h) : terminate with prob $1-\gamma$, $\widetilde{Q}^{\pi_\theta}(s_h, a_h) = \sum_{\tau=h}^{t \geq h} r_\tau$

$$(s_h, a_h) \xrightarrow{\pi_\theta} \underset{h+1}{\cdot} \xrightarrow{\pi_\theta} \underset{h+2}{\cdot}$$

$$r_h + r_{h+1} + r_{h+2}$$

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Unbiased estimate: $\nabla_\theta \ln \pi_\theta(a_h \mid s_h) \widetilde{Q}^{\pi_\theta}(s_h, a_h)$

h: $h+1$ $h+2$ $\ldots$

$(1-\gamma)$ $\gamma(1-\gamma)$ $\gamma^2(1-\gamma)$

r_h $r_h + r_{h+1}$ $r_h + r_{h+1} + r_{h+2}$

Outline for today

1. Two formulations of Policy Gradient

2. Variance Reduction

3. Convergence of SGD

Variance Reduction via Action-Independent Baseline

Unbiased Estimate:

$$\nabla_\theta \ln \pi_\theta(a_h \,|\, s_h)\left(\widetilde{Q}^{\pi_\theta}(s_h, a_h) - b(s_h)\right)$$

Variance Reduction via Action-Independent Baseline

Unbiased Estimate:

$$\nabla_\theta \ln \pi_\theta(a_h \mid s_h)\left(\widetilde{Q}^{\pi_\theta}(s_h, a_h) - b(s_h)\right)$$

$$\mathbb{E}_{s,a\sim d^{\pi_\theta}}\left[\nabla_\theta \ln \pi_\theta(a \mid s)\right] b(s)$$

Variance Reduction via Action-Independent Baseline

Unbiased Estimate:

$$\nabla_\theta \ln \pi_\theta(a_h \mid s_h)\left(\widetilde{Q}^{\pi_\theta}(s_h, a_h) - b(s_h)\right)$$

$$\mathbb{E}_{s,a\sim d^{\pi_\theta}}\left[\nabla_\theta \ln \pi_\theta(a \mid s)\right] b(s) = \mathbb{E}_{s\sim d^{\pi_\theta}} \mathbb{E}_{a\sim\pi_\theta(\cdot\mid s)} \nabla_\theta \ln \pi_\theta(a\mid s) \cdot b(s)$$

$$= \mathbb{E}_{s\sim d^{\pi_\theta}}\left[b(s)\mathbb{E}_{a\sim\pi_\theta(\cdot\mid s)} \nabla_\theta \ln \pi_\theta(a \mid s)\right]$$

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Variance Reduction via Action-Independent Baseline

$$\nabla_\theta \ln \pi_\theta(a_h | s_h)\left(\widetilde{Q}^{\pi_\theta}(s_h, a_h) - b(s_h)\right) := g \qquad \mathbb{E}\left[\|g\|_2^2\right]$$

The best baseline that minimizes variance:

$$\min_b \ \mathbb{E}\left[\left(\nabla_\theta \ln \pi_\theta(a_h | s_h)\left(\widetilde{Q}^{\pi_\theta}(s_h, a_h) - b(s_h)\right)\right)^\top \nabla_\theta \ln \pi_\theta(a_h | s_h)\left(\widetilde{Q}^{\pi_\theta}(s_h, a_h) - b(s_h)\right)\right]$$

Variance Reduction via Action-Independent Baseline

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$$b(s_h) = \frac{\mathbb{E}\left[\nabla_\theta \ln \pi_\theta(a_h \mid s_h)^\top \nabla_\theta \ln \pi_\theta(a_h \mid s_h)\, \widetilde{Q}^{\theta}(s_h, a_h)\right]}{\mathbb{E}\left[\nabla_\theta \ln \pi_\theta(a_h \mid s_h)^\top \nabla_\theta \ln \pi_\theta(a_h \mid s_h)\right]}$$

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In practice:

$$b(s_h) = V^{\pi_\theta}(s)$$

Variance Reduction via Action-Independent Baseline

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In practice:

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$$\nabla_\theta J(\theta) = \frac{1}{1-\gamma} \mathbb{E}_{s,a \sim d^{\pi_\theta}} \left[\nabla_\theta \ln \pi_\theta(a | s) (Q^{\pi_\theta}(s, a) - V^{\pi_\theta}(s)) \right] = \frac{1}{1-\gamma} \mathbb{E}_{s,a \sim d^{\pi_\theta}} \left(\nabla_\theta \ln \pi_\theta(a | s) A^{\pi_\theta}(s, a) \right)$$

Summary so far:

The most commonly used formulation:
Policy Gradient with V^{π_θ} as a baseline:

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Next: Stochastic Gradient Ascent Converges to Stationary Point

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1. Two formulations of Policy Gradient

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Convergence to Stationary Point

$J(\pi_\theta)$ is non-convex (see example in the monograph)

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Def of β-smooth:

Def1 $$\|\nabla_\theta J(\theta) - \nabla_\theta J(\theta_0)\|_2 \leq \beta \|\theta - \theta_0\|_2$$

Def2, $$\left| J(\theta) - J(\theta_0) - \nabla_\theta J(\theta_0)^\top (\theta - \theta_0) \right| \leq \frac{\beta}{2} \|\theta - \theta_0\|_2^2, \forall \theta, \theta_0$$

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[Theorem] If $J(\theta)$ is β-smooth, and we run SGA: $\theta_{t+1} = \theta_t + \eta \widetilde{\nabla}_\theta J(\theta_t)$

where $\mathbb{E}\left[\widetilde{\nabla}_\theta J(\theta_t)\right] = \nabla_\theta J(\theta_t)$, $\mathbb{E}\left[\|\widetilde{\nabla}_\theta J(\theta_t)\|_2^2\right] \leq \sigma^2$,

then:

$$\mathbb{E}\left[\frac{1}{T} \sum_t \|\nabla_\theta J(\theta_t)\|_2^2\right] \leq O\left(\sqrt{\beta \sigma^2 / T}\right)$$

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$\theta_{t+1} - \theta_t = \eta \widetilde{\nabla}_\theta J(\theta_t)$

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$$\Rightarrow \eta\,\nabla_\theta J(\theta_t)^\top \widetilde{\nabla}_\theta J(\theta_t) \le J(\theta_{t+1}) - J(\theta_t) + \frac{\beta}{2}\eta^2\|\widetilde{\nabla}_\theta J(\theta_t)\|_2^2$$

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$J(\theta_{t+1}) \cancel{- J(\theta_t)}$

$\cancel{+ J(\theta_t)} - J(\theta_{t-1})$

Convergence to Stationary Point

[Theorem] If $J(\theta)$ is β-smooth, and we run SGA: $\theta_{t+1} = \theta_t + \eta \widetilde{\nabla}_\theta J(\theta_t)$

where $\mathbb{E}\left[\widetilde{\nabla}_\theta J(\theta_t)\right] = \nabla_\theta J(\theta_t), \quad \mathbb{E}\left[\|\widetilde{\nabla}_\theta J(\theta_t)\|_2^2\right] \le \sigma^2,$

then:

$$\mathbb{E}\left[\frac{1}{T}\sum_t \|\nabla_\theta J(\theta_t)\|_2^2\right] \le O\left(\sqrt{\beta\sigma^2/T}\right)$$

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$$\Rightarrow \left|J(\theta_{t+1}) - J(\theta_t) - \eta\nabla_\theta J(\theta_t)^\top \widetilde{\nabla}_\theta J(\theta_t)\right| \le \frac{\beta}{2}\eta^2\|\widetilde{\nabla}_\theta J(\theta_t)\|_2^2$$

$M : \max_\theta |J(\theta)| \le M$

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Convergence to Stationary Point

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$\eta = \sqrt{M/(\beta\sigma^2 T)}$

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Convergence to Stationary Point

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$$\frac{1}{1-\gamma}$$

$$\mathbb{E}\left[\,\left\| \underline{\ln \pi_\theta(a\,|\,s)\,\widetilde{Q}^{\pi_\theta}(s,a)} \right\|_2^2\right]$$

$$\left|J(\theta_{t+1}) - J(\theta_t) - \nabla_\theta J(\theta_t)^\top(\theta_{t+1} - \theta_t)\right| \le \frac{\beta}{2}\|\theta_{t+1} - \theta_t\|_2^2$$

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Convergence to Stationary Point

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$$\mathbb{E}\left[\left\| \ln \pi_\theta(a\,|\,s)\widetilde{Q}^{\pi_\theta}(s,a) \right\|_2^2\right]$$

$$\le \frac{1}{(1-\gamma)^2}\sup_{s,a}\left\| \nabla_\theta \ln \pi_\theta(a\,|\,s) \right\|_2^2$$

Set $\eta = \sqrt{M/(\beta\sigma^2 T)}$

Summary

$$\nabla_\theta J(\theta) = \frac{1}{1-\gamma}\mathbb{E}_{s,a\sim d^{\pi_\theta}}\left[\nabla_\theta \ln \pi_\theta(a \mid s)\underbrace{\left(Q^{\pi_\theta}(s,a) - V_\theta^{\pi}(s)\right)}_{A^{\pi_\theta}(s,a)}\right]$$

Use unbiased estimate of $\nabla_\theta J(\theta)$, SG ascent converges to stationary point