Convex Parameterization for Linear Dynamical Systems

Sham Kakade and Wen Sun

CS 6789: Foundations of Reinforcement Learning

Recap

LQR:

minimize $E\left[x_H^\top Q x_H + \sum_{t=0}^{H-1} (x_t^\top Q x_t + u_t^\top R u_t)\right]$

such that $x_{t+1} = Ax_t + Bu_t + w_t\,, \quad x_0 \sim D,\ w_t \sim N(0, \sigma^2 I)\,,$

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Set $P_H = Q,\ P_t = A^\top P_{t+1} A + Q - A^\top P_{t+1} B (B^\top P_{t+1} B + R)^{-1} B^\top P_{t+1} A$

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$$V_t^\star(x) = x^\top P_t x + \sigma^2 \sum_{h=t+1}^{H} \mathrm{Trace}(P_h)$$

Today's main Question:

What if the cost function is arbitrary convex function, and how to handle adversarial noise?

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Goal: find the best linear controllers: $\{-K_t^\star\}_{t=0}^{H-1} := \arg\min_{\pi \in \Pi} \mathbb{E}\left[\sum_{t=0}^{H-1} c(x_t, u_t) \,|\, \pi\right]$

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Unfortunately, even with quadratic cost, $\max_{\pi} \mathbb{E}\left[\sum_{h=0}^{H-1} c(x_h, a_h) \,\middle|\, \pi\right]$ is not convex

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Let's search for a different parameterization that is equivalent to this $u = -K_t x$ parameterization

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[Claim] Given controller $\widetilde{\pi} := \left\{ \{M_t, M_{0;t}, \ldots M_{t-1;t}\} \right\}_{t=0}^{H-1}$,

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2. Compute gradient of $\sum_{h=0}^{H-1} c(x_h, u_h)$ wrt all parameters M, perform gradient descent

Extension to adversarial online control

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$$\left(J^k(\pi) = \sum_{h=0}^{H-1} c^k(x_h, u_h), \text{where } x_{h+1} = Ax_h + Bu_t + w_t, u_t = \pi(x_t) \right)$$

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At iteration k:

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1. **Adversary** picks a loss function $\ell_k : \mathcal{Z} \mapsto \mathbb{R}$

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1. **Adversary** picks a loss function $\ell_k : \mathcal{Z} \mapsto \mathbb{R}$

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The OGD Algorithm:

At iteration t, learner proposes $z_k := P_{\mathcal{Z}}\left(z_{k-1} - \eta \nabla \ell_{k-1}(z_{k-1})\right)$

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Assume $\mathcal{X}$ is convex, and bounded $\max_{z,y \in \mathcal{Z}} \|z - y\|_2 \leq F$, and loss is G-Lipschitz, then OGD has the following regret:

$$\sum_{k=1}^{K} \ell_k(z_k) - \min_{x \in \mathcal{Z}} \sum_{k=1}^{K} \ell_k(z) \leq O\left(\left(F^2 + G^2\right)\sqrt{K}\right)$$

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Where:

$$x_0, u_0 = M_0 x_0 \qquad x_1 = Ax_0 + BM_0 x_0 \qquad u_1 = M_1 x_0 + M_{0;1} w_0^k \qquad x_2 = Ax_1 + BM_1 x_0 + BM_{0;1} w_0^k$$

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Thus, we can run any no-regret online learning algorithm (e.g., OGD) on the loss sequence $\{\ell_k\}$

Summary

1. LQR formulation, DP for LQR (Riccati Equation), and SDP formulation

3. Another form of over-parameterized controllers which leads to a convex parameterization (hence we can do gradient descent).

3. Online Control (contrast it w/ min-max robust control)