Planning in Markov Decision Process

Sham Kakade and Wen Sun

CS 6789: Foundations of Reinforcement Learning

Announcements

HW0: due this 9/9 11:59pm ET
Gradescope (please self-enroll)

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Waiting List

Recap: Infinite Horizon MDPs

$$\mathcal{M} = \{S, A, P, r, \gamma\}$$

$$P : S \times A \mapsto \Delta(S), \quad r : S \times A \to [0,1], \quad \gamma \in [0,1)$$

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Value function $V^\pi(s) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \middle| s_0 = s, a_h \sim \pi(s_h), s_{h+1} \sim P(\cdot \mid s_h, a_h)\right]$

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Q function $Q^\pi(s, a) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \middle| (s_0, a_0) = (s, a), a_h \sim \pi(s_h), s_{h+1} \sim P(\cdot \mid s_h, a_h)\right]$

Recap: Bellman Optimality

$$\mathcal{M} = \{S, A, P, r, \gamma\}$$

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Theorem 1: Bellman Optimality (Q-version)

$$Q^\star(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[\max_{a' \in A} Q^\star(s', a') \right]$$

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Theorem 1: Bellman Optimality (Q-version)

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Theorem 2 (Q-version):

For any $Q : S \times A \to \mathbb{R}$, if $Q(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[\max_{a'} Q(s',a') \right]$,

$\forall s, a$, then $Q(s,a) = Q^\star(s,a), \forall s, a$

Main Question for Today:

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Two Approaches:

1. Value Iteration
2. Policy Iteration

Define Bellman Operator $\mathcal{T}$:

Given a function $f: S \times A \mapsto \mathbb{R}$,

function ← $\mathcal{T}f: S \times A \mapsto \mathbb{R}$,

$$(\mathcal{T}f)(s, a) := r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} \max_{a' \in A} f(s', a'), \forall s, a \in S \times A$$

Value Iteration Algorithm:

1. Initialization: $Q^0 : \|Q^0\|_\infty \in (0, \frac{1}{1-\gamma})$

2. Iterate until convergence: $Q^{t+1} = \mathcal{T} Q^t$

$Q^0(\cdot,\cdot) \in \mathbb{R}^{|S| \times |A|}$

$1 + \gamma + \gamma^2 + \gamma^3 + \dots = \frac{1}{1-\gamma}$

$\Rightarrow \forall s,a,$ set $Q^{t+1}(s,a) \Leftarrow r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \max_{a'} Q^t(s',a')$

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Via Bellman optimality theorem:

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$\ell: [a,b] \mapsto [a,b]$

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$$x_0, x_{t+1} = \ell(x_t), t = 0, \ldots,$$

$$|x_t - x^\star| = |\ell(x_{t-1}) - \ell(x^\star)| \leq L|x_{t-1} - x^\star|$$

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$$|x_t - x^\star| = |\ell(x_{t-1}) - \ell(x^\star)| \leq L|x_{t-1} - x^\star| \leq L^2 |x_{t-2} - x^\star|$$

If $L < 1$ (i.e., contraction), then it converges exponentially fast

Convergence of Value Iteration:

Lemma [contraction]: Given any Q, Q', we have:

$$\|\mathcal{T}Q - \mathcal{T}Q'\|_\infty \leq \gamma \|Q - Q'\|_\infty$$

Proof:

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$$\le \gamma^2 \|Q^{t-1} - Q^*\|_\infty$$

$$\dots \le \gamma^{t+1} \|\widehat{Q}^0 - Q^\star\|_\infty$$

$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

$$\pi(s) = \arg\max_a Q^t(s,a)$$

Final Quality of the Policy:

$$\pi^t : \pi^t(s) = \arg\max_a Q^t(s,a)$$

Theorem: $V^{\pi^t}(s) \geq V^\star(s) - \dfrac{2\gamma^t}{1-\gamma}\|Q^0 - Q^\star\|_\infty \forall s \in S$

Proof:

$$\left| V^{\pi^t}(s) - V^{\star}(s) \right| \leq O\left(\frac{\gamma^t}{1-\gamma}\right)$$

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≤ 0

$Q^t(s,\pi^t(s)) = \max_a Q^t(s,a) \geq Q^t(s,\pi^\star(s))$

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$\geq -\gamma^t \|Q^0 - Q^\star\|_\infty$

$$\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) - 2\gamma^t\|Q^0 - Q^\star\|_\infty$$

Apply the same procedure

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Given an MDP $\mathscr{M} = (S, A, P, r, \gamma)$, How to find $\pi^\star$ (stationary & deterministic)

Two Approaches:

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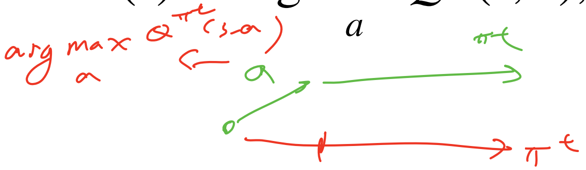
3. Policy Improvement $\pi^{t+1}(s) = \arg\max_{a} Q^{\pi^t}(s, a), \forall s$



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Closed-form for PE
(see 1.1.3 in Monograph)

$\forall s\, a$

$Q^{\pi^t}(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} Q^{\pi^t}(s', \pi^t(s'))$

$|S||A|$ variable & constraints

Analysis of Policy Iteration

Recall: Policy Improvement $\pi^{t+1}(s) = \arg\max_a Q^{\pi^t}(s, a), \forall s$

Lemma: Monotonic improvement $Q^{\pi^{t+1}}(s, a) \geq Q^{\pi^t}(s, a), \forall s, a$

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Bell-Eqn

$$Q^{\pi^{t+1}}(s, a) - Q^{\pi^t}(s, a) = \gamma \mathbb{E}_{s' \sim P(s,a)} \left[Q^{\pi^{t+1}}(s', \pi^{t+1}(s')) - Q^{\pi^t}(s', \pi^t(s')) \right]$$

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$$= \gamma \mathbb{E}_{s'\sim P(s,a)}\left[Q^{\pi^{t+1}}(s',\pi^{t+1}(s')) - Q^{\pi^t}(s',\pi^{t+1}(s')) + Q^{\pi^t}(s',\pi^{t+1}(s')) - Q^{\pi^t}(s',\pi^t(s'))\right]$$

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≥ 0

$$\geq \gamma \mathbb{E}_{s' \sim P(s,a)} \left[Q^{\pi^{t+1}}(s', \pi^{t+1}(s')) - Q^{\pi^t}(s', \pi^{t+1}(s')) \right]$$

$$\geq \gamma \mathbb{E}_{s'' \sim P(s', \pi^{t+1}(s'))} \left[Q^{\pi^{t+1}}(s'', \pi^{t+1}(s'')) - Q^{\pi^t}(s'', \pi^{t+1}(s'')) \right]$$

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$$\geq \gamma \mathbb{E}_{s' \sim P(s,a)} \left[Q^{\pi^{t+1}}(s', \pi^{t+1}(s')) - Q^{\pi^t}(s', \pi^{t+1}(s')) \right] \geq \ldots, \geq -\gamma^\infty/(1-\gamma) = 0$$

$\in \left[-\frac{1}{1-\gamma}, \frac{1}{1-\gamma} \right]$

$\geq -\gamma/(1-\gamma)$

Analysis of Policy Iteration

Recall: Policy Improvement $\pi^{t+1}(s) = \arg\max_a Q^{\pi^t}(s,a), \forall s$

Lemma: Monotonic improvement $Q^{\pi^{t+1}}(s,a) \geq Q^{\pi^t}(s,a), \forall s, a$

$$Q^{\pi^{t+1}}(s,a) - Q^{\pi^t}(s,a) = \gamma \mathbb{E}_{s' \sim P(s,a)} \left[Q^{\pi^{t+1}}(s', \pi^{t+1}(s')) - Q^{\pi^t}(s', \pi^t(s')) \right]$$

$$= \gamma \mathbb{E}_{s' \sim P(s,a)} \left[Q^{\pi^{t+1}}(s', \pi^{t+1}(s')) - Q^{\pi^t}(s', \pi^{t+1}(s')) + Q^{\pi^t}(s', \pi^{t+1}(s')) - Q^{\pi^t}(s', \pi^t(s')) \right]$$

$$\geq \gamma \mathbb{E}_{s' \sim P(s,a)} \left[Q^{\pi^{t+1}}(s', \pi^{t+1}(s')) - Q^{\pi^t}(s', \pi^{t+1}(s')) \right] \geq \ldots, \geq -\gamma^{\infty}/(1-\gamma) = 0$$

$$V^{\pi^{t+1}}(s) \geq V^{\pi^t}(s), \forall s$$

Analysis of Policy Iteration

Recall: Policy Improvement $\pi^{t+1}(s) = \arg\max_a Q^{\pi^t}(s, a), \forall s$

Theorem: Convergence $||V^{\pi^{t+1}} - V^\star||_\infty \leq \gamma||V^{\pi^t} - V^\star||_\infty$

$\cdots \leq \gamma^t ||V^{\pi^0} - V^*||_\infty$

Analysis of Policy Iteration

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Theorem: Convergence $\|V^{\pi^{t+1}} - V^\star\|_\infty \leq \gamma \|V^{\pi^t} - V^\star\|_\infty$

$$V^\star(s) - V^{\pi^{t+1}}(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V^\star(s') \right] - \left[r(s, \pi^{t+1}(s)) + \gamma \mathbb{E}_{s' \sim P(s, \pi^{t+1}(s))} V^{\pi^{t+1}}(s') \right]$$

bell-ope

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$$= \max_a (r(s,a) + \mathbb{E}_{s' \sim P(s,a)} \gamma V^\star(s')) - \max_a (r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V^{\pi^t}(s'))$$

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$$\le \max_a \left[r(s,a) + \gamma \mathbb{E}_{s'\sim P(s,a)} V^\star(s') \right] - \left[r(s, \pi^{t+1}(s)) + \gamma \mathbb{E}_{s'\sim P(s,\pi^{t+1}(s))} V^{\pi^t}(s') \right]$$

$$:= Q^{\pi^t}(s, \pi^{t+1}(s))$$

$$= \max_a (r(s,a) + \mathbb{E}_{s'\sim P(s,a)} \gamma V^\star(s')) - \max_a (r(s,a) + \gamma \mathbb{E}_{s'\sim P(s,a)} V^{\pi^t}(s'))$$

$$\le \max_a \left(r(s,a) + \gamma \mathbb{E}_{s'\sim P(s,a)} V^\star(s') - \left(r(s,a) + \gamma \mathbb{E}_{s'\sim P(s,a)} V^{\pi^t}(s') \right) \right)$$

$$\max_x f(x) - \max_x g(x) \le \max_x [f(x) - g(x)]$$

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$$\le \gamma \|V^\star - V^{\pi^t}\|_\infty$$

$\Rightarrow \|V^\star - V^{\pi^{t+1}}\|_\infty \le \gamma \|V^\star - V^{\pi^t}\|_\infty$

Value Iteration vs Policy Iteration?

Which one is faster?

How many iterations (computation complexity) need to find the EXACT optimal policy?

Trajectory distribution and state-action distribution

Q: what is the probability of π generating trajectory $\tau = \{s_0, a_0, s_1, a_1, \ldots, s_h, a_h\}$?

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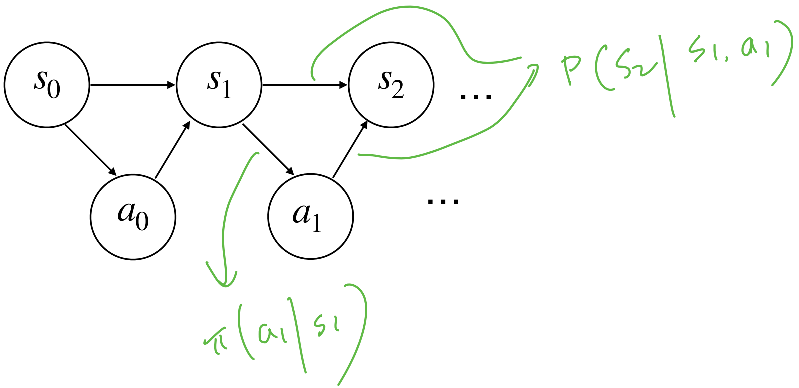

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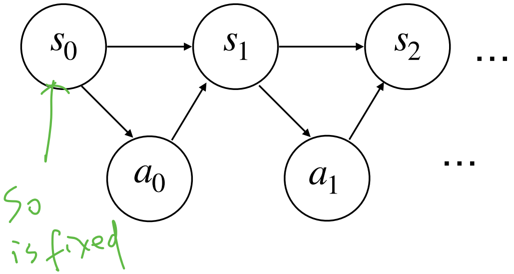

$$\mathbb{P}^{\pi}(s_0, a_0, \ldots, s_h, a_h)$$

$$= \pi(a_0 \mid s_0) P(s_1 \mid s_0, a_0) \pi(a_1 \mid s_1) P(s_2 \mid s_1, a_1) \ldots P(s_h \mid s_{h-1}, a_{h-1}) \pi(a_h \mid s_h)$$

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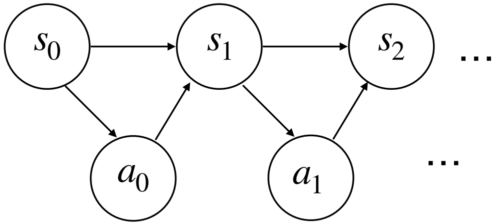

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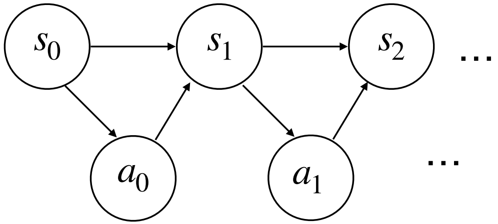

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Q: what's the probability of π visiting state (s,a) at time step h?

$$\mathbb{P}_h(s, a; s_0, \pi) = \sum_{a_0, s_1, a_1, \ldots, s_{h-1}, a_{h-1}} \mathbb{P}^\pi(s_0, a_0, \ldots, s_{h-1}, a_{h-1} s_h = s, a_h = a)$$

State action occupancy measure

$\mathbb{P}_h(s, a; s_0, \pi)$: probability of π visiting (s, a) at time step $h \in \mathbb{N}$, starting at s_0

$$d^{\pi}_{s_0}(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h(s, a; s_0, \pi)$$

$\sum_{sa} \mathbb{P}_h(s,a) = 1$

normalizing

$\sum_{sa} d^{\pi}_{s_0}(s,a) = 1$

State action occupancy measure

$\mathbb{P}_h(s, a; s_0, \pi)$: probability of π visiting (s, a) at time step $h \in \mathbb{N}$, starting at s_0

$$d^{\pi}_{s_0}(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h(s, a; s_0, \pi)$$

$$V^{\pi}(s_0) = \frac{1}{1 - \gamma} \sum_{s,a} d^{\pi}_{s_0}(s, a) r(s, a)$$

$$= \frac{1}{1-\gamma} \mathbb{E}_{s,a \sim d^{\pi}_{s_0}} [r(s,a)]$$

Summary for today

Two planning algorithms (no learning so far):

VI: fixed point iteration $Q^{t+1} = \mathcal{T} Q^t$

Key property: it's a contraction map

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PI: $\pi^{t+1}(s) = \arg\max_a Q^{\pi^t}(s,a)$

Key property: monotonic improvement $V^{\pi^{t+1}}(s) \geq V^{\pi^t}(s), \forall s$