Fitted Q Iteration

Sham Kakade and Wen Sun

CS 6789: Foundations of Reinforcement Learning

Recap: Value Iteration (Planning)

$$f_{t+1} = \mathcal{T} f_t$$

$\mathcal{T}$

$\forall s, a, \quad \mathcal{T} f_t(s, a)$

$= r(s, a) + \gamma \, \mathbb{E}_{s' \sim P(s,a)} \left(\max_{a'} f_t(s', a') \right)$

Recap: Value Iteration (Planning)

$$f_{t+1} = \mathcal{T} f_t$$

1. We have point-wise accuracy (via the contraction property):

$$\|f_t - Q^\star\|_\infty \leq \gamma^k/(1-\gamma)$$

Recap: Value Iteration (Planning)

$$f_{t+1} = \mathcal{T} f_t$$

1. We have point-wise accuracy (via the contraction property):

$$\|f_t - Q^\star\|_\infty \leq \gamma^k / (1 - \gamma)$$

2. Turn f_t's **point-wise** approximation error to policy's performance (error amplification):

$$\pi^t(s) = \arg\max_a f_t(s, a), \forall s$$

$$V^\star - V^{\pi^t} \leq \frac{2}{1-\gamma} \frac{\gamma^k}{1-\gamma}$$

Recap: Linear Bellman Completion

Given feature ϕ, take any linear function $w^\top \phi(s,a)$:

$$\forall h, \exists \theta \in \mathbb{R}^d, s.t., \theta^\top \phi(s,a) = r(s,a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} w^\top \phi(s', a'), \forall s, a$$

(It implies that $Q_h^\star$ is linear in ϕ: $Q_h^\star = (\theta_h^\star)^\top \phi, \forall h$

Theorem: There exists a way to construct datasets $\{\mathscr{D}_h\}_{h=0}^{H-1}$, such that with probability at least $1-\delta$, we have:

$$V^{\hat{\pi}} - V^\star \leq \epsilon$$

w/ total number of samples in these datasets scaling $\widetilde{O}\left(d^2 + H^6 d^2/\epsilon^2\right)$

Recap: Least-Square Value Iteration

Using D-optimal design, we construct a linear regression dataset such that at all h:

$$\max_{s,a} \left| \theta_h^\top \phi(s,a) - \mathcal{T}_h(\theta_{h+1})^\top \phi(s,a) \right| \leq O\left(Hd/\sqrt{N}\right)$$

Which implies that $Q_t := \theta_t^\top \phi$ is **point-wise accurate:**

$$\|Q_t - Q^\star\|_\infty \leq H^2 d/\sqrt{N}$$

$$\theta_h^\top \phi(s,a)$$

Today's Question:

what happens when we do nonlinear function regression?

Point-wise prediction error guarantee is not possible anymore

Today's Question:

what happens when we do nonlinear function regression?

Point-wise prediction error guarantee is not possible anymore

Instead of aiming for point-wise guarantee,
We will focus on the average case (i.e., average over some distributions)

Outline

1. Setting: Assumptions

2. Algorithm: Fitted Q Iteration

2. Guarantee and Proof sketch

Setting

1. Infinite horizon Discounted MDPs $\gamma \in (0,1)$

s,a ~ ν

2. A given sampling distribution $\nu \in \Delta(S \times A)$ reset system to (s,a)

3. Function class $\mathscr{F} = \{f : S \times A \mapsto [0,1/(1-\gamma)]\}$

Q^*

Key Assumptions

1. Sampling distribution ν has full coverage (i.e., diverse):

$$\max_{\pi} \max_{s,a} \frac{d^{\pi}(s,a)}{\nu(s,a)} \leq C < \infty$$

Key Assumptions

1. Sampling distribution ν has full coverage (i.e., diverse):

Necessary for today's Alg

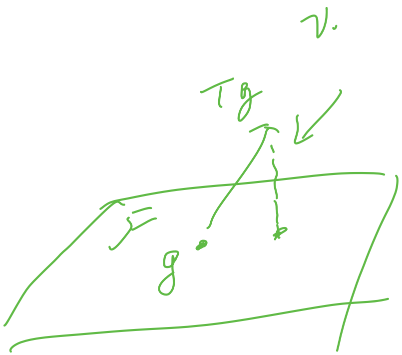
$$\max_{\pi} \max_{s,a} \frac{d^{\pi}(s,a)}{\nu(s,a)} \leq C < \infty$$

Key Assumptions

1. Sampling distribution ν has full coverage (i.e., diverse):

Necessary for today's Alg

$$\max_{\pi} \max_{s,a} \frac{d^{\pi}(s,a)}{\nu(s,a)} \leq C < \infty$$



2. Small inherent Bellman error, i.e., near Bellman Completion (note it's averaged over ν):

$$\max_{g \in \mathcal{F}} \min_{f \in \mathcal{F}} \mathbb{E}_{s,a \sim \nu} \left(f(s,a) - \mathcal{T} g(s,a) \right)^2 \leq \epsilon_{approx,\nu}$$

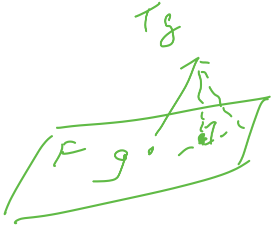
$$\forall g \in \mathcal{F}: \min_{f \in \mathcal{F}} \mathbb{E}_{s,a \sim \nu} \left(f(s,a) - \mathcal{T} g(s,a) \right)^2 \leq \epsilon_{app,\nu}$$

Key Assumptions

1. Sampling distribution ν has full coverage (i.e., diverse):

Necessary for today's Alg

$$\max_{\pi} \max_{s,a} \frac{d^{\pi}(s,a)}{\nu(s,a)} \leq C < \infty$$



2. Small inherent Bellman error, i.e., near Bellman Completion (note it's averaged over ν):

$$\max_{g \in \mathcal{F}} \min_{f \in \mathcal{F}} \mathbb{E}_{s,a \sim \nu} \left(f(s,a) - \mathcal{T} g(s,a) \right)^2 \leq \epsilon_{approx,\nu}$$

Necessary in general (we saw realizability itself is not enough)

Outline

✓ 1. Setting: Assumptions

2. Algorithm: Fitted Q Iteration

2. Guarantee and Proof sketch

The FQI Algorithm

1. Sample data points in an i.i.d fashion:

$$\mathscr{D} = \{s, a, r, s'\}, \quad (s, a) \sim \nu, r = r(s, a), s' \sim P(\cdot \mid s, a)$$

The FQI Algorithm

1. Sample data points in an i.i.d fashion:

$$\mathscr{D} = \{s, a, r, s'\}, \quad (s, a) \sim \nu, r = r(s, a), s' \sim P(\cdot \mid s, a)$$

2. Initialize $f_0 \in \mathscr{F}$, and iterate:

$$f_{t+1} = \arg\min_{f \in \mathscr{F}} \sum_{s,a,r,s' \in \mathscr{D}} \left(f(s, a) - r - \gamma \max_{a'} f_t(s', a') \right)^2$$

t

$\approx Q^*$

$$\approx r(s,a) + \gamma \mathop{E}_{s' \sim P(s,a)} \max_{a'} Q^*(s', a')$$

The FQI Algorithm

1. Sample data points in an i.i.d fashion:

$$\mathscr{D} = \{s, a, r, s'\}, \quad (s, a) \sim \nu, r = r(s, a), s' \sim P(\cdot \mid s, a)$$

2. Initialize $f_0 \in \mathscr{F}$, and iterate:

$$f_{t+1} = \arg\min_{f \in \mathscr{F}} \sum_{s,a,r,s' \in \mathscr{D}} \left(f(s, a) - r - \gamma \max_{a'} f_t(s', a') \right)^2$$

3. After K iterations, return $\pi(s) = \arg\max_a f_K(s, a), \forall s$

The FQI Algorithm

1. Sample data points in an i.i.d fashion:

$$\mathcal{D} = \{s, a, r, s'\}, \quad (s, a) \sim \nu, r = r(s, a), s' \sim P(\cdot \,|\, s, a)$$

2. Initialize $f_0 \in \mathcal{F}$, and iterate:

$$f_{t+1} = \arg\min_{f \in \mathcal{F}} \sum_{s,a,r,s' \in \mathcal{D}} \left(f(s, a) - r - \gamma \max_{a'} f_t(s', a') \right)^2$$

3. After K iterations, return $\pi(s) = \arg\max_{a} f_K(s, a), \forall s$

(Note: the algorithmic idea here is similar to DQNs [Deepmind 15]

Why we could expect it to work…

$$f_{t+1} = \arg\min_{f \in \mathcal{F}} \sum_{s,a,r,s' \in \mathcal{D}} \left(f(s,a) - r - \gamma \max_{a'} f_t(s',a') \right)^2$$

Why we could expect it to work…

$$f_{t+1} = \arg\min_{f \in \mathcal{F}} \sum_{s,a,r,s' \in \mathcal{D}} \left(f(s,a) - r - \gamma \max_{a'} f_t(s',a') \right)^2$$

$$y := r(s,a) + \gamma \max_{a'} f_t(s',a')$$

Why we could expect it to work…

$$f_{t+1} = \arg\min_{f\in\mathcal{F}} \sum_{s,a,r,s'\in\mathcal{D}} \left(f(s,a) - r - \gamma \max_{a'} f_t(s',a') \right)^2$$

$$y := r(s,a) + \gamma \max_{a'} f_t(s',a')$$

Bayes optimal: $\underbrace{r(s,a) + \gamma \mathbb{E}_{s'\sim P(\cdot|s,a)} \max_{a} f_t(s',a')}_{(\mathcal{T}f_t)(s,a)}$

Why we could expect it to work…

$$f_{t+1} = \arg\min_{f \in \mathcal{F}} \sum_{s,a,r,s' \in \mathcal{D}} \left(f(s,a) - r - \gamma \max_{a'} f_t(s',a') \right)^2$$

$$y := r(s,a) + \gamma \max_{a'} f_t(s',a')$$

Bayes optimal: $\underbrace{r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} \max_{a'} f_t(s',a')}_{(\mathcal{T} f_t)(s,a)}$

$\mathcal{T} f_t \in \mathcal{F}$

1. Near Bellman completion means regression target $\mathcal{T} f_t$ nearly belongs to $\mathcal{F}$

$$\mathbb{E}_{s,a \sim \nu} \left(f_{t+1}(s,a) - \mathcal{T} f_t(s,a) \right)^2 \approx \frac{1}{N} + \epsilon_{approx,\nu}$$

Why we could expect it to work…

$$f_{t+1} = \arg\min_{f\in\mathcal{F}} \sum_{s,a,r,s'\in\mathcal{D}} \left(f(s,a) - r - \gamma \max_{a'} f_t(s',a') \right)^2$$

$$y := r(s,a) + \gamma \max_{a'} f_t(s',a')$$

Bayes optimal: $\underbrace{r(s,a) + \gamma \mathbb{E}_{s'\sim P(\cdot|s,a)} \max_{a} f_t(s',a')}_{(\mathcal{T}f_t)(s,a)}$

1. Near Bellman completion means regression target $\mathcal{T}f_t$ nearly belongs to $\mathcal{F}$

$$\mathbb{E}_{s,a\sim\nu} \left(f_{t+1}(s,a) - \mathcal{T}f_t(s,a) \right)^2 \approx \frac{1}{N} + \epsilon_{approx,\nu}$$

2. $f_{t+1} \approx \mathcal{T}f_t$ (under the diverse ν), i.e., it's like Value Iteration, we could hope for a convergence

Outline

1. Setting: Assumptions

2. Algorithm: Fitted Q Iteration

2. Guarantee and Proof sketch

Theorem

Theorem: Fix iteration number K, w/ probability at least $1-\delta$,

$$V^\star - V^\pi \le O\left(\frac{1}{(1-\gamma)^4}\sqrt{\frac{C\ln(|\mathcal{F}|K/\delta)}{N}} + \frac{1}{(1-\gamma)^3}\sqrt{C\epsilon_{approx,\nu}}\right) + \frac{2\gamma^K}{(1-\gamma)^2}$$

Theorem

Theorem: Fix iteration number K, w/ probability at least $1 - \delta$,

$$V^\star - V^\pi \leq O\left(\frac{1}{(1-\gamma)^4} \sqrt{\frac{C \ln(|\mathscr{F}| K/\delta)}{N}} + \frac{1}{(1-\gamma)^3} \sqrt{C \epsilon_{approx,\nu}} \right) + \frac{2\gamma^K}{(1-\gamma)^2}$$

Statistical error related to regression

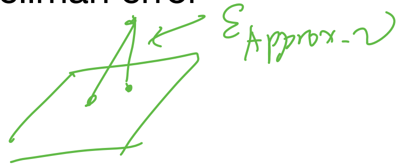
Theorem

Theorem: Fix iteration number K, w/ probability at least $1 - \delta$,

$$V^\star - V^\pi \leq O\left(\frac{1}{(1-\gamma)^4} \sqrt{\frac{C \ln(|\mathcal{F}| K/\delta)}{N}} + \frac{1}{(1-\gamma)^3} \sqrt{C \epsilon_{approx,\nu}} \right) + \frac{2\gamma^K}{(1-\gamma)^2}$$

Statistical error related to regression

Inherent Bellman error



Theorem

Theorem: Fix iteration number K, w/ probability at least $1 - \delta$,

$$V^\star - V^\pi \leq O\left(\frac{1}{(1-\gamma)^4} \sqrt{\frac{C \ln(|\mathcal{F}| K/\delta)}{N}} + \frac{1}{(1-\gamma)^3} \sqrt{C \epsilon_{approx,\nu}} \right) + \frac{2\gamma^K}{(1-\gamma)^2}$$

Statistical error related to regression

Inherent Bellman error

VI-style Convergence rate

$$C := \max_{\pi} \max_{s,a} \frac{d^\pi(s,a)}{\nu(s,a)}$$

Step 1:

Least Squares regression ensure near Bellman consistency (averaged over ν)

Standard Generalization Bound for regression:

Step 1:

Least Squares regression ensure near Bellman consistency (averaged over ν)

Standard Generalization Bound for regression:

Given $\{x_i, y_i\}_{i=1}^{N}$, $(x_i, y_i) \sim \nu$, $y_i = f^\star(x_i) + \epsilon_i$, where $|y_i| \leq Y$, $\|f^\star\|_\infty \leq Y$,

Step 1:

Least Squares regression ensure near Bellman consistency (averaged over ν)

Standard Generalization Bound for regression:

Given $\{x_i, y_i\}_{i=1}^{N}$, $(x_i, y_i) \sim \nu$, $y_i = f^\star(x_i) + \epsilon_i$, where $|y_i| \leq Y$, $\|f^\star\|_\infty \leq Y$,

a function class $\mathcal{F} = \{f : \mathcal{X} \mapsto [-Y, Y]\}$, where $\min_{f \in \mathcal{F}} \mathbb{E}_{x\sim\nu}(f(x) - f^\star(x))^2 \leq \varepsilon$

Step 1:

Least Squares regression ensure near Bellman consistency (averaged over ν)

$\mathbb{E}[\epsilon_i] = 0$

Standard Generalization Bound for regression:

Given $\{x_i, y_i\}_{i=1}^{N}$, $(x_i, y_i) \sim \nu$, $y_i = f^\star(x_i) + \epsilon_i$, where $|y_i| \leq Y$, $\|f^\star\|_\infty \leq Y$,

a function class $\mathcal{F} = \{f : \mathcal{X} \mapsto [-Y, Y]\}$, where $\min_{f \in \mathcal{F}} \mathbb{E}_{x \sim \nu}(f(x) - f^\star(x))^2 \leq \varepsilon$

Denote $\hat{f} := \arg\min_{f \in \mathcal{F}} \sum_{i=1}^{N} (f(x_i) - y_i)^2$ as the least square minimizer, then w/ prob $1 - \delta$:

$$\mathbb{E}_{x \sim \nu}\left(\hat{f}(x) - f^\star(x)\right)^2 \leq O\left(\frac{Y^2 \ln(|\mathcal{F}|/\delta)}{N} + \varepsilon\right)$$

Step 1:

Least Squares regression ensure near Bellman consistency (averaged over ν)

1. Recall FQI's regression problem:

$$f_{t+1} = \arg\min_{f \in \mathcal{F}} \sum_{s,a,r,s' \in \mathcal{D}} \left(f(s,a) - r - \gamma \max_{a'} f_t(s',a') \right)^2$$

$x := (s,a)$

y

Step 1:

Least Squares regression ensure near Bellman consistency (averaged over ν)

1. Recall FQI's regression problem:

$$f_{t+1} = \arg\min_{f \in \mathcal{F}} \sum_{s,a,r,s' \in \mathcal{D}} \left(f(s,a) - r - \gamma \max_{a'} f_t(s',a') \right)^2$$

$$r + \gamma \mathbb{E}_{s' \sim P_{s,a}} \max_{a'} f_t(s',a')$$

2. Here Bayes optimal is $f^\star := \mathcal{T} f_t$,

Step 1:

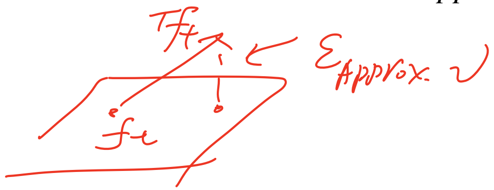
Least Squares regression ensure near Bellman consistency (averaged over ν)

1. Recall FQI's regression problem:

$$f_{t+1} = \arg\min_{f \in \mathcal{F}} \sum_{s,a,r,s' \in \mathcal{D}} \left(f(s,a) - r - \gamma \max_{a'} f_t(s',a') \right)^2$$

2. Here Bayes optimal is $f^\star := \mathcal{T} f_t$,

3. Via small inherent BE, we know that $\min_{f \in \mathcal{F}} \mathbb{E}_{s,a \sim \nu} (f(s,a) - \mathcal{T} f_t(s,a))^2 \le \epsilon_{approx,\nu}$



Step 1:

Least Squares regression ensure near Bellman consistency (averaged over ν)

1. Recall FQI's regression problem:

$$f_{t+1} = \arg\min_{f\in\mathcal{F}} \sum_{s,a,r,s'\in\mathcal{D}} \left(f(s,a) - \left(r - \gamma \max_{a'} f_t(s',a')\right)\right)^2$$

$\left[-1-\frac{\gamma}{1-\gamma},\ 1+\frac{\gamma}{1-\gamma}\right]$

2. Here Bayes optimal is $f^\star := \mathcal{T} f_t$,

3. Via small inherent BE, we know that $\min_{f\in\mathcal{F}} \mathbb{E}_{s,a\sim\nu}(f(s,a) - \mathcal{T} f_t(s,a))^2 \le \epsilon_{approx,\nu}$

1+2+3 => $\mathbb{E}_{s,a\sim\nu}(f_{t+1}(s,a) - \mathcal{T} f_t(s,a))^2 \le \frac{1}{(1-\gamma)^2} \frac{\ln(|\mathcal{F}|/\delta)}{N} + \epsilon_{approx,\nu}$

$f^\star$

Step 1:

Least Squares regression ensure near Bellman consistency (averaged over ν)

1. Recall FQI's regression problem:

$$f_{t+1} = \arg\min_{f \in \mathcal{F}} \sum_{s,a,r,s' \in \mathcal{D}} \left(f(s,a) - r - \gamma \max_{a'} f_t(s',a') \right)^2$$

2. Here Bayes optimal is $f^\star := \mathcal{T} f_t$,

3. Via small inherent BE, we know that $\min_{f \in \mathcal{F}} \mathbb{E}_{s,a\sim\nu}(f(s,a) - \mathcal{T} f_t(s,a))^2 \le \epsilon_{approx,\nu}$

1+2+3 => $\mathbb{E}_{s,a\sim\nu}(f_{t+1}(s,a) - \mathcal{T} f_t(s,a))^2 \le \frac{1}{(1-\gamma)^2} \frac{\ln(|\mathcal{F}|/\delta)}{N} + \epsilon_{approx,\nu}$

$$\mathbb{E}_{s,a\sim\nu} |f_{t+1}(s,a) - \mathcal{T} f_t(s,a)| \le \underbrace{\sqrt{\frac{1}{(1-\gamma)^2} \frac{\ln(|\mathcal{F}|/\delta)}{N} + \epsilon_{approx,\nu}}}_{:=\epsilon_{regress}}$$

$\left(E|x| \le \sqrt{E[x^2]} \right)$

Step 2:

Near Bellman consistency implies convergence

Consider any state-action distribution $\beta(s, a)$ (induced by some policy)

$$\sqrt{\mathbb{E}_{s,a\sim\beta}(f_t(s,a) - Q^\star(s,a))^2} := \|f_t - Q^\star\|_{\beta,2}$$

Step 2:

Near Bellman consistency implies convergence

Consider any state-action distribution $\beta(s, a)$ (induced by some policy)

$$\sqrt{\mathbb{E}_{s,a\sim\beta}(f_t(s,a) - Q^\star(s,a))^2} := \|f_t - Q^\star\|_{\beta,2}$$

$$\leq \|f_t - \mathcal{T} f_{t-1}\|_{2,\beta} + \|\mathcal{T} f_{t-1} - Q^\star\|_{2,\beta}$$

β

Step 2:

Near Bellman consistency implies convergence

Consider any state-action distribution $\beta(s, a)$ (induced by some policy)

$$\sqrt{\mathbb{E}_{s,a\sim\beta}(f_t(s,a) - Q^\star(s,a))^2} := \|f_t - Q^\star\|_{\beta,2}$$

$\to 2,\beta$

$$\leq \|f_t - \mathcal{T} f_{t-1}\|_{2,\beta} + \|\mathcal{T} f_{t-1} - Q^\star\|_{2,\beta}$$

β instead of ν

$\hookrightarrow$ Regression at Round $t-1$

Dist-change and Coverage condition

$$\mathbb{E}_{s,a\sim\beta}(f_t(s,a) - \mathcal{T} f_{t-1}(s,a))^2 = \mathbb{E}_{s,a\sim\nu} \frac{\beta(s,a)}{\nu(s,a)} (f_t(s,a) - \mathcal{T} f_{t-1}(s,a))^2$$

$$\leq \max_{s,a} \frac{\beta(s,a)}{\nu(s,a)} \mathbb{E}_{s,a\sim\nu}(f_t - \mathcal{T} f_{t-1})^2$$

$\max_{s,a} \frac{\beta(s,a)}{\nu(s,a)} \leq C$

Step 2:

Near Bellman consistency implies convergence

Consider any state-action distribution $\beta(s, a)$ (induced by some policy)

$$\sqrt{\mathbb{E}_{s,a\sim\beta}(f_t(s,a) - Q^\star(s,a))^2} := \|f_t - Q^\star\|_{\beta,2}$$

$$\leq \|f_t - \mathcal{T}f_{t-1}\|_{2,\beta} + \|\mathcal{T}f_{t-1} - Q^\star\|_{2,\beta}$$

$$\leq \sqrt{C}\|f_t - \mathcal{T}f_{t-1}\|_{2,\nu} + \|\mathcal{T}f_{t-1} - Q^\star\|_{2,\beta}$$

Dist-change and Coverage condition

$\sqrt{C}\,\varepsilon_{regress}$

$\mathcal{T}Q^\star = Q^\star$

$\mathcal{T}f_{t-1}(s,a) - \mathcal{T}Q^\star(s,a)$

$= \gamma \mathbb{E}_{s'\sim P(s,a)}\left(\max_{a'} f_{t-1}(s',a') - \max_{a'} Q^\star(s',a')\right)$

Step 2:

Near Bellman consistency implies convergence

Consider any state-action distribution $\beta(s,a)$ (induced by some policy)

$$\sqrt{\mathbb{E}_{s,a\sim\beta}(f_t(s,a) - Q^\star(s,a))^2} := \|f_t - Q^\star\|_{\beta,2}$$

$$\leq \|f_t - \mathcal{T} f_{t-1}\|_{2,\beta} + \|\mathcal{T} f_{t-1} - Q^\star\|_{2,\beta}$$

$$\leq \sqrt{C}\|f_t - \mathcal{T} f_{t-1}\|_{2,\nu} + \|\mathcal{T} f_{t-1} - Q^\star\|_{2,\beta}$$

Dist-change and Coverage condition

$$\leq \sqrt{C}\epsilon_{regress} + \gamma\sqrt{\mathbb{E}_{s,a\sim\beta}\left(\mathbb{E}_{s'\sim P(\cdot|s,a)}\left(\max_{a'} f_{t-1}(s',a') - \max_{a'} Q^\star(s',a')\right)\right)^2}$$

Step 2:

Near Bellman consistency implies convergence

Consider any state-action distribution $\beta(s, a)$ (induced by some policy)

$$\sqrt{\mathbb{E}_{s,a\sim\beta}(f_t(s,a) - Q^\star(s,a))^2} := \|f_t - Q^\star\|_{\beta,2}$$

$$\leq \|f_t - \mathcal{T} f_{t-1}\|_{2,\beta} + \|\mathcal{T} f_{t-1} - Q^\star\|_{2,\beta}$$

$$\leq \sqrt{C}\|f_t - \mathcal{T} f_{t-1}\|_{2,\nu} + \|\mathcal{T} f_{t-1} - Q^\star\|_{2,\beta}$$

Dist-change and Coverage condition

$$\left(E(x)\right)^2 \leq E\left(x^2\right)$$

$$\leq \sqrt{C}\epsilon_{regress} + \gamma \sqrt{\mathbb{E}_{s,a\sim\beta}\left(\mathbb{E}_{s'\sim P(\cdot|s,a)}\left(\max_{a'} f_{t-1}(s',a') - \max_{a'} Q^\star(s',a')\right)\right)^2}$$

$$\leq \sqrt{C}\epsilon_{regress} + \gamma \sqrt{\underbrace{\mathbb{E}_{s,a\sim\beta}\mathbb{E}_{s'\sim P(\cdot|s,a)} \max_{a'}}_{:=\beta'(s',a')} \left(f_{t-1}(s',a') - Q^\star(s',a')\right)^2}$$

Step 2:

Near Bellman consistency implies convergence

Consider any state-action distribution $\beta(s,a)$ (induced by some policy)

$$\sqrt{\mathbb{E}_{s,a\sim\beta}(f_t(s,a) - Q^\star(s,a))^2} := \|f_t - Q^\star\|_{\beta,2}$$

$$\leq \|f_t - \mathcal{T}f_{t-1}\|_{2,\beta} + \|\mathcal{T}f_{t-1} - Q^\star\|_{2,\beta}$$

$$\leq \sqrt{C}\|f_t - \mathcal{T}f_{t-1}\|_{2,\nu} + \|\mathcal{T}f_{t-1} - Q^\star\|_{2,\beta}$$

Dist-change and Coverage condition

$$\leq \sqrt{C}\epsilon_{regress} + \gamma\sqrt{\mathbb{E}_{s,a\sim\beta}\left(\mathbb{E}_{s'\sim P(\cdot|s,a)}\left(\max_{a'} f_{t-1}(s',a') - \max_{a'} Q^\star(s',a')\right)\right)^2}$$

$$\leq \sqrt{C}\epsilon_{regress} + \gamma\sqrt{\underbrace{\mathbb{E}_{s,a\sim\beta}\mathbb{E}_{s'\sim P(\cdot|s,a)}\max_{a'}}_{:=\beta'(s',a')}\left(f_{t-1}(s',a') - Q^\star(s',a')\right)^2} = \sqrt{C}\epsilon_{regress} + \gamma\|f_{t-1} - Q^\star\|_{2,\beta'}$$

Recursion

Step 2:

Near Bellman consistency implies convergence

Consider ANY state-action distribution $\beta(s, a)$ (induced by some policy)

$$\sqrt{\mathbb{E}_{s,a\sim\beta}(f_t(s,a) - Q^\star(s,a))^2} := \|f_t - Q^\star\|_{\beta,2} \quad \leq \sqrt{C}\epsilon_{regress} + \gamma\|f_{t-1} - Q^\star\|_{2,\beta'}$$

Step 2:

Near Bellman consistency implies convergence

Consider ANY state-action distribution $\beta(s, a)$ (induced by some policy)

$$\sqrt{\mathbb{E}_{s,a\sim\beta}(f_t(s,a) - Q^\star(s,a))^2} := \|f_t - Q^\star\|_{\beta,2} \leq \sqrt{C}\epsilon_{regress} + \gamma\|f_{t-1} - Q^\star\|_{2,\beta'}$$

$$\leq \sqrt{C}\epsilon_{regress} + \gamma\left[\sqrt{C}\epsilon_{regress} + \gamma\|f_{t-2} - Q^\star\|_{2,\beta''}\right]$$

Step 2:

Near Bellman consistency implies convergence

Consider ANY state-action distribution $\beta(s,a)$ (induced by some policy)

$$\sqrt{\mathbb{E}_{s,a\sim\beta}(f_t(s,a) - Q^\star(s,a))^2} := \|f_t - Q^\star\|_{\beta,2} \leq \sqrt{C}\epsilon_{regress} + \gamma\|f_{t-1} - Q^\star\|_{2,\beta'}$$

$$\leq \sqrt{C}\epsilon_{regress} + \gamma\left[\sqrt{C}\epsilon_{regress} + \gamma\|f_{t-2} - Q^\star\|_{2,\beta''}\right]$$

$$\leq \sqrt{C}\epsilon_{regress}\left(1 + \gamma + \ldots + \gamma^k\right) + \gamma^k\|f_0 - Q^\star\|_{2,\tilde{\beta}}$$

$\leq \frac{1}{1-\gamma}$

Step 2:

Near Bellman consistency implies convergence

Consider ANY state-action distribution $\beta(s,a)$ (induced by some policy)

$$\sqrt{\mathbb{E}_{s,a\sim\beta}(f_t(s,a) - Q^\star(s,a))^2} := \|f_t - Q^\star\|_{\beta,2} \leq \sqrt{C}\epsilon_{regress} + \gamma\|f_{t-1} - Q^\star\|_{2,\beta'}$$

$$\leq \sqrt{C}\epsilon_{regress} + \gamma\left[\sqrt{C}\epsilon_{regress} + \gamma\|f_{t-2} - Q^\star\|_{2,\beta''}\right]$$

$$\leq \sqrt{C}\epsilon_{regress}\left(1 + \gamma + \ldots + \gamma^k\right) + \gamma^k\|f_0 - Q^\star\|_{2,\tilde{\beta}}$$

$$\leq \frac{\sqrt{C}\epsilon_{regress}}{1-\gamma} + \gamma^k/(1-\gamma)$$

Step 3:

Turn in error $\|f_k - Q^\star\|_{2,\beta}$ to policy π^k performance

Denote $\pi^k(s) = \arg\max_a f_k(s, a)$

$$V^\star - V^{\pi^k}$$

Step 3:

Turn in error $\|f_k - Q^\star\|_{2,\beta}$ to policy π^k performance

Denote $\pi^k(s) = \arg\max_a f_k(s, a)$

$$V^\star - V^{\pi^k} = \mathbb{E}_{s_0 \sim \mu}\left[Q^\star(s_0, \pi^\star(s_0)) - Q^{\pi^k}(s_0, \pi^k(s_0))\right]$$

Step 3:

Turn in error $\|f_k - Q^\star\|_{2,\beta}$ to policy π^k performance

Denote $\pi^k(s) = \arg\max_a f_k(s, a)$

$$V^\star - V^{\pi^k} = \mathbb{E}_{s_0 \sim \mu}\left[Q^\star(s_0, \pi^\star(s_0)) - Q^{\pi^k}(s_0, \pi^k(s_0))\right]$$

$$= \mathbb{E}_{s_0 \sim \mu}\left[Q^\star(s_0, \pi^\star(s_0)) - Q^\star(s_0, \pi^k(s_0)) + Q^\star(s_0, \pi^k(s_0)) - Q^{\pi^k}(s_0, \pi^k(s_0))\right]$$

Step 3:

Turn in error $\|f_k - Q^\star\|_{2,\beta}$ to policy π^k performance

Denote $\pi^k(s) = \arg\max_a f_k(s, a)$

$$V^\star - V^{\pi^k} = \mathbb{E}_{s_0 \sim \mu} \left[Q^\star(s_0, \pi^\star(s_0)) - Q^{\pi^k}(s_0, \pi^k(s_0)) \right]$$

$$= \mathbb{E}_{s_0 \sim \mu} \left[Q^\star(s_0, \pi^\star(s_0)) - Q^\star(s_0, \pi^k(s_0)) + Q^\star(s_0, \pi^k(s_0)) - Q^{\pi^k}(s_0, \pi^k(s_0)) \right]$$

$$= \mathbb{E}_{s_0 \sim \mu} \left[Q^\star(s_0, \pi^\star(s_0)) - Q^\star(s_0, \pi^k(s_0)) \right] + \mathbb{E}_{s_0 \sim \mu, a_0 \sim \pi^k(s_0)} \left[Q^\star(s_0, a_0) - Q^{\pi^k}(s_0, a_0) \right]$$

Step 3:

Turn in error $\|f_k - Q^\star\|_{2,\beta}$ to policy π^k performance

Denote $\pi^k(s) = \arg\max_a f_k(s, a)$

$$V^\star - V^{\pi^k} = \mathbb{E}_{s_0 \sim \mu}\left[Q^\star(s_0, \pi^\star(s_0)) - Q^{\pi^k}(s_0, \pi^k(s_0))\right]$$

$$= \mathbb{E}_{s_0 \sim \mu}\left[Q^\star(s_0, \pi^\star(s_0)) - Q^\star(s_0, \pi^k(s_0)) + Q^\star(s_0, \pi^k(s_0)) - Q^{\pi^k}(s_0, \pi^k(s_0))\right]$$

$$= \mathbb{E}_{s_0 \sim \mu}\left[Q^\star(s_0, \pi^\star(s_0)) - Q^\star(s_0, \pi^k(s_0))\right] + \mathbb{E}_{s_0 \sim \mu, a_0 \sim \pi^k(s_0)}\left[Q^\star(s_0, a_0) - Q^{\pi^k}(s_0, a_0)\right]$$

$$= \mathbb{E}_{s_0 \sim \mu}\left[Q^\star(s_0, \pi^\star(s_0)) - Q^\star(s_0, \pi^k(s_0))\right] + \gamma \mathbb{E}_{s_0 \sim \mu, a_0 \sim \pi^k(s_0)}\left[\mathbb{E}_{s_1 \sim P(s_0, a_0)}\left(V^\star(s_1) - V^{\pi^k}(s_1)\right)\right]$$

Step 3:

Turn in error $\|f_k - Q^\star\|_{2,\beta}$ to policy π^k performance

Denote $\pi^k(s) = \arg\max_a f_k(s,a)$

$$V^\star - V^{\pi^k} = \mathbb{E}_{s_0\sim\mu}\left[Q^\star(s_0,\pi^\star(s_0)) - Q^{\pi^k}(s_0,\pi^k(s_0))\right]$$

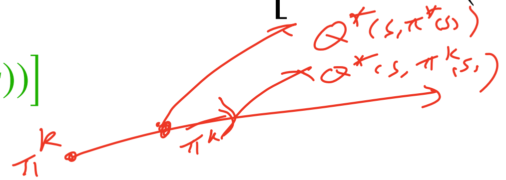
$$= \mathbb{E}_{s_0\sim\mu}\left[Q^\star(s_0,\pi^\star(s_0)) - Q^\star(s_0,\pi^k(s_0)) + Q^\star(s_0,\pi^k(s_0)) - Q^{\pi^k}(s_0,\pi^k(s_0))\right]$$

$$= \mathbb{E}_{s_0\sim\mu}\left[Q^\star(s_0,\pi^\star(s_0)) - Q^\star(s_0,\pi^k(s_0))\right] + \mathbb{E}_{s_0\sim\mu, a_0\sim\pi^k(s_0)}\left[Q^\star(s_0,a_0) - Q^{\pi^k}(s_0,a_0)\right]$$

$$= \mathbb{E}_{s_0\sim\mu}\left[Q^\star(s_0,\pi^\star(s_0)) - Q^\star(s_0,\pi^k(s_0))\right] + \gamma\mathbb{E}_{s_0\sim\mu, a_0\sim\pi^k(s_0)}\left[\mathbb{E}_{s_1\sim P(s_0,a_0)}\left(V^\star(s_1) - V^{\pi^k}(s_1)\right)\right]$$

$$= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi^k}}\left[Q^\star(s,\pi^\star(s)) - Q^\star(s,\pi^k(s))\right]$$

$s \sim d^{\pi^k}$



Step 3:

Turn in error $\|f_k - Q^\star\|_{2,\beta}$ to policy π^k performance

Denote $\pi^k(s) = \arg\max_a f_k(s,a)$

We know f_k is close to $Q^\star$ (averaged over any distribution):

$$V^\star - V^{\pi^k} = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi^k}}\left[Q^\star(s,\pi^\star(s)) - Q^\star(s,a)\right]$$

Step 3:

Turn in error $\|f_k - Q^\star\|_{2,\beta}$ to policy π^k performance

Denote $\pi^k(s) = \arg\max_a f_k(s, a)$

We know f_k is close to $Q^\star$ (averaged over any distribution):

$$V^\star - V^{\pi^k} = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi^k}}\left[Q^\star(s, \pi^\star(s)) - Q^\star(s, a)\right]$$

$$\le \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi^k}}\left[Q^\star(s, \pi^\star(s)) - f_k(s, \pi^\star(s)) + f_k(s, \pi^k(s)) - Q^\star(s, a)\right]$$

≥ 0

Step 3:

Turn in error $\|f_k - Q^\star\|_{2,\beta}$ to policy π^k performance

Denote $\pi^k(s) = \arg\max_a f_k(s, a)$

We know f_k is close to $Q^\star$ (averaged over any distribution):

$$V^\star - V^{\pi^k} = \frac{1}{1-\gamma} \mathbb{E}_{s \sim d^{\pi^k}} \left[Q^\star(s, \pi^\star(s)) - Q^\star(s, a) \right]$$

$$\leq \frac{1}{1-\gamma} \mathbb{E}_{s \sim d^{\pi^k}} \left[Q^\star(s, \pi^\star(s)) - f_k(s, \pi^\star(s)) + f_k(s, \pi^k(s)) - Q^\star(s, a) \right]$$

$$\leq \frac{1}{1-\gamma} \left[\sqrt{\mathbb{E}_{s \sim d^{\pi^k}} \left(Q^\star(s, \pi^\star(s)) - f_k(s, \pi^\star(s)) \right)^2} + \sqrt{\mathbb{E}_{s \sim d^{\pi^k}} \left(f_k(s, \pi^k(s)) - Q^\star(s, \pi^k(s)) \right)^2} \right]$$

$$E[|X|] \leq \sqrt{E X^2}$$

Step 3:

Turn in error $\|f_k - Q^\star\|_{2,\beta}$ to policy π^k performance

Denote $\pi^k(s) = \arg\max_a f_k(s,a)$

We know f_k is close to $Q^\star$ (averaged over any distribution):

$$V^\star - V^{\pi^k} = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi^k}}\left[Q^\star(s,\pi^\star(s)) - Q^\star(s,a)\right]$$

$$\leq \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi^k}}\left[Q^\star(s,\pi^\star(s)) - f_k(s,\pi^\star(s)) + f_k(s,\pi^k(s)) - Q^\star(s,a)\right]$$

$$\leq \frac{1}{1-\gamma}\left[\sqrt{\mathbb{E}_{s\sim d^{\pi^k}}\left(Q^\star(s,\pi^\star(s)) - f_k(s,\pi^\star(s))\right)^2} + \sqrt{\mathbb{E}_{s\sim d^{\pi^k}}\left(f_k(s,\pi^k(s)) - Q^\star(s,\pi^k(s))\right)^2}\right]$$

$$\leq \frac{2}{1-\gamma}\left(\frac{\sqrt{C}\epsilon_{regress}}{1-\gamma} + \frac{\gamma^k}{1-\gamma}\right)$$

To conclude:

$$V^\star - V^{\pi^k} \leq \frac{2}{1-\gamma}\left(\frac{\sqrt{C}\epsilon_{regress}}{1-\gamma} + \frac{\gamma^k}{1-\gamma}\right) \quad \text{where } \epsilon_{regress} = \sqrt{\frac{1}{(1-\gamma)^2}\frac{\ln(|\mathcal{F}|/\delta)}{N} + \epsilon_{approx,\nu}}$$

To conclude:

$$V^\star - V^{\pi^k} \leq \frac{2}{1-\gamma}\left(\frac{\sqrt{C}\epsilon_{regress}}{1-\gamma} + \frac{\gamma^k}{1-\gamma}\right)$$ where $\epsilon_{regress} = \sqrt{\frac{1}{(1-\gamma)^2}\frac{\ln(|\mathcal{F}|/\delta)}{N} + \epsilon_{approx,\nu}}$

1. Least square ensures we have near Bellman consistency under ν:

$$\|f_t - \mathcal{T} f_{t-1}\|_{2,\nu} \leq \epsilon_{regress}$$

To conclude:

$$V^\star - V^{\pi^k} \leq \frac{2}{1-\gamma}\left(\frac{\sqrt{C}\epsilon_{regress}}{1-\gamma} + \frac{\gamma^k}{1-\gamma}\right) \quad \text{where } \epsilon_{regress} = \sqrt{\frac{1}{(1-\gamma)^2}\frac{\ln(|\mathscr{F}|/\delta)}{N} + \epsilon_{approx,\nu}}$$

1. Least square ensures we have near Bellman consistency under ν:

$$\|f_t - \mathscr{T} f_{t-1}\|_{2,\nu} \leq \epsilon_{regress}$$

2. Near Bellman consistency under $\nu + \nu$ covers all other possible distributions β:

$$\|f_t - Q^\star\|_{2,\beta} \leq \left(\sqrt{C\epsilon_{regress}} + \gamma^k\right) \cdot \text{poly}(1/(1-\gamma))$$

To conclude:

$$V^\star - V^{\pi^k} \leq \frac{2}{1-\gamma}\left(\frac{\sqrt{C}\epsilon_{regress}}{1-\gamma} + \frac{\gamma^k}{1-\gamma}\right) \quad \text{where } \epsilon_{regress} = \sqrt{\frac{1}{(1-\gamma)^2}\frac{\ln(|\mathcal{F}|/\delta)}{N} + \epsilon_{approx,\nu}}$$

1. Least square ensures we have near Bellman consistency under ν:

$$\|f_t - \mathcal{T} f_{t-1}\|_{2,\nu} \leq \epsilon_{regress}$$

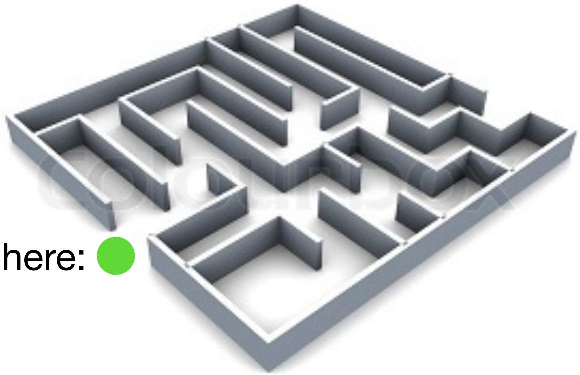
2. Near Bellman consistency under ν + ν covers all other possible distributions β:

$$\|f_t - Q^\star\|_{2,\beta} \leq \left(\sqrt{C\epsilon_{regress}} + \gamma^k\right) \cdot \text{poly}(1/(1-\gamma))$$

3. Like what we did in VI, turn f_t's approximation error to its policy's performance $(1/(1-\gamma)$ amplification):

Starting this Thursday:

Exploration!



You can only start/reset here:

1. Cannot reset everywhere (i.e., no generative model)

2. No such diverse ν distribution