Exploration in Linear MDPs

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CS 6789: Foundations of Reinforcement Learning

Recap: linear MDP definition

Finite horizon time-dependent episodic MDP $\mathscr{M} = \{S, A, H, \{r\}_h, \{P\}_h, s_0\}$

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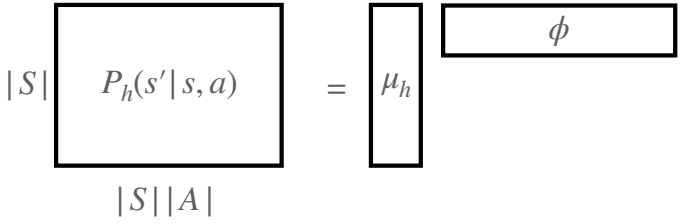
Low-Rank Decomposition:



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Finite horizon time-dependent episodic MDP $\mathscr{M} = \{S, A, H, \{r\}_h, \{P\}_h, s_0\}$

$$P_h(s' \mid s, a) = \mu_h^\star(s') \cdot \phi(s, a), \quad \mu_h^\star \in S \mapsto \mathbb{R}^d, \phi \in S \times A \mapsto \mathbb{R}^d$$

$$r(s, a) = \theta_h^\star \cdot \phi(s, a), \quad \theta_h^\star \in \mathbb{R}^d$$

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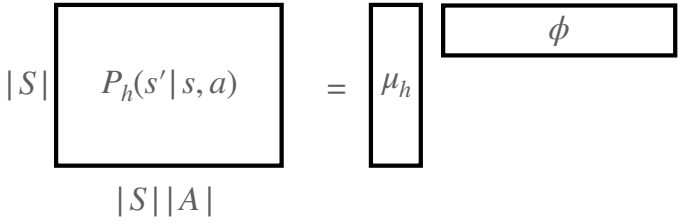

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Given dataset $\{s_h^i, a_h^i, s_{h+1}^i\}_{i=1}^{n-1}$

Ridge Linear Regression:

$$\min_{\mu} \sum_{i=1}^{n-1} \|\mu\phi(s_h^i, a_h^i) - \delta(s_{h+1}^i)\|_2^2 + \lambda\|\mu\|_F^2$$

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$$\widehat{\mu}_h^n = \sum_{i=1}^{n-1} \delta(s_{h+1}^i)\phi(s_h^i, a_h^i)^\top (\Lambda_h^n)^{-1}; \qquad \widehat{P}_h^n(\cdot \mid s,a) = \widehat{\mu}_h^n \phi(s,a)$$

Today:

Regret bound for the UCBVI algorithm for Linear MDP and its proof sketch

Outline:

1. Model fitting and its guarantee $(\hat{P}(\,.\,|\,s,a) - P(\,.\,|\,s,a))^{\top} V,$ for some fixed V

2. Covering argument to bound $(\hat{P}(\,.\,|\,s,a) - P(\,.\,|\,s,a))^{\top} V$, for ALL $V \in \mathcal{F}$

$\approx b(s,a)$

3. UCBVI revisit and its guarantee

1. Model Learning in Linear MDPs

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Cannot directly bound $\| \widehat{P}^n(\cdot \mid s,a) - P(\cdot \mid s,a) \|_1$ since $P(. \mid s,a)$ is in $\mathbb{R}^{|S|}$

$\approx \sqrt{\frac{S}{N(s,a)}}$

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As in tabular-UCBVI and Generative Model, we care **average model error:**

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As in tabular-UCBVI and Generative Model, we care **average model error:**

Consider a fixed function $V : S \mapsto [0,H]$, we can bound:

$$\left| \left(\widehat{P}_h^n(\cdot \mid s,a) - P_h(\cdot \mid s,a) \right) \cdot V \right|$$

1. Model Learning in Linear MDPs

Ridge Linear Regression: $\min_{\mu} \sum_{i=1}^{n-1} \|\mu\phi(s_h^i, a_h^i) - \delta(s_{h+1}^i)\|_2^2 + \|\mu\|_F^2$

$$\widehat{\mu}_h^n = \sum_{i=1}^{n-1} \delta(s_{h+1}^i)\phi(s_h^i, a_h^i)^\top (\Lambda_h^n)^{-1}, \quad \Lambda_h^n = \sum_{i=1}^{n-1} \phi(s_h^i, a_h^i)\phi(s_h^i, a_h^i)^\top + \lambda I$$

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Lemma [Model Average Error under a fixed V]:

Consider a fixed $V : S \to [0, H]$. With probability at least $1 - \delta$, for any s, a, h, n, we have:

$$\left| \left(\widehat{P}_h^n(\cdot \mid s, a) - P_h(\cdot \mid s, a) \right) \cdot V \right| \le \|\phi(s, a)\|_{(\Lambda_h^n)^{-1}} \times \left(2H \sqrt{\ln \frac{H \det(\Lambda_h^n)^{1/2} \det(\lambda I)^{-1/2}}{\delta}} + H\sqrt{\lambda d} \right)$$

$\ln\det(\Lambda_h^n) \approx d \ln(n)$

1. Model Learning in Linear MDPs: proof

$$\widehat{\mu}_h^n - \mu_h^\star = -\lambda \mu_h^\star \left(\Lambda_h^n\right)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top \left(\Lambda_h^n\right)^{-1}$$

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$\Rightarrow P_h(\cdot \mid s_h^i, a_h^i) + \varepsilon_h^i$

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$\mu_h^\star \phi(s_h^i, a_h^i)$

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$$= \mu_h^\star \left(\sum_{i=1}^{n-1} \phi(s_h^i, a_h^i) \phi(s_h^i, a_h^i)^\top \right) (\Lambda_h^n)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top (\Lambda_h^n)^{-1}$$

$+\lambda I - \lambda I$

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$$= \mu_h^\star - \lambda \mu_h^\star (\Lambda_h^n)^{-1} + \sum_{i=1}^{n-1} \epsilon_h^i \phi(s_h^i, a_h^i)^\top (\Lambda_h^n)^{-1}$$

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$\forall\ \phi(s,a)$:

$$\left((\widehat{\mu}_h^n - \mu_h^\star)\phi(s,a)\right)^\top V$$

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$(\Lambda_h^n)^{-\frac{1}{2}} \cdot (\Lambda_h^n)^{-\frac{1}{2}}$

1. Model Learning in Linear MDPs

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1. Model Learning in Linear MDPs

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$\forall v \in \mathbb{R}^{|S|}, \|v\|_\infty \le 1$

$\Rightarrow \|\mu_h^{*\top} v\|_2 \le \sqrt{d}$

Normalization assumption on $\mu_h^\star$

1. Model Learning in Linear MDPs

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1. Model Learning in Linear MDPs

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$\Lambda^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}}$

$= \|\phi(s,a)\|_{(\Lambda_h^n)^{-1}}$

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$$\left| \sum_{i=1}^{n-1} \phi(s,a)^\top (\Lambda_h^n)^{-1} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right| \leq \|\phi(s,a)^\top (\Lambda_h^n)^{-1/2}\|_2 \left\| \sum_{i=1}^{n-1} (\Lambda_h^n)^{-1/2} \phi(s_h^i, a_h^i)(\epsilon_h^i)^\top V \right\|_2$$

$$= \|\phi(s,a)\|_{(\Lambda_h^n)^{-1}} \left\| \sum_{i=1}^{n-1} \phi(s_h^i, a_h^i) \left((\epsilon_h^i)^\top V \right) \right\|_{(\Lambda_h^n)^{-1}}$$

1. Model Learning in Linear MDPs

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$$\mathbb{E}\left[(\epsilon_h^i)^\top V \,|\, \mathscr{H}_{i,h}\right] = 0, \quad |(\epsilon_h^i)^\top V| \le \|\epsilon_h^i\|_1 \|V\|_\infty \le 2H$$

1. Model Learning in Linear MDPs

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With prob $1-\delta$, $\forall n$

$$\mathbb{E}\left[(\epsilon_h^i)^\top V \,|\, \mathscr{H}_{i,h}\right] = 0, \quad |(\epsilon_h^i)^\top V| \le \|\epsilon_h^i\|_1 \|V\|_\infty \le 2H$$

1. Model Learning in Linear MDPs

Lemma [Model Average Error under a fixed V]:

Consider a fixed $V : S \to [0, H]$. With probability at least $1 - \delta$, for all s, a, n, h, we have:

$$\left| \left(\widehat{P}_h^n(\cdot \mid s,a) - P_h(\cdot \mid s,a) \right) \cdot V \right| \leq \|\phi(s,a)\|_{(\Lambda_h^n)^{-1}} \times \left(2H\sqrt{\ln \frac{H \det(\Lambda_h^n)^{1/2} \det(\lambda I)^{-1/2}}{\delta}} + H\sqrt{\lambda d} \right)$$

$$\approx 2H\sqrt{d} + H\sqrt{\lambda d}$$

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$$\det(\Lambda_h^n)^{1/2} \det(\lambda I)^{-1/2} \leq (n+\lambda)^{d/2} \lambda^{-d/2} = (N/\lambda + 1)^{d/2}$$

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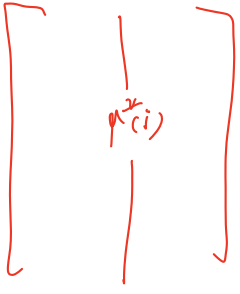
$$= \widetilde{O}\left(H\sqrt{d} + H\sqrt{\ln(1/\delta)} \right) \left\| \phi(s,a) \right\|_{(\Lambda_h^n)^{-1}}$$

1. Model learning: summary

$\mu^* \in \mathbb{R}^{|S| \times d}$



Lemma [Model Average Error under a fixed V]:

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$\phi(s,a)$ is one-hot

$$= \sqrt{\frac{1}{N(s,a)}}$$

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Lemma [Model Average Error under a fixed V]:

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Q: Can we get a uniform convergence argument for a function class $\mathcal{F}$?

Outline:

1. Model fitting and its guarantee $(\hat{P}(\,.\,|\,s,a) - P(\,.\,|\,s,a))^{\top} V,$ for some fixed V

2. Covering argument to bound $(\hat{P}(\,.\,|\,s,a) - P(\,.\,|\,s,a))^{\top} V$, for ALL $V \in \mathscr{F}$

3. UCBVI revisit and its guarantee

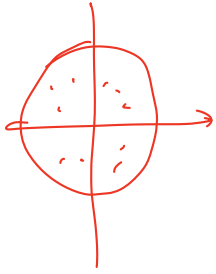
Detour: Covering Number

Consider the ball $\Theta = \{\theta : \theta \in \mathbb{R}^d, \|\theta\|_2 \leq R\}$.

Denote ϵ-**Net** as a subset $\mathcal{N}_\epsilon \subseteq \Theta$, such that $\forall \theta \in \Theta$:

$$\exists \theta' \in \mathcal{N}_\epsilon, \text{ s.t. } \|\theta' - \theta\|_2 \leq \epsilon.$$

Denote ϵ-**cover** as the smallest $\mathcal{N}_\epsilon$



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Lemma [Covering of Θ] We have $|\mathcal{N}_\epsilon| \leq (1 + 2R/\epsilon)^d$, and $\ln(|\mathcal{N}_\epsilon|) \leq d \ln(1 + 2R/\epsilon)$

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and for any $f_{\theta_1}, f_{\theta_2} \in \mathcal{F}$, $\|f_{\theta_1} - f_{\theta_2}\|_\infty \leq L\|\theta_1 - \theta_2\|_2$

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Then $(\epsilon/L)-$Net on Θ gives us an $\epsilon-$Net on $\mathcal{F}$ with $d(f_{\theta_1}, f_{\theta_2}) := \|f_{\theta_1} - f_{\theta_2}\|_\infty$

Detour: Covering Number and An Example

Consider a specific parameterization $\theta = (w, \beta, \Lambda)$,
$\Theta = \{(w, \beta, \Lambda) : \|w\|_2 \leq L, \beta \in [0,B], \sigma_{\min}(\Lambda) \geq \lambda\}$

$w \in \mathbb{R}^d$

$\beta \in \mathbb{R}$

$\Lambda \in \mathbb{R}^{d \times d}$

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$$f_{w,\beta,\Lambda}(s) := \min \left\{ \max_a \left(w^\top \phi(s,a) + \beta \sqrt{\phi(s,a)^\top \Lambda^{-1} \phi(s,a)} \right), H \right\}$$

$\approx Q^*$ $\quad$ $b(s,a)$

$\min_a Q^*(s,a)$

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Denote $\mathcal{F} = \{f_{w,\beta,\Lambda} : \|w\|_2 \le L, \beta \in [0,B], \sigma_{\min}(\Lambda) \ge \lambda\}$, **what's the covering number of $\mathcal{F}$ under ℓ_∞**

Detour: Covering Number

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Lemma: Denote $\mathscr{F} = \{f_{w,\beta,\Lambda} : \|w\|_2 \le L, \beta \in [0,B], \sigma_{\min}(\Lambda) \ge \lambda\}$,

under ℓ_∞ we have: $\ln|\mathscr{N}_\epsilon| \le d\ln(1+6L/\epsilon) + 2d^2\ln(1+18B^2\sqrt{d}/(\lambda\epsilon^2)) = \widetilde{O}\left(d^2\right)$

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Key step in the proof:

$$\left|f_\theta(s) - f_{\hat{\theta}}(s)\right| \le \|w-\hat{w}\|_2 + |\beta-\hat{\beta}|/\sqrt{\lambda} + B\sqrt{\|\Lambda^{-1}-\hat{\Lambda}^{-1}\|_F}$$

Detour: Uniform Convergence

$$f_{w,\beta,\Lambda} \because f_{w,\beta,\Lambda}(s) := \min\left\{ \max_a \left(w^\top \phi(s,a) + \beta\sqrt{\phi(s,a)^\top \Lambda^{-1} \phi(s,a)} \right), H \right\}$$

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Lemma [uniform convergence]: With probability at least $1-\delta$, for all s, a, h, n, **and ALL** $f \in \mathscr{F}$:

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Bonus b(s,a)

Proof sketch: the model error we had for a fixed V + ϵ-Net argument
(Same high level steps as the ones we used for HW1's last question)

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This will be our bonus term

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Summary of Covering Argument

Covering allows us to build a uniform convergence result (i.e., $\forall f \in \mathscr{F}$)
over a infinite hypothesis class
(Intuitively, log of covering number scales w.r.t to the # of parameters)

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3. UCBVI revisit and its guarantee

Algorithm: UCBVI in Linear MDPs

At the beginning of iteration n:

1. Learn transition model $\{\widehat{P}_h^n\}_{h=0}^{H-1}$ from all previous data via Ridge linear regression

comment: $\min_{\mu} \sum_{i=1}^{n-1} \|\mu\phi(s_h^i, a_h^i) - \delta(s_{h+1}^i)\|_2^2 + \lambda\|\mu\|_F^2$

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$O(Hd)$ $\|\phi\|_{\Lambda^{-1}}$

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4. Execute π^{n+1} for H steps

Regret bound for UCBVI in linear MDP

$$\mathbb{E}\left[\sum_{n=1}^{N}\left(V^{\star}-V^{\pi^{n}}\right)\right] \leq \widetilde{O}(H^{2}d^{1.5}\sqrt{N})$$

No S or A polynomial dependence!

2. Value Iteration in the Learned Model w/ Reward Bonus

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2. Value Iteration in the Learned Model w/ Reward Bonus

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Lemma [Optimism]: with high probability, for all n, h, s:

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$b(s,a)$

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4. Regret Decomposition

Conditioned on history up to the end of episode n-1:

$$V_0^\star(s_0) - V_0^{\pi^n}(s_0) \leq \widehat{V}_0^n(s_0) - V_0^{\pi^n}(s_0)$$

$\triangleq$

optimism

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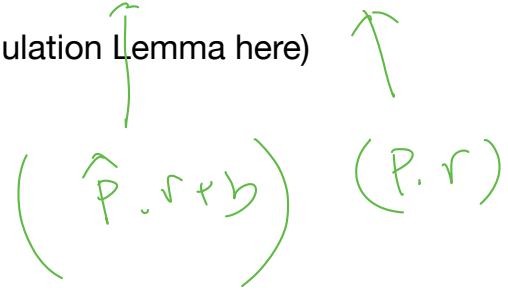

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4. Concluding the Regret Computation

$$\mathbb{E}\left[\sum_{n=1}^{N}\left(V_0^{\star}(s_0)-V_0^{\pi^n}(s_0)\right)\right]=\mathbb{E}\left[\mathbf{1}[\text{good event holds}]\sum_{n=1}^{N}\left(V_0^{\star}(s_0)-V_0^{\pi^n}(s_0)\right)\right]+\mathbb{E}\left[\mathbf{1}[\text{good event doesn't hold}]\sum_{n=1}^{N}\left(V_0^{\star}(s_0)-V_0^{\pi^n}(s_0)\right)\right]$$

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$\mathcal{O}(d\ln(N))$

4 ↑

CS $\sum_{n=1}^{N} a_n \le \sqrt{N}\sqrt{\sum_{n=1}^{N} a_n^2}$

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$$\lesssim \widetilde{O}(H^2d^{1.5}\sqrt{N})$$

$\leq d\log(N)$

$\|\phi\|^2_{\Lambda^{-1}}$

Summary

W/ optimism (thanks to bonus which offsets errors in estimated models):

$$V_0^\star(s_0) - V_0^{\pi^n}(s_0) \leq \widehat{V}_0^n(s_0) - V_0^{\pi^n}(s_0) \lesssim \sum_{h=0}^{H-1} \mathbb{E}_{s_h, a_h \sim d_h^{\pi^n}} \left[b_h^n(s_h, a_h) \right]$$

$V^* - V^{\pi^n} \leq \varepsilon$

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Since we live in a d-dim space, eventually we will explore all possible directions.