Learning with Linear Bellman Completion & Generative Model

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CS 6789: Foundations of Reinforcement Learning

Recap: DP in Finite Horizon MDPs

$$\mathscr{M} = \{S, A, P_h, r, H\}$$

$$P_h : S \times A \mapsto \Delta(S), \quad r : S \times A \to [0,1], \quad \gamma \in [0,1)$$

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2. At h, set $Q^\star_h(s,a) = r(s,a) + \mathbb{E}_{s' \sim P_h(\cdot|s,a)} V^\star_{h+1}(s')$,
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Similar results hold in finite horizon, with the effective horizon $1/(1-\gamma)$ being replaced by H

Question for Today:

We know that $Q_h^\star(s, a)$ being linear in some feature $\phi(s, a)$ is not enough for efficient learning… 😔

Q: what structural conditions can permit efficient learning?

Outline:

1. The Linear Bellman Completion Condition

2. The Least Square Value Iteration Algorithm

3. Guarantee and the proof sketch

Linear Bellman Completion

Given feature ϕ, take any linear function $w^\top \phi(s, a)$:

$$\forall h, \exists \theta \in \mathbb{R}^d, s.t., \theta^\top \phi(s, a) = r(s, a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} w^\top \phi(s', a'), \forall s, a$$

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Notation: we will denote such $\theta := \mathcal{T}_h(w)$, where $\mathcal{T}_h : \mathbb{R}^d \mapsto \mathbb{R}^d$

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reward $r(s,a)$ is linear in ϕ, i.e., $Q_{H-1}^\star(s,a)$ is linear,

now recursively show that $Q_h^\star$ is linear

Why this is a reasonable assumption?

It captures at least two special cases: tabular MDP and linear dynamical systems

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Set $\phi(s,a)$ to be a one-hot encoding vector in $\mathbb{R}^{SA}$, i.e., $\phi(s,a) = [0,\ldots,0,1,0,\ldots 0]^\top$

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(we will see the details when we get to the LQR lectures)

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Assume the given feature ϕ has linear Bellman completion, i.e.,

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This is counter-intuitive: in SL (e.g., linear regression), adding elements to features is ok!

What we will show today:

1. Generative Model

(i.e., we can reset system to any (s, a), query $r(s, a), s' \sim P(\,.\,|\,s, a)$)

$+$

2. Linear Bellman Completion

$=$

Sample efficient Learning

(finding near optimal policy in poly time)

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Sample complexity of LSVI

Theorem: There exists a way to construct datasets $\{\mathscr{D}_h\}_{h=0}^{H-1}$, such that with probability at least $1-\delta$, we have:

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Plans: (1) OLS and D-optimal design; (2) construct $\mathscr{D}_h$ using D-optimal design; (3) transfer regression error to $\|\theta_h^\top \phi - Q_h^\star\|_\infty$

Detour: Ordinary Linear Squares

Consider a dataset $\{x_i, y_i\}_{i=1}^N$, where $y_i = (\theta^\star)^\top x_i + \epsilon_i, \quad \mathbb{E}[\epsilon_i \,|\, x_i] = 0$, ϵ_i are independent with $|\epsilon_i| \leq \sigma$, assume $\Lambda = \sum_{i=1}^N x_i x_i^\top / N$ is full rank;

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Standard OLS guarantee: with probability at least $1 - \delta$, we have:

$$(\hat{\theta} - \theta^\star)^\top \Lambda (\hat{\theta} - \theta^\star) \leq O\left(\frac{\sigma^2 d \ln(1/\delta)}{N}\right)$$

Detour: Issues in Ordinary Linear Squares

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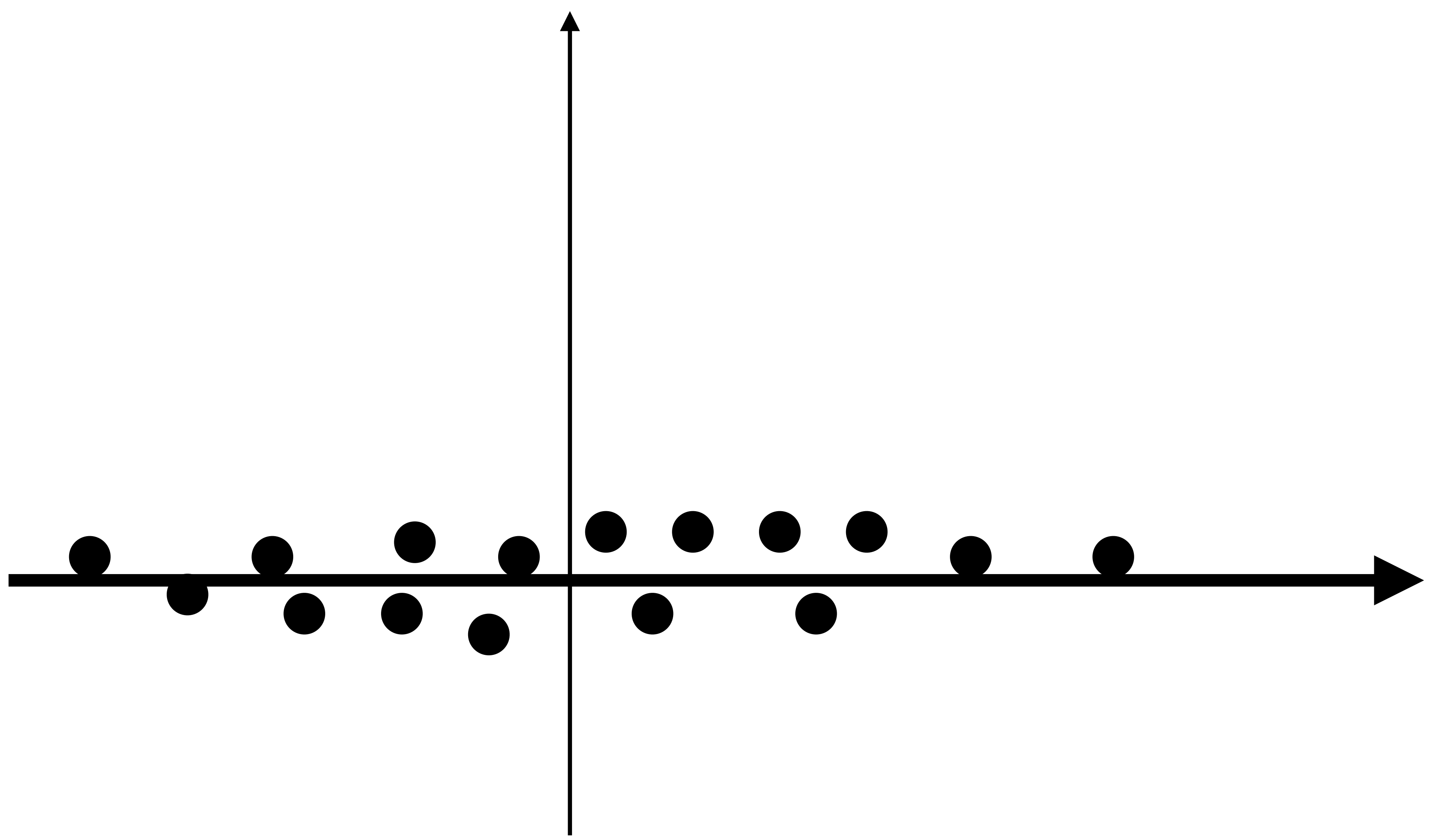

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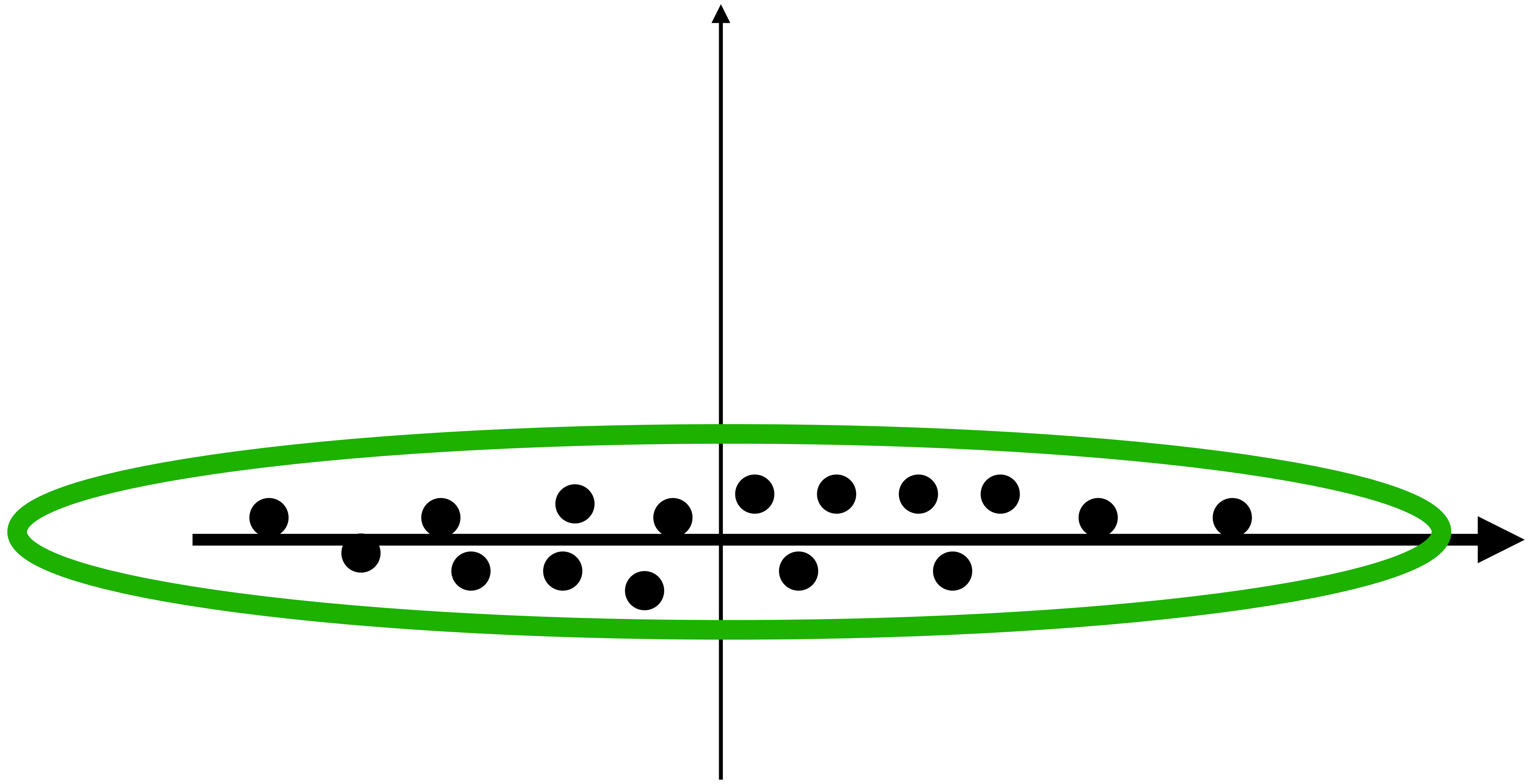

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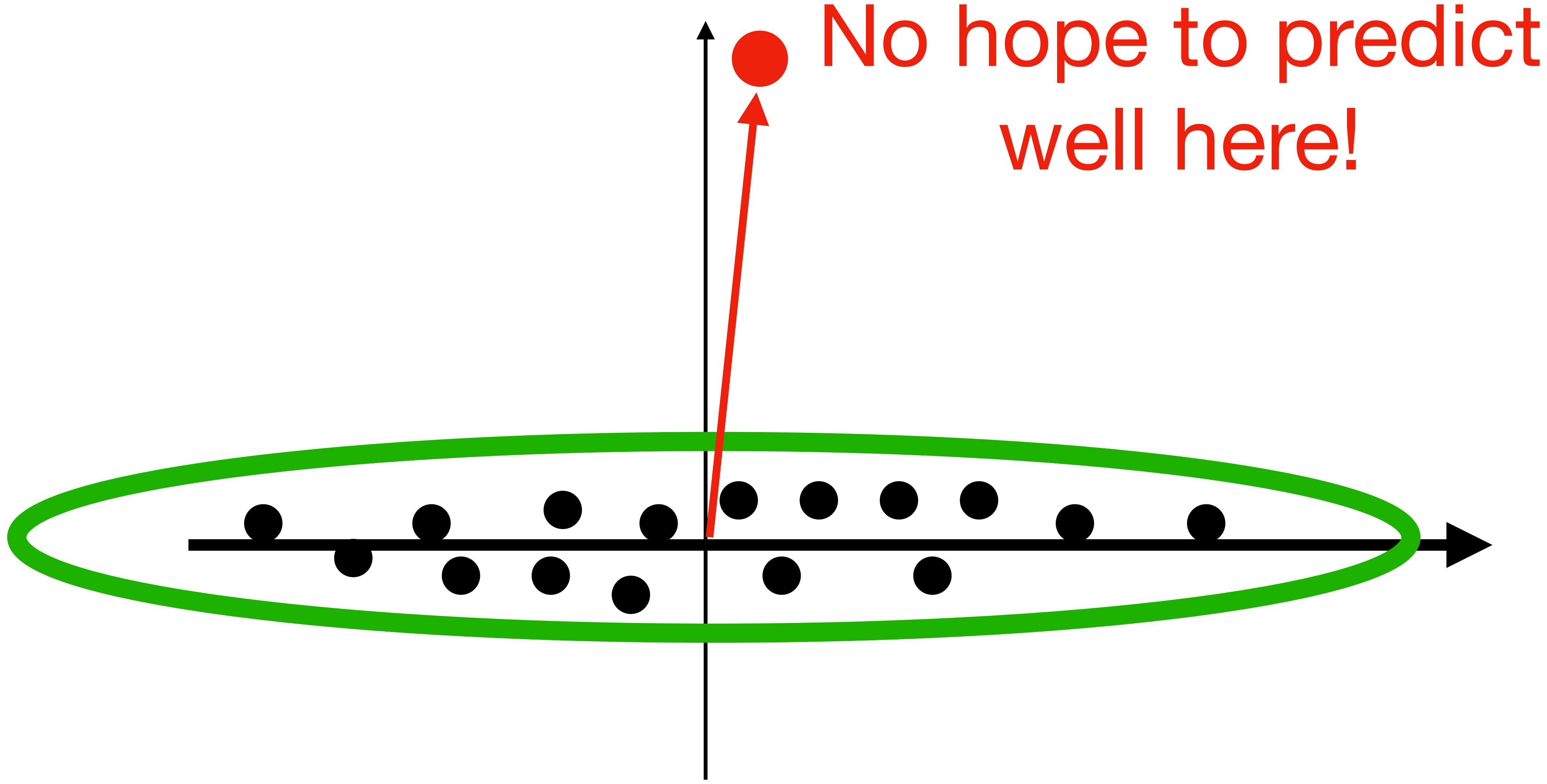

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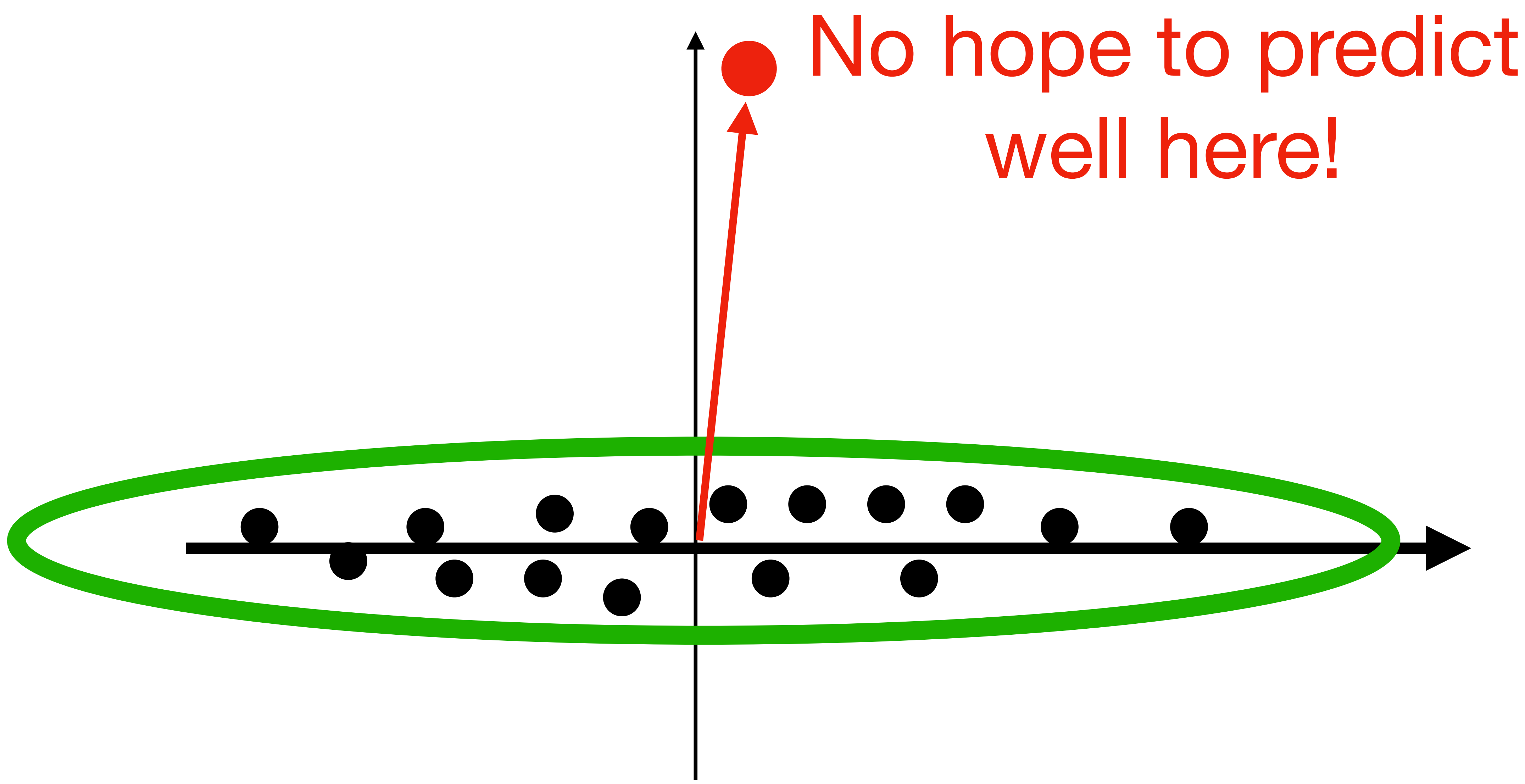
Let's actively design a diverse dataset !
(D-optimal Design)



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$$\max_{y\in\mathscr{X}} y^\top \left[\mathbb{E}_{x\sim\rho} xx^\top\right]^{-1} y \leq d$$

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The OLS solution $\hat{\theta}$ on $\mathscr{D}$ has the following point-wise guarantee: w/ prob $1-\delta$

$$\max_{x\in\mathscr{X}} \left| \langle \hat{\theta} - \theta^\star, x\rangle \right| \leq \frac{\sigma d \ln(1/\delta)}{\sqrt{N}}$$

Summary so far on OLS & D-optimal Design

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D-optimal design allows us to **actively** construct a dataset $\mathcal{D} = \{x, y\}$,
such that OLS solution is **POINT-WISE** accurate:

$$\max_{x\in\mathcal{X}} \left|\langle\hat{\theta} - \theta^\star, x\rangle\right| \leq \frac{\sigma d \ln(1/\delta)}{\sqrt{N}}$$

Using D-optimal design to construct $\mathscr{D}_h$ in LSVI

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OLS /w D-optimal design implies that $\hat{\theta}_h$ is point-wise accurate:

$$\max_{s,a} \left| \theta_h^\top \phi(s,a) - \mathscr{T}_h(\theta_{h+1})^\top \phi(s,a) \right| \leq \widetilde{O}\left(Hd/\sqrt{N}\right).$$

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