Learning with Linear Bellman Completion & Generative Model

Sham Kakade and Wen Sun

CS 6789: Foundations of Reinforcement Learning

Recap: Linear Bellman Completion

Given feature ϕ, take any linear function $w^\top \phi(s,a)$:

$$\forall h, \exists \theta \in \mathbb{R}^d, s.t., \theta^\top \phi(s,a) = r(s,a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} w^\top \phi(s',a'), \forall s, a$$

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But adding additional elements may just break the condition

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BC always ensures linear regression is realizable:

i.e., our regression target

$$r(s, a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} \theta_{h+1}^\top \phi(s', a')$$

is always linear:

Outline for Today

1. Proof Sketch of LSVI

2. LSVI in Offline RL

Theorem

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$$V^{\hat{\pi}} - V^{\star} \leq \epsilon$$

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2. Show that our estimators are near-bellman consistent: $\|\theta_h^\top \phi - \mathscr{T}_h(\theta_{h+1}^\top \phi)\|_\infty$ is small
3. Near-Bellman consistency implies near optimal performance (s.t. H error amplification)

Detour: Ordinary Linear Squares

Consider a dataset $\{x_i, y_i\}_{i=1}^N$, where $y_i = (\theta^\star)^\top x_i + \epsilon_i, \quad \mathbb{E}[\epsilon_i \mid x_i] = 0,$ ϵ_i are independent with $|\epsilon_i| \leq \sigma,$ assume $\Lambda = \sum_{i=1}^{N} x_i x_i^\top / N$ is full rank;

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Standard OLS guarantee: with probability at least $1 - \delta$:

$$(\hat{\theta} - \theta^\star)^\top \Lambda (\hat{\theta} - \theta^\star) \leq O\left(\frac{\sigma^2 d \ln(1/\delta)}{N}\right)$$

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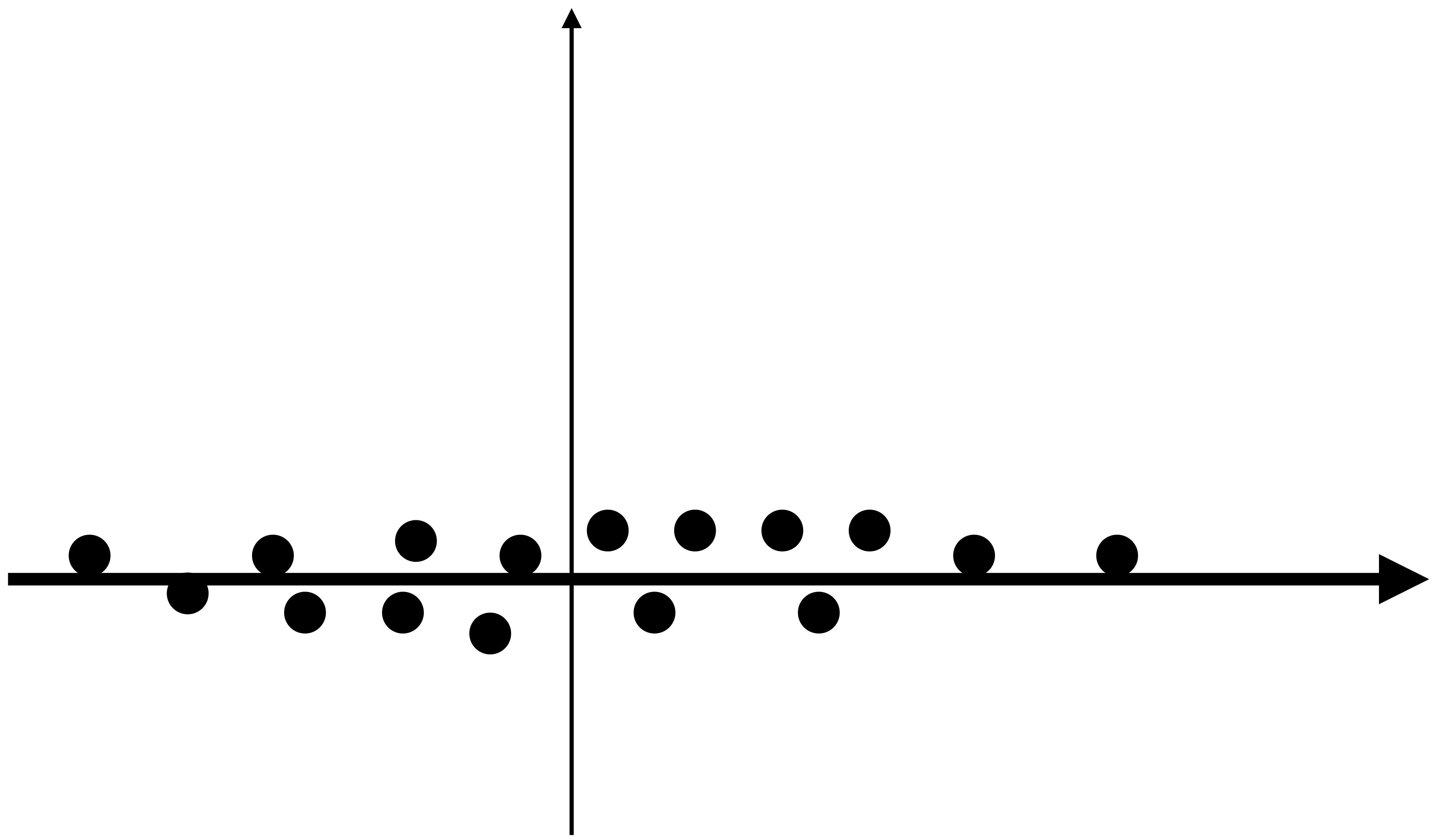

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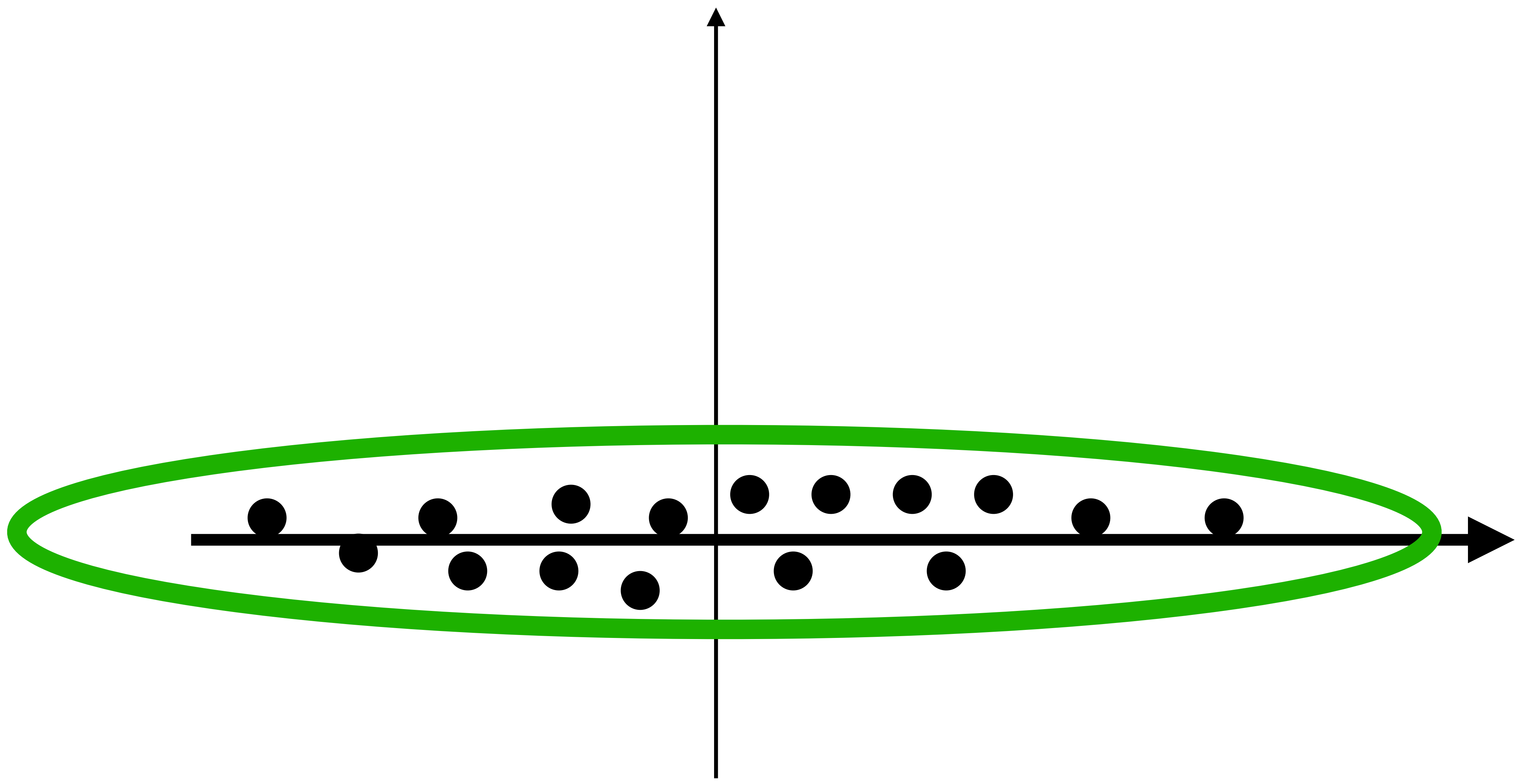

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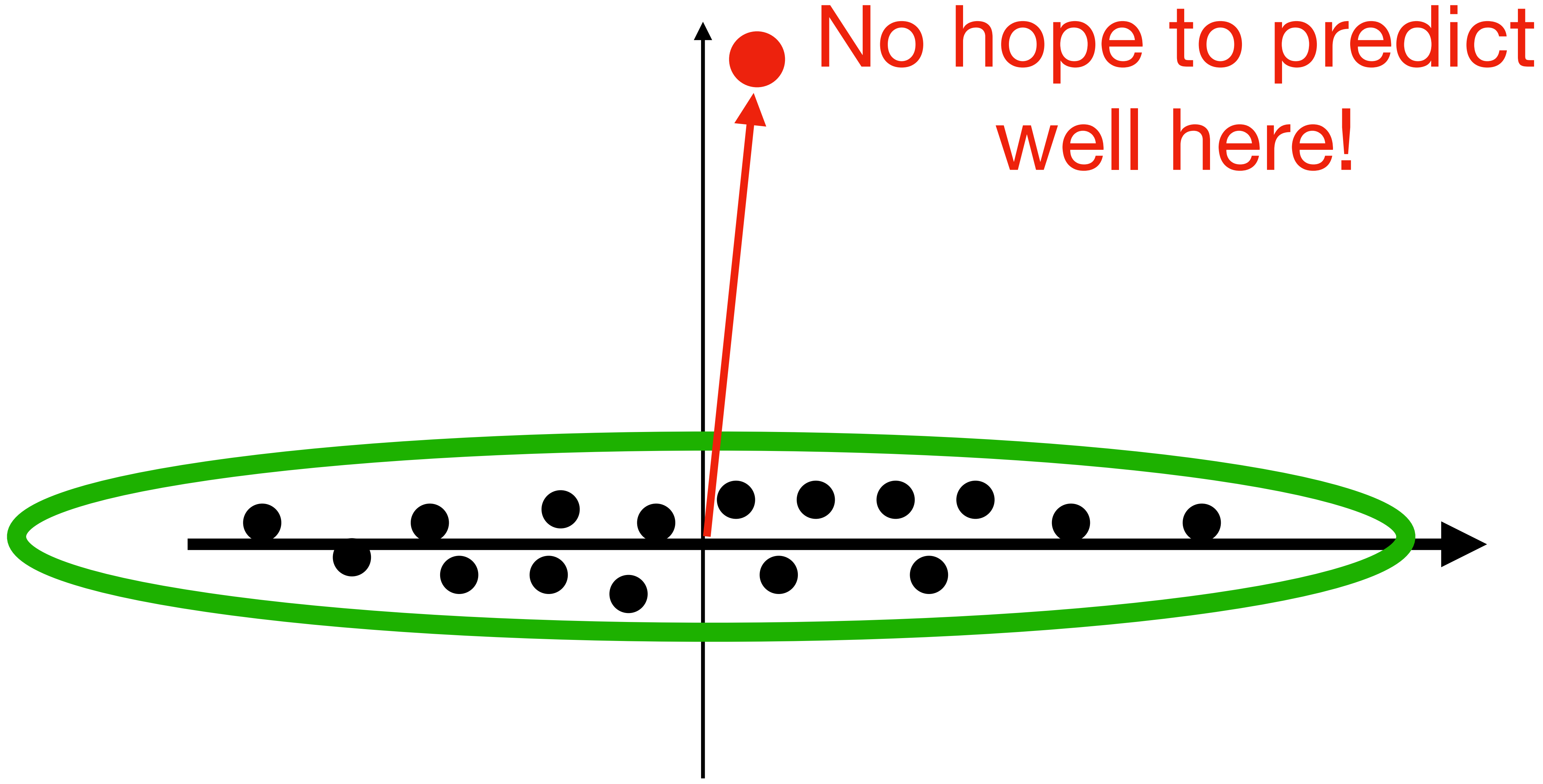

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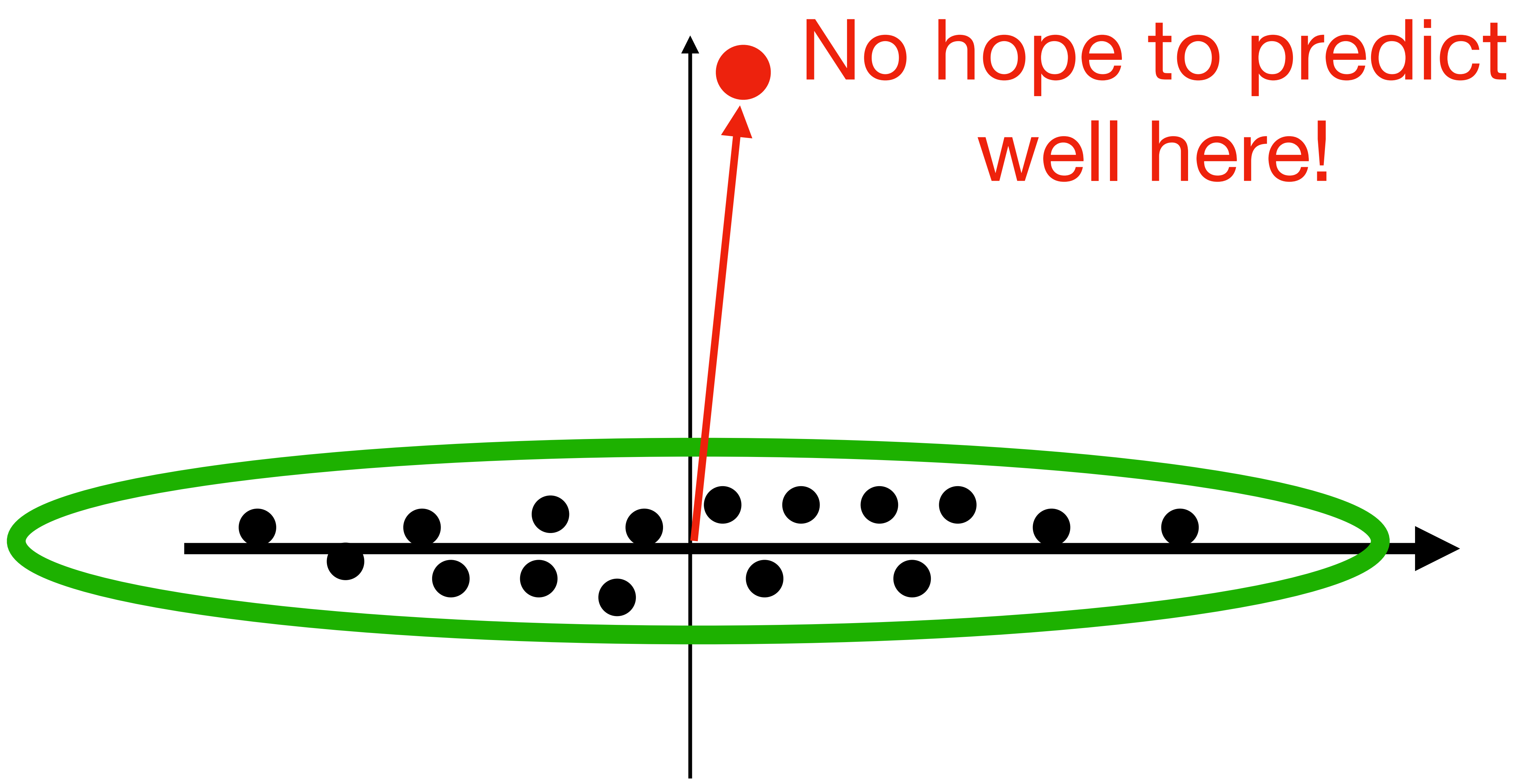

Let's actively design a diverse dataset !
(D-optimal Design)

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$$\left| (\hat\theta - \theta^\star)^\top x \right| \leq \left\| \Lambda^{1/2}(\hat\theta - \theta^\star) \right\|_2 \left\| \Lambda^{-1/2} x \right\|_2$$

Summary so far on OLS & D-optimal Design

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D-optimal design allows us to **actively** construct a dataset $\mathcal{D} = \{x, y\}$, such that OLS solution is **POINT-WISE** accurate:

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$$\Rightarrow V^\star - V^{\hat{\pi}} \leq \widetilde{O}(H^3 d/\sqrt{N})$$

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2. LSVI in **Offline RL**

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Offline RL is promising for safety critical applications
(i.e., learning from logged data for health applications…)

Recall Least-Square Value Iteration

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Set $V_h(s) := \max_a \theta_h^\top \phi(s, a), \forall s$

Return $\hat{\pi}_h(s) = \arg\max_a \theta_h^\top \phi(s, a), \forall h$

Recall Least-Square Value Iteration

Datasets $\mathcal{D}_0, \ldots, \mathcal{D}_{H-1}$, w/

$\mathcal{D}_h = \{s, a, r, s'\}, r = r(s, a), s' \sim P_h(\cdot \mid s, a)$

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LSVI directly can directly operate in offline model!

Least-Square Value Iteration Guarantee

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Assumptions

1. Full offline data coverage: $\sigma_{\min}\left(\mathbb{E}_{s,a\sim\nu}\phi(s, a)\phi(s, a)^\top\right) \geq \kappa$
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Then, with probability at least $1 - \delta$, LSVI return $\hat{\pi}$ with $V^\star - V^{\hat{\pi}} \leq \epsilon$, using at most poly $\left(H, 1/\epsilon, 1/\kappa, d, \ln(1/\delta)\right)$

The proof for the offline set is almost identical

Key step:

Linear Bellman completion + Linear Regression w/ full data coverage

=> Near-Bellman consistency, **i.e.,** $\|Q_h - \mathcal{T}_h Q_{h+1}\|_\infty$ **is small**

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e.g., with N training examples where $(s,a) \sim \nu$, and $r = r(s,a), s' \sim P_h(\cdot \mid s,a)$, we have

$\mathbb{E}_{s,a\sim\nu}\left(\theta_h^\top \phi(s,a) - \mathcal{T}_h(\theta_{h+1})^\top \phi(s,a)\right)^2 \leq$ poly$(H, d, 1/N)$

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(we will give a HW question on a related topic)

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4. Near-Bellman consistency implies small approximation error of Q_h (holds in general)

Next week

Fitted Dynamic Programming — can we extend linear function approx to general function approx (e.g., neural network, decision tree, etc)?

Exploration: Multi-armed Bandits and Stochastic Linear Bandits (Bandits = MDP w/ $H = 1$)