Learning with Linear Bellman Completion & Generative Model

Sham Kakade and Wen Sun

CS 6789: Foundations of Reinforcement Learning

Recap: Linear Bellman Completion

Given feature ϕ, take any linear function $w^\top \phi(s, a)$:

$$\forall h, \exists \theta \in \mathbb{R}^d, s.t., \theta^\top \phi(s, a) = r(s, a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} w^\top \phi(s', a'), \forall s, a$$

$T_h(w)$

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Captures Tabular MDPs, and Linear Quadratic Regulators

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But adding additional elements may just break the condition

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$\mathscr{D}_h = \{s, a, r, s'\}, r = r(s, a), s' \sim P_h(\cdot \mid s, a)$

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$$\theta_h = \arg\min_\theta \sum_{\mathcal{D}_h} \left(\theta^T \phi(s, a) - \left(r + V_{h+1}(s')\right)\right)^2$$

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$\approx Q_h^*$

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For h = H-1 to 0:

$\approx V^\star_{h+1}(s')$

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BC always ensures linear regression is realizable:

i.e., our regression target

$$r(s, a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} \theta_{h+1}^\top \phi(s', a')$$

is always linear:

Outline for Today

1. Proof Sketch of LSVI

2. LSVI in Offline RL

Theorem

Theorem: There exists a way to construct datasets $\{\mathcal{D}_h\}_{h=0}^{H-1}$, such that with probability at least $1 - \delta$, we have:

$$V^{\hat{\pi}} - V^\star \leq \epsilon$$

w/ total number of samples in these datasets scaling $\widetilde{O}\left(d^2 + H^6 d^2/\epsilon^2\right)$

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2. Show that our estimators are near-bellman consistent: $\|\theta_h^\top \phi - \mathscr{T}_h(\theta_{h+1}^\top \phi)\|_\infty$ is small
3. Near-Bellman consistency implies near optimal performance (s.t. H error amplification)

Detour: Ordinary Linear Squares

Consider a dataset $\{x_i, y_i\}_{i=1}^N$, where $y_i = (\theta^\star)^\top x_i + \epsilon_i$, $\quad \mathbb{E}[\epsilon_i | x_i] = 0$, ϵ_i are independent

with $|\epsilon_i| \leq \sigma$, assume $\Lambda = \sum_{i=1}^N x_i x_i^\top / N$ is full rank;

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$$\text{OLS}: \hat{\theta} = \arg\min_{\theta} \sum_{i=1}^{N} (\theta^\top x_i - y_i)^2$$

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Standard OLS guarantee: with probability at least $1 - \delta$:

$$\underbrace{(\hat{\theta} - \theta^\star)^\top \Lambda (\hat{\theta} - \theta^\star)}_{\Rightarrow \frac{1}{N} \sum_{i=1}^{n} \left((\hat{\theta} - \theta^\star)^\top x_i \right)^2} \leq O\left(\frac{\sigma^2 d \ln(1/\delta)}{N} \right)$$

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If the test point x is not covered by the training data, i.e., $x^\top \Lambda^{-1} x$ is huge, then we cannot guarantee $\hat{\theta}^\top x$ is close to $(\theta^\star)^\top x$

$x^\top \Lambda^{-1} x$

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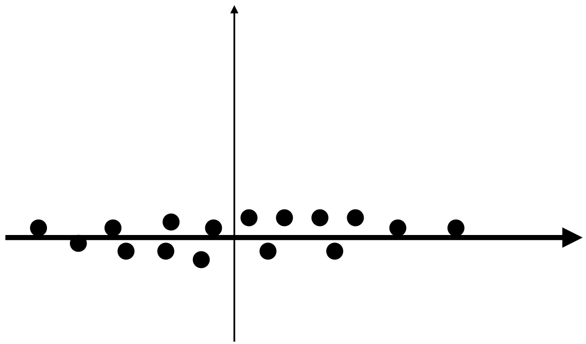

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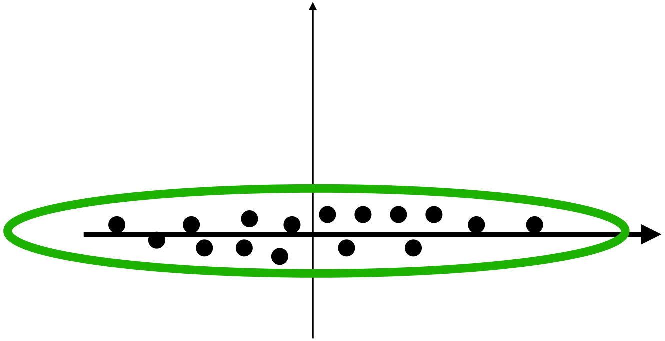

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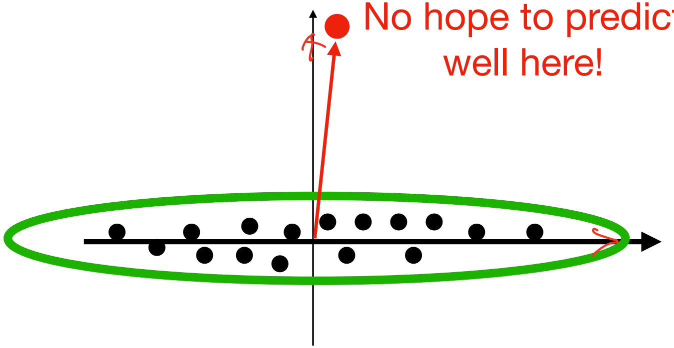

$\Lambda = U \Sigma U^T$

λ_i

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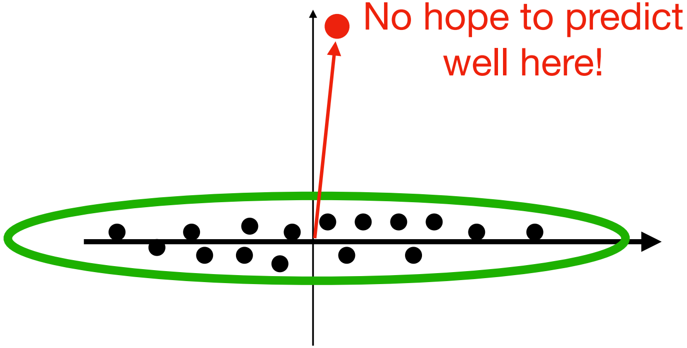

Let's actively design a diverse dataset !
(D-optimal Design)

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support$(\rho^\star) \leq d(d+1)/2$

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$$\max_{y\in\mathcal{X}} y^\top \left[\mathbb{E}_{x\sim\cancel{\rho}\,\rho^\star} xx^\top\right]^{-1} y \leq d$$

$$\Sigma := \mathbb{E}_{x\sim\rho^\star} xx^\top$$

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For each $x \in \mathscr{D}$, query y (noisy measure);

$y = \theta^{\star\top} x + \varepsilon$

$\sum_x \rho^*(x) N = N$

$\Lambda = \frac{1}{N}\sum_{x\in D} xx^T$

$= \frac{1}{N}\sum_{x\in \text{support}(\rho^*)} \rho(x) N = \mathbb{E}_{x\sim\rho^*} xx^T$

$\forall x \in \mathcal{X}$

$x^T \Lambda^{-1} x = x^T \left[\mathbb{E}_{x\sim\rho^*} xx^T\right]^{-1} x \leq d$

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$$\max_{x\in\mathscr{X}} \left| \langle \hat{\theta} - \theta^\star, x \rangle \right| \leq \frac{\sigma d \ln(1/\delta)}{\sqrt{N}}$$

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$\Lambda^{\frac{1}{2}} \Lambda^{-\frac{1}{2}} := I$

$$\left|(\hat\theta - \theta^\star)^\top x\right| \le \underbrace{\left\| \Lambda^{1/2}(\hat\theta - \theta^\star)\right\|_2}_{①} \underbrace{\left\|\Lambda^{-1/2} x\right\|_2}_{②}$$

$|a^\top b| \le \|a\|_2 \|b\|_2$

①: $(\hat\theta - \theta^\star)\Lambda(\hat\theta - \theta^\star)$

②: $x^\top \Lambda^{-1} x \le d$

Summary so far on OLS & D-optimal Design

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D-optimal design allows us to **actively** construct a dataset $\mathscr{D} = \{x, y\}$, such that OLS solution is **POINT-WISE** accurate:

$$\max_{x\in\mathscr{X}} \left|\langle \hat{\theta} - \theta^\star, x\rangle\right| \leq \frac{\sigma d \ln(1/\delta)}{\sqrt{N}}$$

Using D-optimal design to construct $\mathscr{D}_h$ in LSVI

Consider the space $\Phi = \{\phi(s, a) : s, a \in S \times A\}$

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for each $\phi(s,a)$, **query** $y := r(s,a) + V_{h+1}(s'), s' \sim \underbrace{P_h(\,.\,|\,s,a)}_{\text{Generative model}}$

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What's the Bayes optimal $\mathbb{E}[y\,|\,s,a]$?

$V_{h+1}(s') = \max_{a'} \theta_{h+1}^\top \phi(s',a')$

Bayes optimal: $\mathcal{T}_h(\theta_{h+1})^\top \phi(s,a)$

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$$\|Q_h - Q_h^\star\|_\infty \le O(H^2 d/\sqrt{N})$$

$$\Rightarrow V^\star - V^{\hat{\pi}} \le \widetilde{O}(H^3 d/\sqrt{N}) \Rightarrow \varepsilon$$

$$\Rightarrow N \approx \frac{H^6 d^2}{\varepsilon^2}$$

$\Rightarrow$ VI

$\widetilde{O}$: ignore $(\ln 1/\delta)$

Outline for Today

✓ 1. Proof Sketch of LSVI

2. LSVI in **Offline RL**

Offline Reinforcement Learning

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Learner **cannot interact** with the environment, instead, learner is given **static** datasets:

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Offline RL is promising for safety critical applications
(i.e., learning from logged data for health applications…)

Recall Least-Square Value Iteration

Datasets $\mathscr{D}_0, \ldots, \mathscr{D}_{H-1}$, w/

$\mathscr{D}_h = \{s, a, r, s'\}, r = r(s,a), s' \sim P_h(\cdot \mid s, a)$

Set $V_H(s) = 0, \forall s$

For h = H-1 to 0:

$$\theta_h = \arg\min_{\theta} \sum_{\mathscr{D}_h} \left(\theta^T \phi(s,a) - \left(r + V_{h+1}(s')\right)\right)^2$$

$\Rightarrow r + \max_{a'} \theta_{h+1}^T \phi(s', a')$

Set $V_h(s) := \max_a \theta_h^\top \phi(s,a), \forall s$

$Q^*_h(s,a) = \theta^{*T} \phi(s,a)$

Return $\hat{\pi}_h(s) = \arg\max_a \theta_h^\top \phi(s,a), \forall h$

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LSVI directly can directly operate in offline model!

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Recall $\mathscr{D}_h = \{s, a, r, s'\}, s, a \sim \nu, r = r(s, a), s' \sim P_h(\cdot \mid s, a)$

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Assumptions

1. Full offline data coverage: $\sigma_{\min}\left(\mathbb{E}_{s,a\sim\nu}\phi(s,a)\phi(s,a)^{\top}\right) \geq \kappa$
2. Linear Bellman completion

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Then, with probability at least $1 - \delta$, LSVI return $\hat{\pi}$ with $V^\star - V^{\hat{\pi}} \leq \epsilon$, using at most poly $\left(H, 1/\epsilon, 1/\kappa, d, \ln(1/\delta)\right)$

4

The proof for the offline set is almost identical

Key step:

Linear Bellman completion + Linear Regression w/ full data coverage

=> Near-Bellman consistency, **i.e.,** $\|Q_h - \mathcal{T}_h Q_{h+1}\|_\infty$ **is small**

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e.g., with N training examples where $(s, a) \sim \nu$, and $r = r(s, a), s' \sim P_h(\cdot \mid s, a)$, we have

$$\mathbb{E}_{s,a\sim\nu}\left(\theta_h^\top \phi(s, a) - \mathcal{T}_h(\theta_{h+1})^\top \phi(s, a)\right)^2 \leq \text{poly}(H, d, 1/N)$$

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(we will give a HW question on a related topic)

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4. Near-Bellman consistency implies small approximation error of Q_h (holds in general)

Next week

Fitted Dynamic Programming — can we extend linear function approx to general function approx (e.g., neural network, decision tree, etc)?

Exploration: Multi-armed Bandits and Stochastic Linear Bandits (Bandits = MDP w/ $H = 1$)