

Planning in MDPs

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CS 6789: Foundations of Reinforcement Learning

Announcements

HW0 is due Feb 1st

Recap: Value iteration

$$Q^{t+1} = \mathcal{T} Q^t$$

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Q: when will π^t be the optimal policy?

Outline

1. Policy Iteration
2. Computation complexity of VI and PI
3. Linear Programming formulation

Policy Iteration Algorithm:

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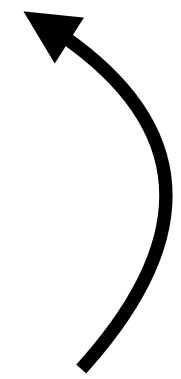
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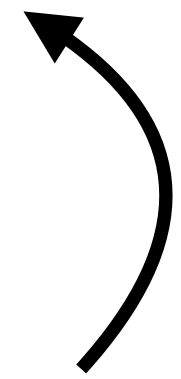
Policy Iteration Algorithm:

Closed-form for PE
(see 1.1.3 in Monograph)

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Analysis of Policy Iteration

Recall: Policy Improvement $\pi^{t+1}(s) = \arg \max_a Q^{\pi^t}(s, a), \forall s$

Lemma: Monotonic improvement $Q^{\pi^{t+1}}(s, a) \geq Q^{\pi^t}(s, a), \forall s, a$

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$$V^{\pi^{t+1}}(s) \geq V^{\pi^t}(s), \forall s$$

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Theorem: Convergence $\|V^{\pi^{t+1}} - V^{\star}\|_{\infty} \leq \gamma \|V^{\pi^t} - V^{\star}\|_{\infty}$

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Analysis of Policy Iteration

Q: what happens when π^{t+1} and π^t are exactly the same?

Show that π^t is an optimal policy π^\star

Q: does this imply that the algorithm will terminate?

Outline

1. Policy Iteration

2. Computation complexity of VI and PI

3. Linear Programming formulation

Computation complexity of VI and PI

Given an MDP $\mathcal{M} = (S, A, P, r, \gamma)$ can we **exactly** compute Q^* (or find π^*)
in time polynomial wrt $S, A, 1/(1 - \gamma)$?

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What about poly(S, A) algs?

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The primal linear programming

Recall the Bellman consistency:

$$V(s) = \max_a \left\{ r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot | s, a)} [V(s')] \right\}, \forall s$$

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$$\min \sum_s \mu(s) V(s)$$

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(Proof in HW1)

LP Runtime

[Ye, '05]: there is an interior point algorithm (CIPA)
which is (“nearly”) **strongly polynomial, i.e., no poly dependence on $1/(1 - \gamma)$**

$$S^4 A^4 \ln \left(\frac{S}{1 - \gamma} \right)$$

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 - It is also very helpful conceptually.
 - In some cases, it also provides a reasonable algorithmic approach
- Let us start by understanding the dual variables

State action occupancy measure

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$$V^\pi(s_0) = \frac{1}{1 - \gamma} \sum_{s,a} d_{s_0}^\pi(s, a) r(s, a)$$

A Bellman equation like property for $d_{s_0}^\pi(s, a)$

$$\sum_a d_\mu^\pi(s, a) = (1 - \gamma)\mu(s) + \gamma \sum_{\bar{s}, \bar{a}} P(s | \bar{s}, \bar{a}) d_\mu^\pi(\bar{s}, \bar{a})$$

Proof:

The “State-Action” Polytope

- Let us define the **state-action polytope** K as follows:

$$K_{\mu} := \left\{ d \mid d \geq 0 \text{ and } \sum_a d(s, a) = (1 - \gamma)\mu(s) + \gamma \sum_{s', a'} P(s \mid s', a') d(s', a') \right\}$$

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Lemma: $d \in K_{\mu}$ if and only if there exists a (possibly randomized) policy π
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(Proof in HW1)

The Dual LP

$$\begin{aligned} \max \quad & \sum_{s,a} d(s,a)r(s,a) \\ \text{s.t.} \quad & d \in K_\mu \end{aligned}$$

- One can verify that this is the dual of the primal LP.

Summary

Notations: Value / Q functions, state-action occupant measures,
Bellman equation / optimality

Planning algorithms: VI, PI, LP (primal and dual)