Planning in MDPs

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CS 6789: Foundations of Reinforcement Learning

Announcements

HW0 is due Feb 1st

Recap: Value iteration

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Q: when will π^t be the optimal policy?

Outline

1. Policy Iteration

2. Computation complexity of VI and PI

3. Linear Programming formulation

Policy Iteration Algorithm:

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Policy Iteration Algorithm:

Closed-form for PE
(see 1.1.3 in Monograph)

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Analysis of Policy Iteration

Recall: Policy Improvement $\pi^{t+1}(s) = \arg\max_a Q^{\pi^t}(s, a), \forall s$

Lemma: Monotonic improvement $Q^{\pi^{t+1}}(s, a) \geq Q^{\pi^t}(s, a), \forall s, a$

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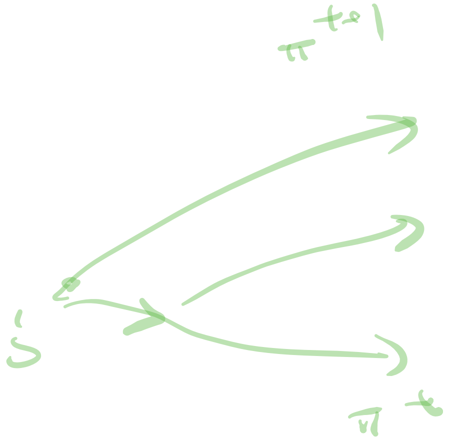
$= r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} Q^{\pi^t}(s', \pi^t(s'))$ ← Bell-Eqn



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$$= \gamma \mathbb{E}_{s' \sim P(s,a)} \left[Q^{\pi^{t+1}}(s', \pi^{t+1}(s')) - Q^{\pi^t}(s', \pi^{t+1}(s')) + Q^{\pi^t}(s', \pi^{t+1}(s')) - Q^{\pi^t}(s', \pi^t(s')) \right]$$

≥ 0

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$$\geq \gamma \mathbb{E}_{s' \sim P(s,a)} \left[\gamma \mathbb{E}_{s'' \sim P(s', \pi^{t+1}(s'))} \left[Q^{\pi^{t+1}}(s'', \pi^{t+1}(s'')) - Q^{\pi^t}(s'', \pi^{t+1}(s'')) \right] \right]$$

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$$\in \left[-\frac{1}{1-\gamma}, \frac{1}{1-\gamma} \right]$$

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$V^{\pi^{t+1}}(s) \geq V^{\pi^t}(s), \forall s$

$$Q^{\pi^{t+1}}(s, \pi^{t+1}(s)) \geq Q^{\pi^t}(s, \pi^{t+1}(s))$$

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Theorem: Convergence $\|V^{\pi^{t+1}} - V^\star\|_\infty \leq \gamma \|V^{\pi^t} - V^\star\|_\infty$

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$$V^\star(s) - V^{\pi^{t+1}}(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V^\star(s') \right] - \left[r(s, \pi^{t+1}(s)) + \gamma \mathbb{E}_{s' \sim P(s, \pi^{t+1}(s))} V^{\pi^{t+1}}(s') \right]$$

V^{π^t}

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$$\le \gamma \|V^\star - V^{\pi^t}\|_\infty$$

$\ge$ $\le \gamma \|V^{\pi^{t+1}} - V^\star\|_\infty$

$\le \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} (V^\star(s') - V^{\pi^t}(s'))$

$\le \max_a \max_{s'} \gamma (V^\star(s') - V^{\pi^t}(s'))$

Analysis of Policy Iteration

Q: what happens when π^{t+1} and π^t are exactly the same?

Show that π^t is an optimal policy $\pi^\star$

$V^{\pi^t}(s) = V^\star(s), \forall s$

Q: does this imply that the algorithm will terminate?

A^S

Outline

1. Policy Iteration

2. Computation complexity of VI and PI

3. Linear Programming formulation

Computation complexity of VI and PI

Given an MDP $\mathscr{M} = (S, A, P, r, \gamma)$ can we exactly compute $Q^\star$ (or find $\pi^\star$)
in time polynomial wrt $S, A, 1/(1-\gamma)$?

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Complexity per Iteration

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What about poly(S, A) algs?

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$$A^{\pi_t}(s,a) = Q^{\pi_t}(s,a) - V^{\pi_t}(s)$$

$$\max_a A^{\pi_t}(s,a)$$

The primal linear programming

Recall the Bellman consistency:

$$V(s) = \max_a \left\{ r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[V(s') \right] \right\}, \forall s$$

$\Rightarrow V(s) = V^\star(s), \forall s$

$V(s) \geq r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s'), \quad \forall a$

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$$\min \sum_s \mu(s) V(s)$$

$\mu(s) > 0$

s.t. $V(s) \geq r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \quad \forall s, a \in S \times A$

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(Proof in HW1)

LP Runtime

[Ye, ’05]: there is an interior point algorithm (CIPA) which is (“nearly”) strongly polynomial, i.e., no poly dependence on $1/(1-\gamma)$

$$S^4 A^4 \ln\left(\frac{S}{1-\gamma}\right)$$

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  - It is also very helpful conceptually.
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- Let us start by understanding the dual variables

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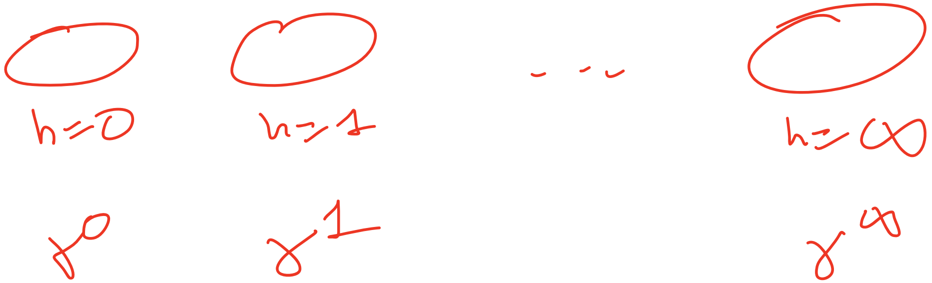
$$d^{\pi}_{s_0}(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h(s, a; s_0, \pi)$$

$$\sum_{sa} d^{\pi}_{s_0}(s,a) = 1$$



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$$d^{\pi}_{s_0}(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h(s, a; s_0, \pi)$$

$$V^{\pi}(s_0) = \frac{1}{1 - \gamma} \sum_{s,a} d^{\pi}_{s_0}(s, a) r(s, a)$$

$$= \frac{1}{1-\gamma} \mathbb{E}_{s,a \sim d^{\pi}_{s_0}(s,a)} \left[r(s,a) \right]$$

$s_0 \sim \mu$

$$d^{\pi}_{\mu}(s,a) = \mathbb{E}_{s_0 \sim \mu} \left[d^{\pi}_{s_0}(s,a) \right]$$

A Bellman equation like property for $d^{\pi}_{s_0}(s, a)$

$$\sum_a d^{\pi}_{\mu}(s,a) = (1-\gamma)\mu(s) + \gamma \sum_{\bar{s},\bar{a}} P(s \mid \bar{s}, \bar{a}) d^{\pi}_{\mu}(\bar{s},\bar{a})$$

Proof:

$$d^{\pi}_{\mu}(s,a) = (1-\gamma)\left[\mu(s)\pi(a|s) + \gamma P_1(s,a;\mu,\pi) + \gamma^2 P_2(s,a;\mu,\pi) \cdots \right]$$

$$= (1-\gamma)\left[\mu(s)\pi(a|s) + \gamma \mathop{\mathbb{E}}_{\bar{s},\bar{a} \sim P_0(\cdot,\cdot;\mu,\pi)} P(s|\bar{s},\bar{a})\pi(a|s) + \gamma^2 \mathop{\mathbb{E}}_{\bar{s},\bar{a} \sim P(\cdot;\pi,\mu)} P(s|\bar{s},\bar{a})\pi(a|s)\right]$$

$$= (1-\gamma)\Big[\mu(s)\pi(a|s) + \Big(\gamma \mathop{\mathbb{E}}_{P_0} + \gamma^2 \mathop{\mathbb{E}}_{P_1} + \gamma^3 \mathop{\mathbb{E}}_{P_2} \cdots\Big) P(s|\bar{s},\bar{a})\pi(a|s)$$

The “State-Action” Polytope

- Let us define the state-action polytope K as follows:

$$K_\mu := \left\{ d \,|\, d \geq 0 \text{ and } \sum_a d(s,a) = (1-\gamma)\mu(s) + \gamma \sum_{s',a'} P(s\,|\,s',a')d(s',a') \right\}$$

$d \in \mathbb{R}^{|S \times A|}$ $\quad d(s,a) \geq 0$

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Lemma: $d \in K_\mu$ if and only if there exists a (possibly randomized) policy π s.t. $d_\mu^\pi = d$

(Proof in HW1)

The Dual LP

$$\max_{d} \sum_{s,a} d(s,a) r(s,a)$$

$$\text{s.t.} \quad d \in K_{\mu}$$

d^{π}_{μ}.

- One can verify that this is the dual of the primal LP.

Summary

Notations: Value / Q functions, state-action occupant measures, Bellman equation / optimality

Planning algorithms: VI, PI, LP (primal and dual)