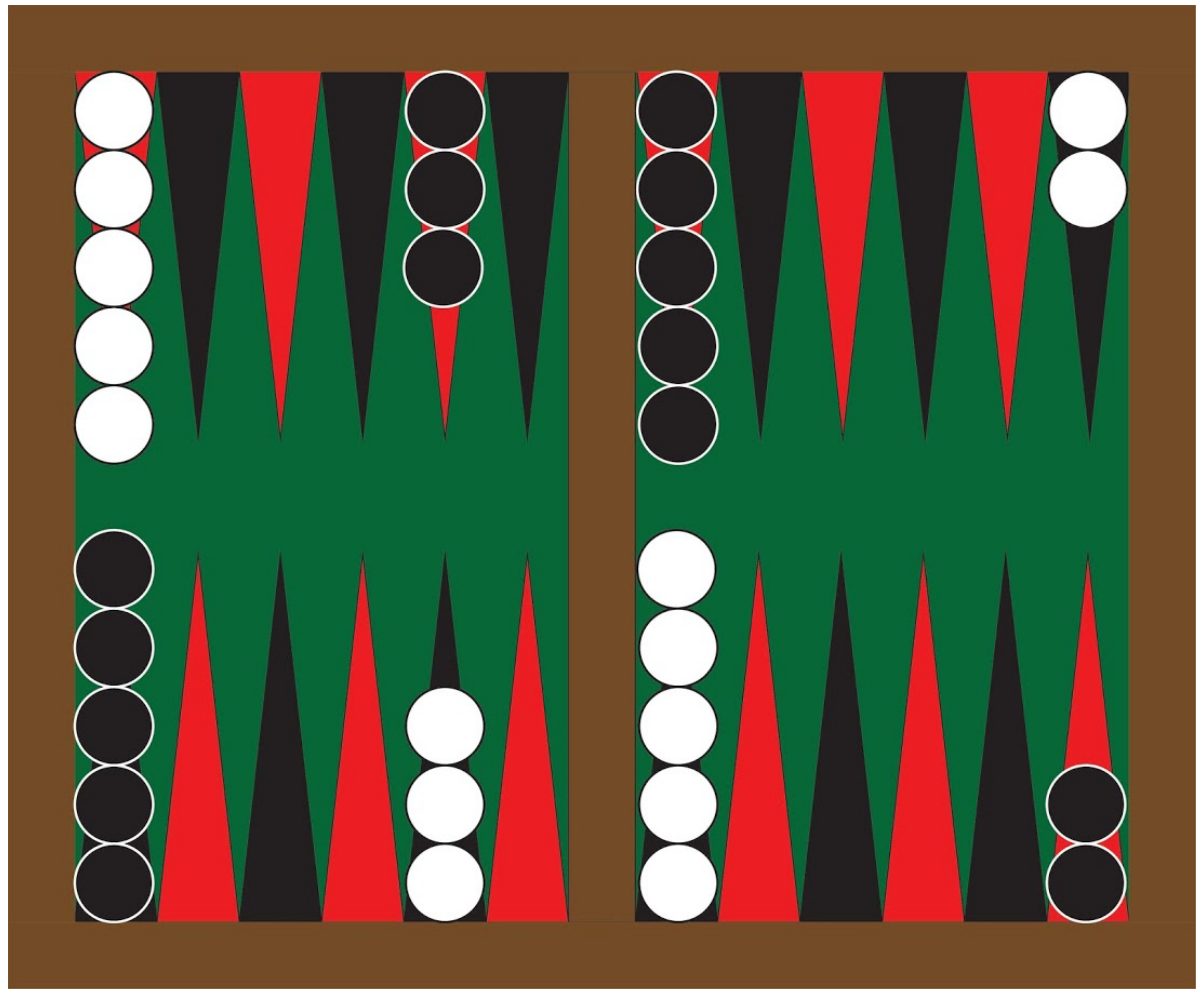
Introduction and Basics of Markov Decision Process

Wen Sun

CS 6789: Foundations of Reinforcement Learning

Progress of RL in Practice



TD GAMMON [Tesauro 95]

[AlphaZero, Silver et.al, 17]

[OpenAI Five, 18]

RL in Real World:

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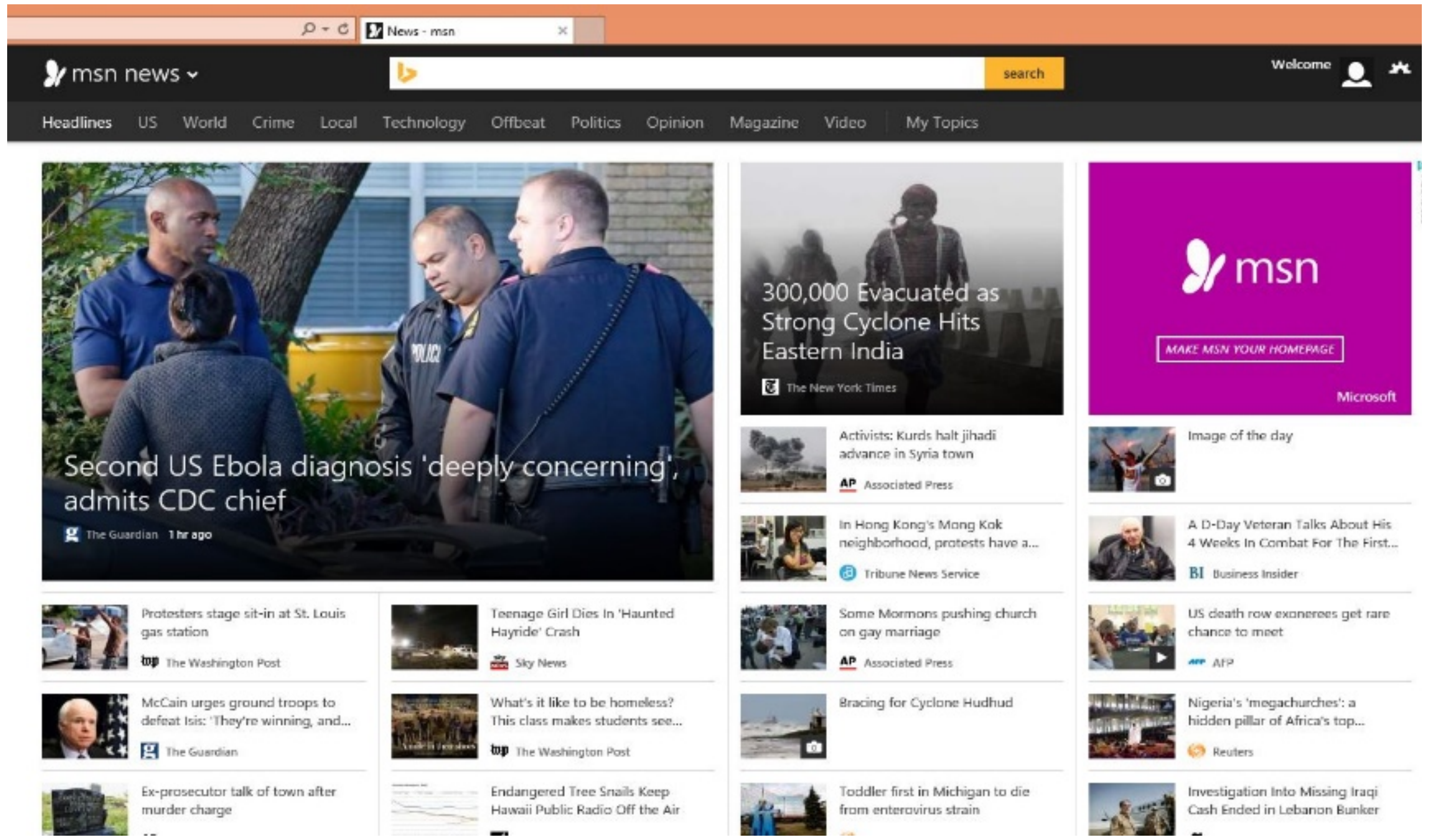

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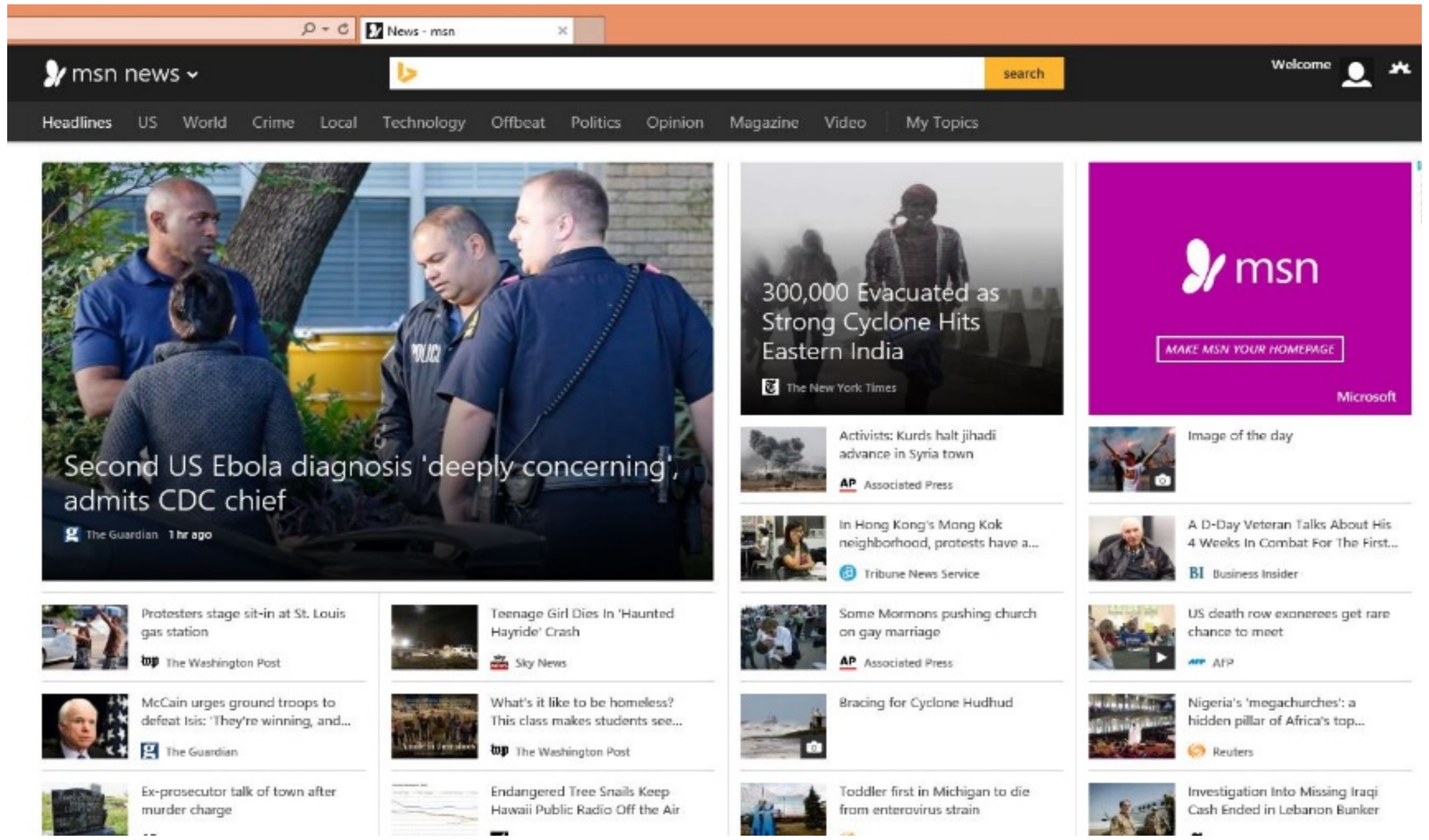

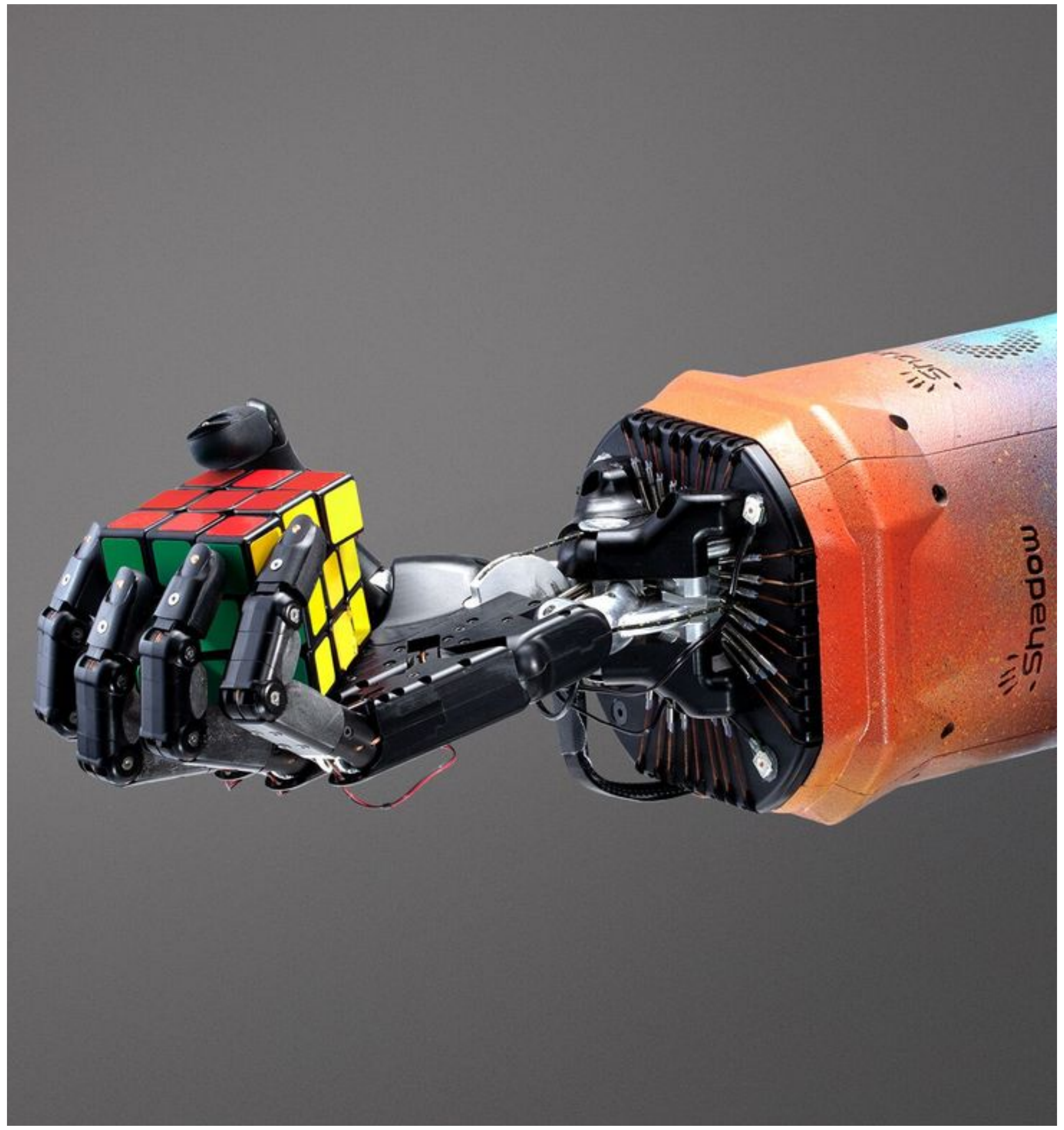

RL in Real World:

USER this code is not working like i expect — how do i fix it?

```
resultWorkerErr := make(chan error)
defer close(resultWorkerErr)
go func() {
	defer cancel()
	resultWorkerErr <- b.resultWorker(ctx)
}()

err := b.worker(ctx)
cancel()
if err == nil {
	return <-resultWorkerErr
}
return multierror.Append(err, <-resultWorkerErr)
```

CHATGPT It's difficult to say what's wrong with the code without more context. Can you provide more information about what the code is supposed to do and what isn't working as expected? Also, is this the entire code or just a part of it?

USER **just part of it — the error never surfaces. I think there is something wrong with the channel**

CHATGPT It's difficult to say without more information about what the code is supposed to do and what's happening when it's executed. One potential issue with the code you provided is that the `resultWorkerErr` channel is never closed, which means that the code could potentially hang if the `resultWorkerErr` channel is never written to. This

Training Language models using RL, e.g., chatGPT

This course focuses on RL Theory

When and Why RL works!
(Convergence, sample / computation complexity, etc)

Four main themes we will cover in this course:

1. Fundamentals (MDPs and Optimal planning)
2. Exploration (sample complexity)
3. Policy Gradient (global convergence)
4. Learning Partially observable models

Logistics

Four (HW0-HW3) assignments (total 55%), Course Project (40%), Reading (5%)

(HW0 10%, HW1-3 15% each)

HW0 is out today and due in one week

Prerequisites (HW0)

Deep understanding of Machine Learning, Optimization, Statistics

ML: sample complexity analysis for supervised learning (PAC)

Opt: Convex (linear) optimization, e.g., gradient decent for convex functions

Stats: basics of concentration (e.g., Hoeffding's), tricks such as union bound

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Check out HW0 asap!

Course projects (40%)

- Team work: size 3
- Midterm report (5%), Final presentation (15%), and Final report (20%)

- Basics: **survey** of a set of similar RL theory papers. Reproduce analysis and provide a coherent story
- Advanced: **identify** extensions of existing RL papers, **formulate** theory questions, and **provide** proofs

Course Notes: Reinforcement Learning Theory & Algorithms

- Book website: https://rltheorybook.github.io/

- Many lectures will correspond to chapters in Version 2.
- Reading assignment (5%) is from this book and additional notes

- Please let us know if you find typos/errors in the book!
  We appreciate it!

Outline

1. Definition of infinite horizon discounted MDPs

2. Bellman Optimality

3. State-action distribution

Supervised Learning

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Given i.i.d examples at training:

( ,cat) ( ,cat) ( ,dog)

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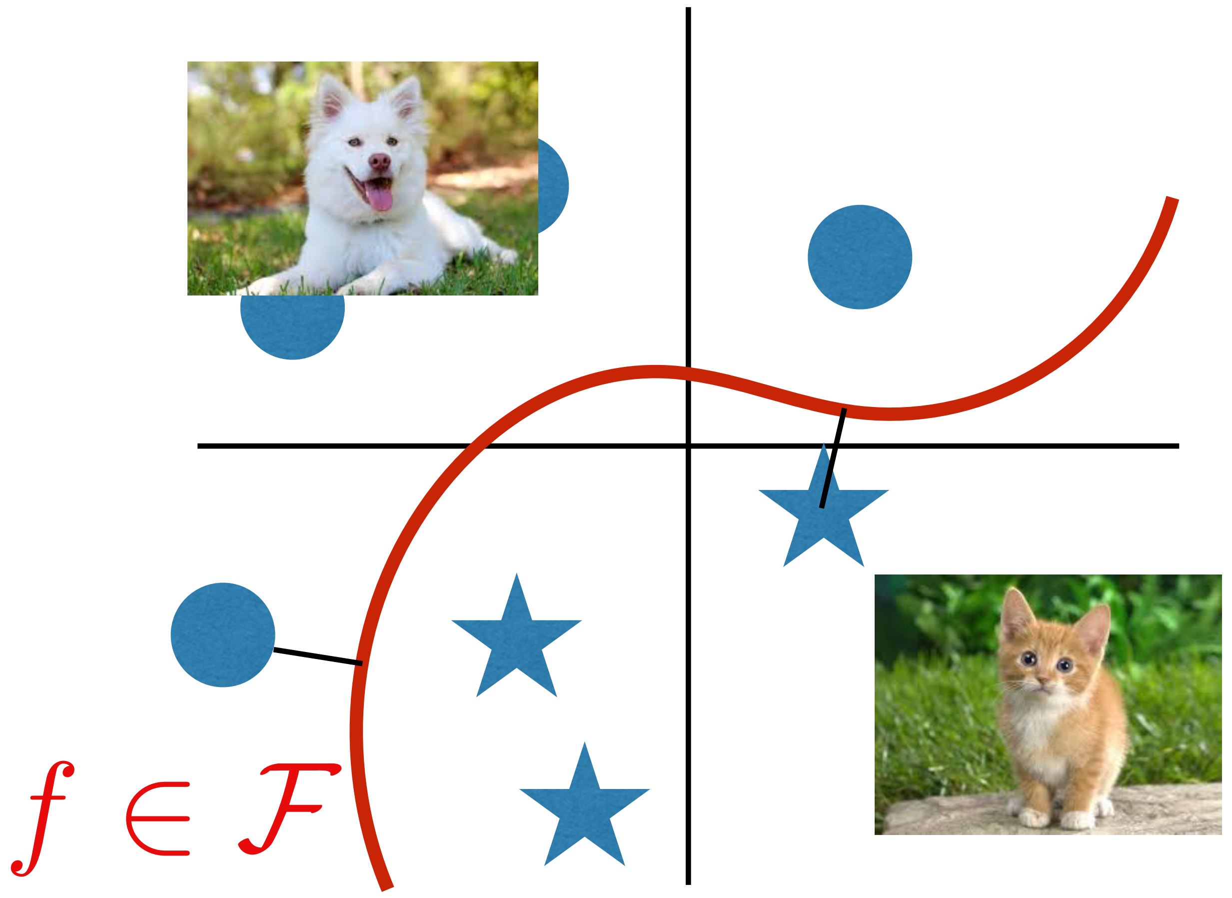

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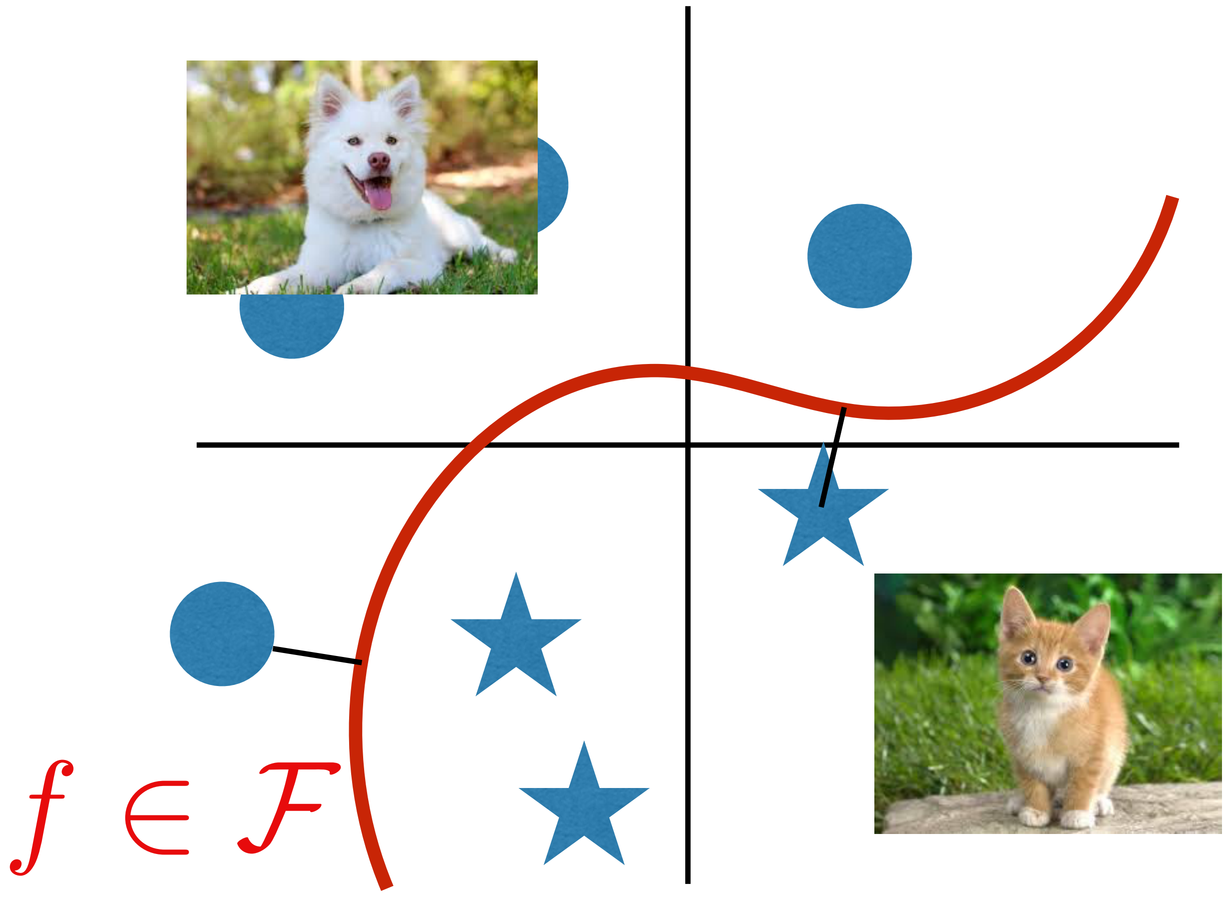
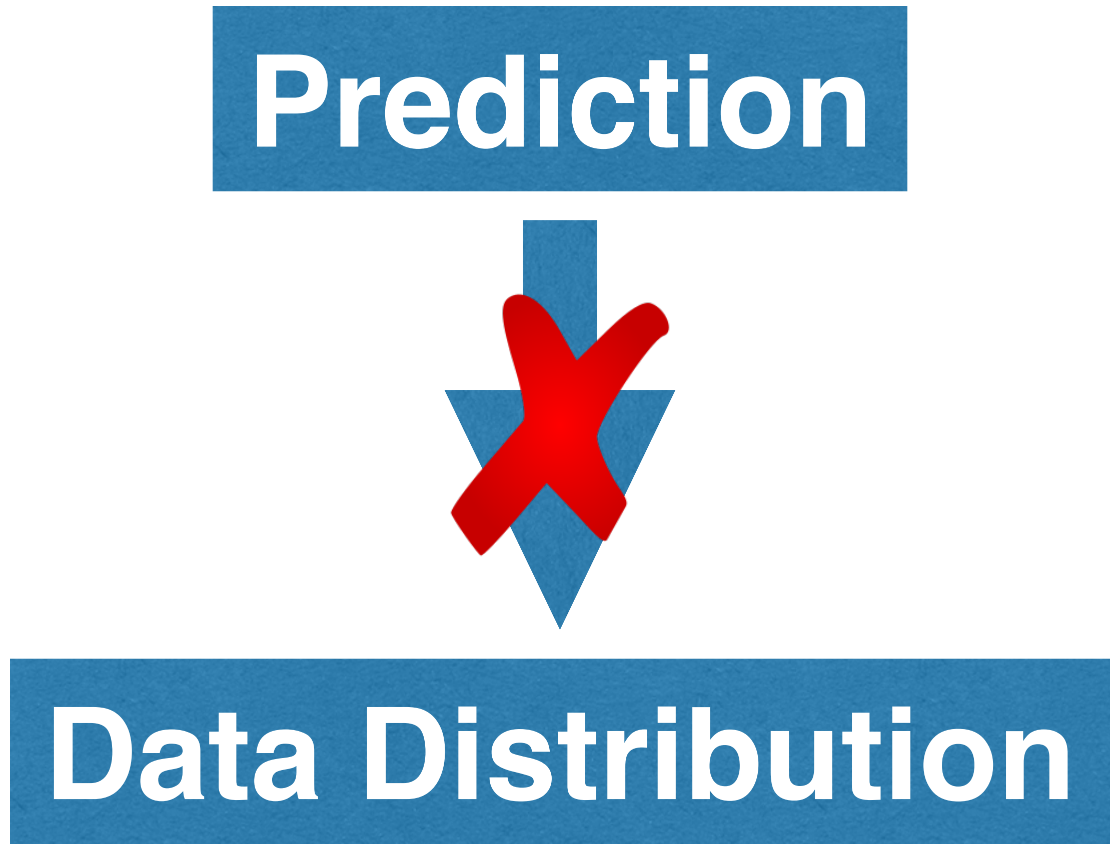


Passive:



AgentLinear

Selected Actions:

RIGHT SPEED

Active: Decisions → Data Distribution

AgentLinear
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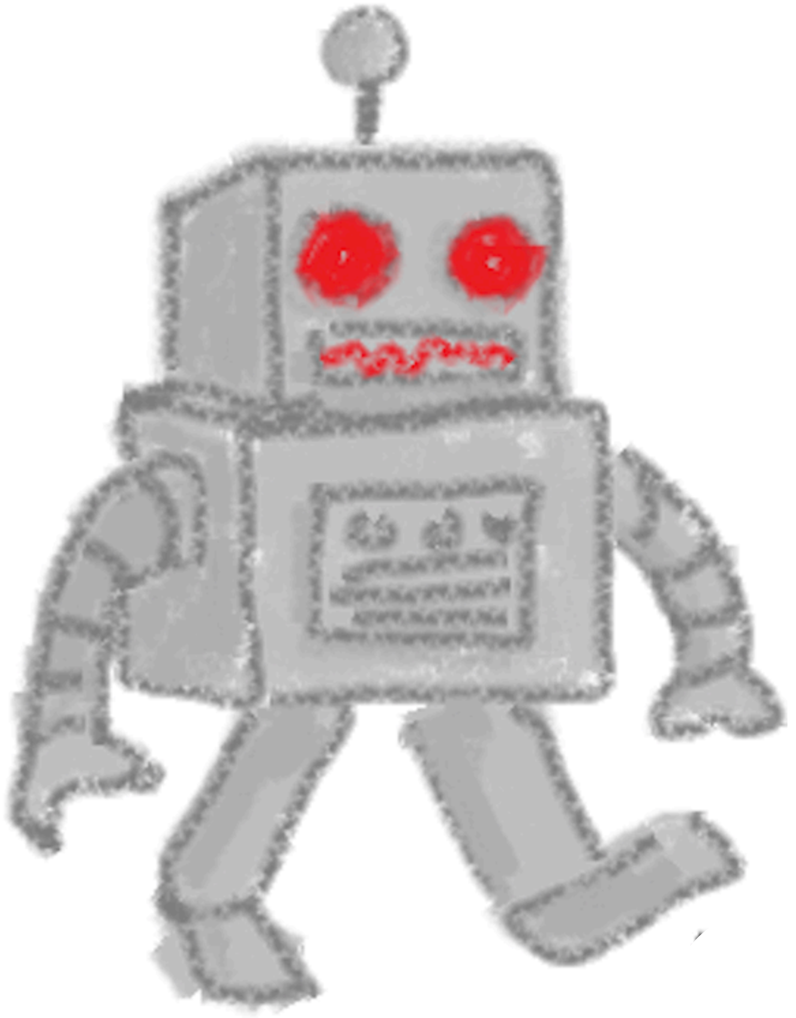
Markov Decision Process

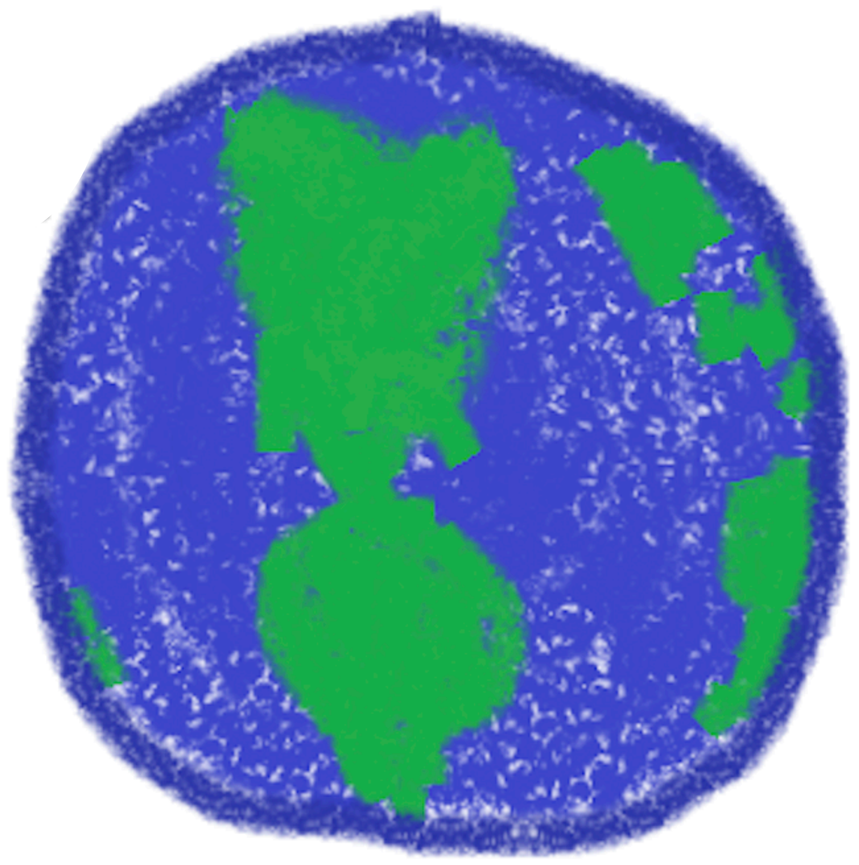
Learning Agent



$$a \sim \pi(s)$$

Policy: determine **action** based on **state**

Environment



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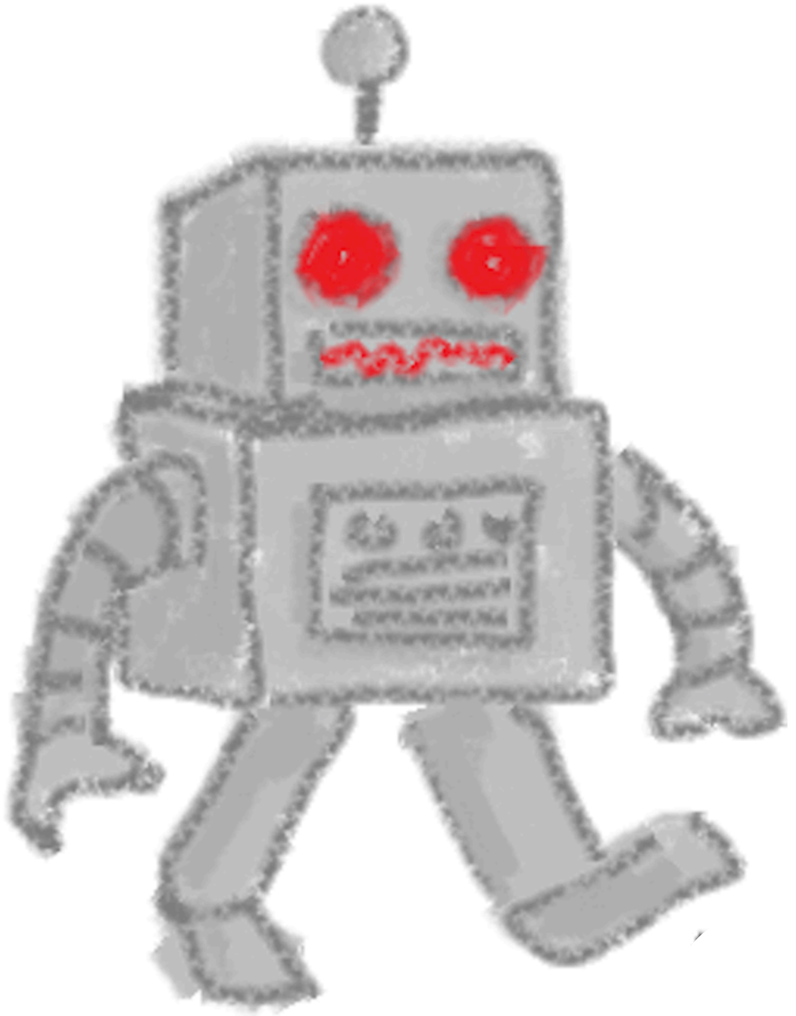

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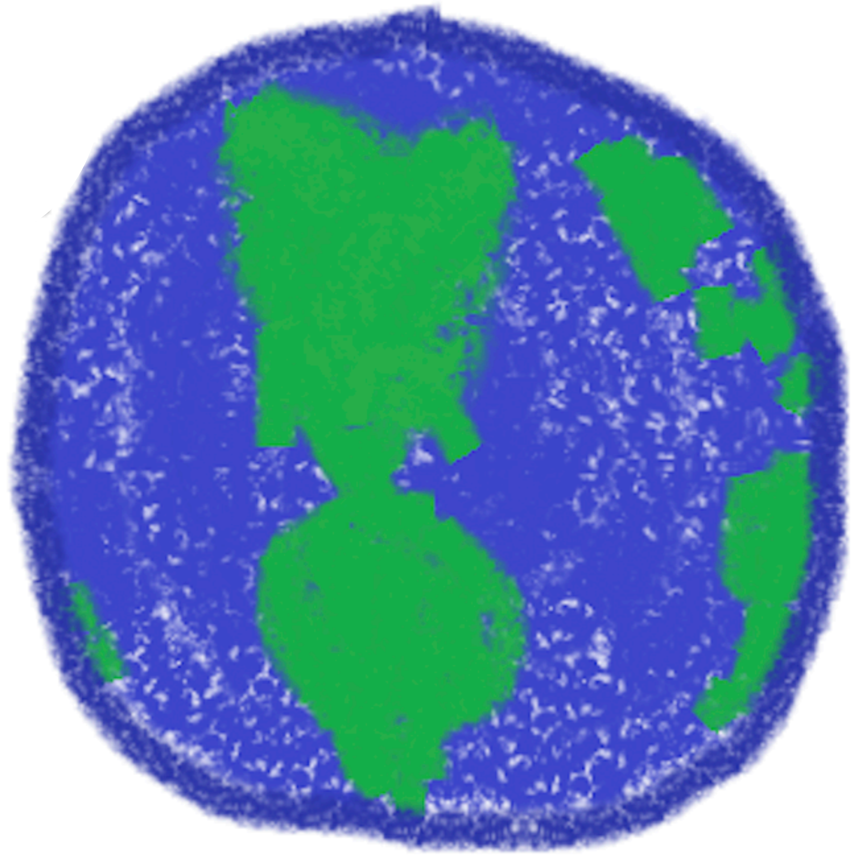
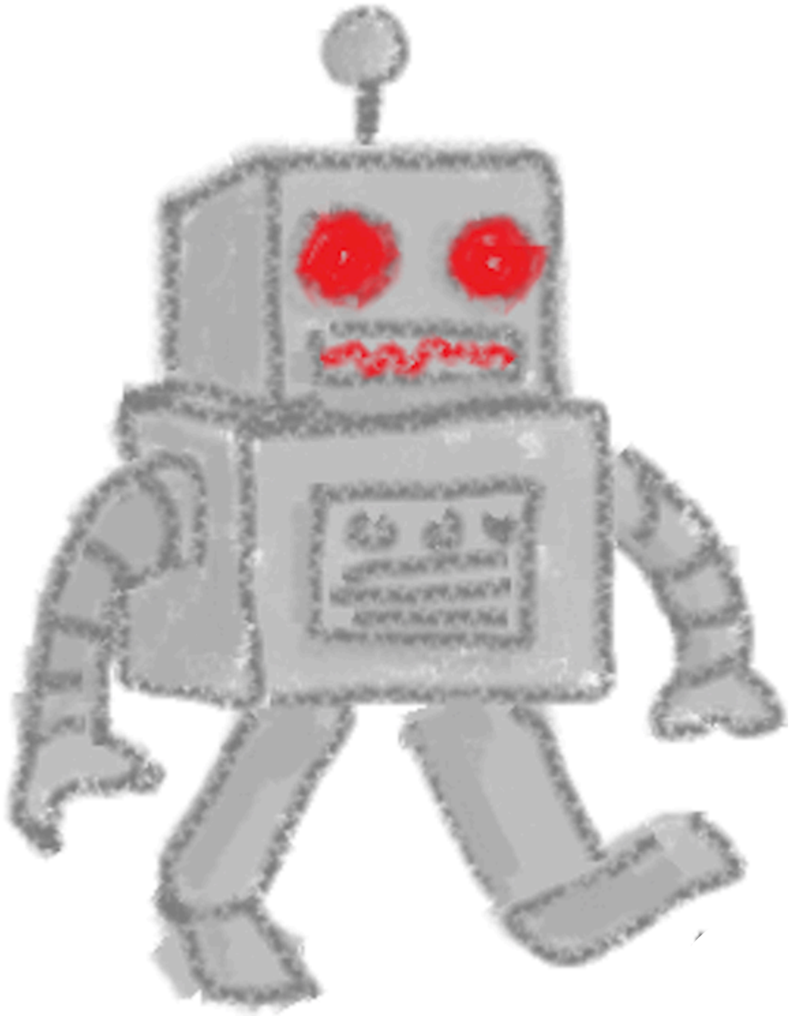


Send **reward** and **next state** from a Markovian transition dynamics

$$r(s, a), s' \sim P(\,\cdot\,|\,s, a)$$

Markov Decision Process

Learning Agent



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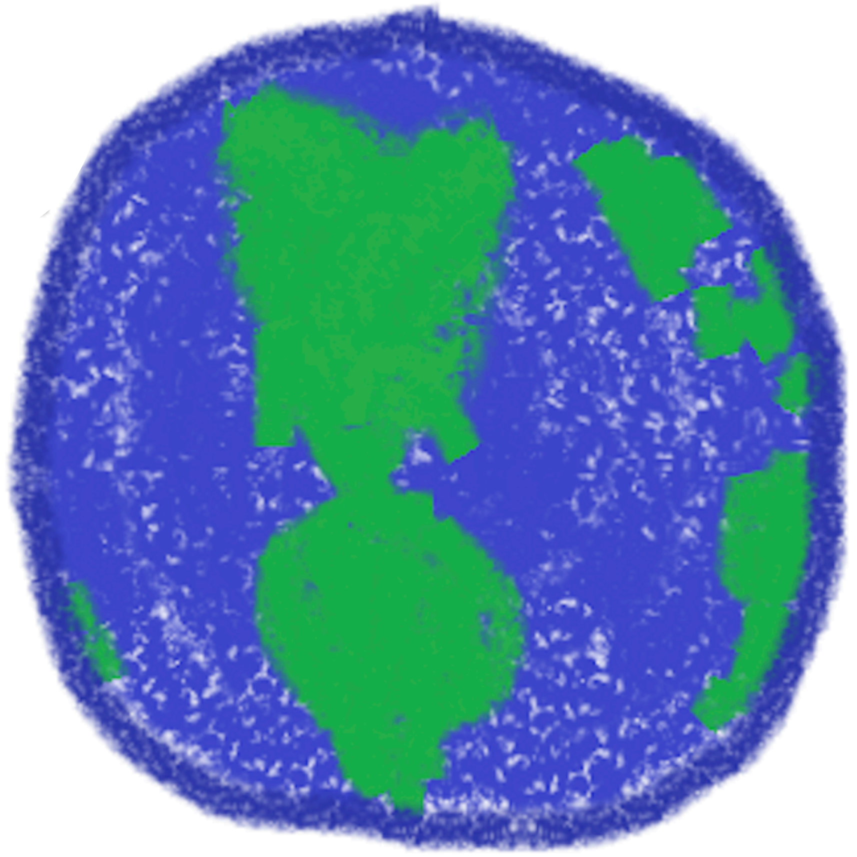

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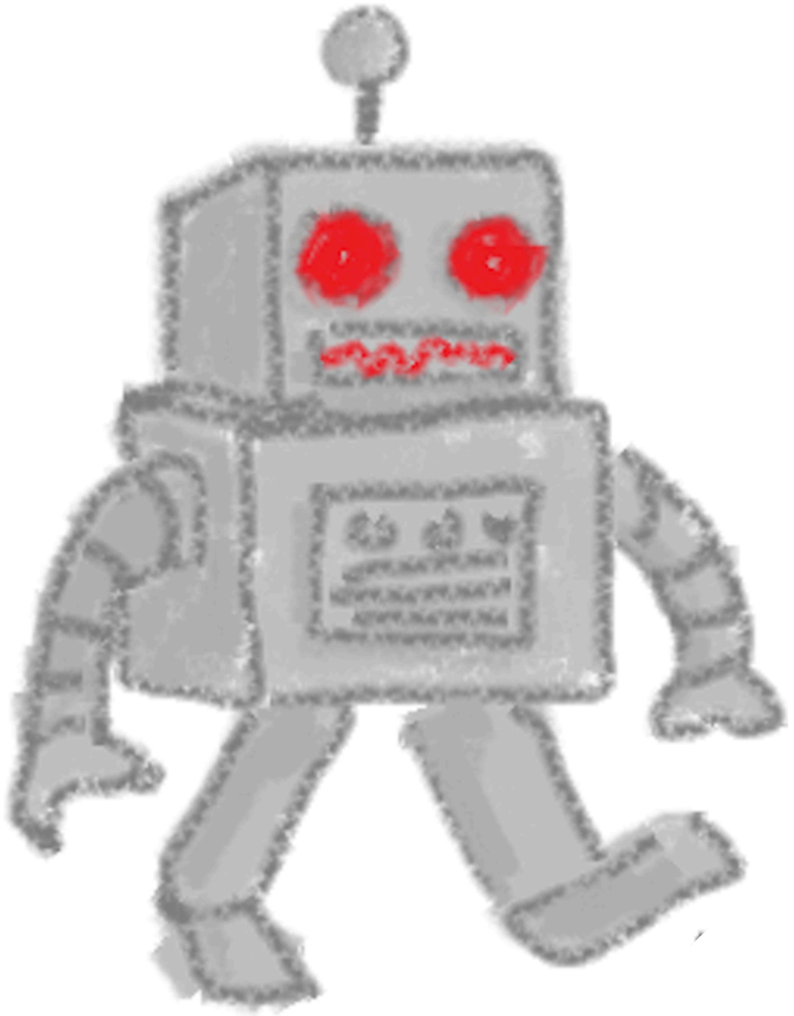

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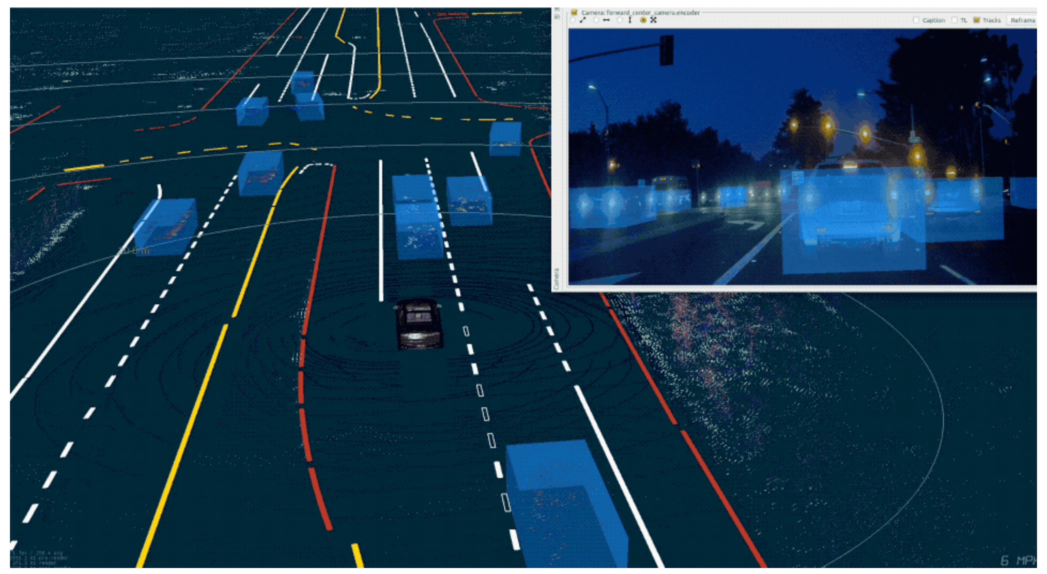

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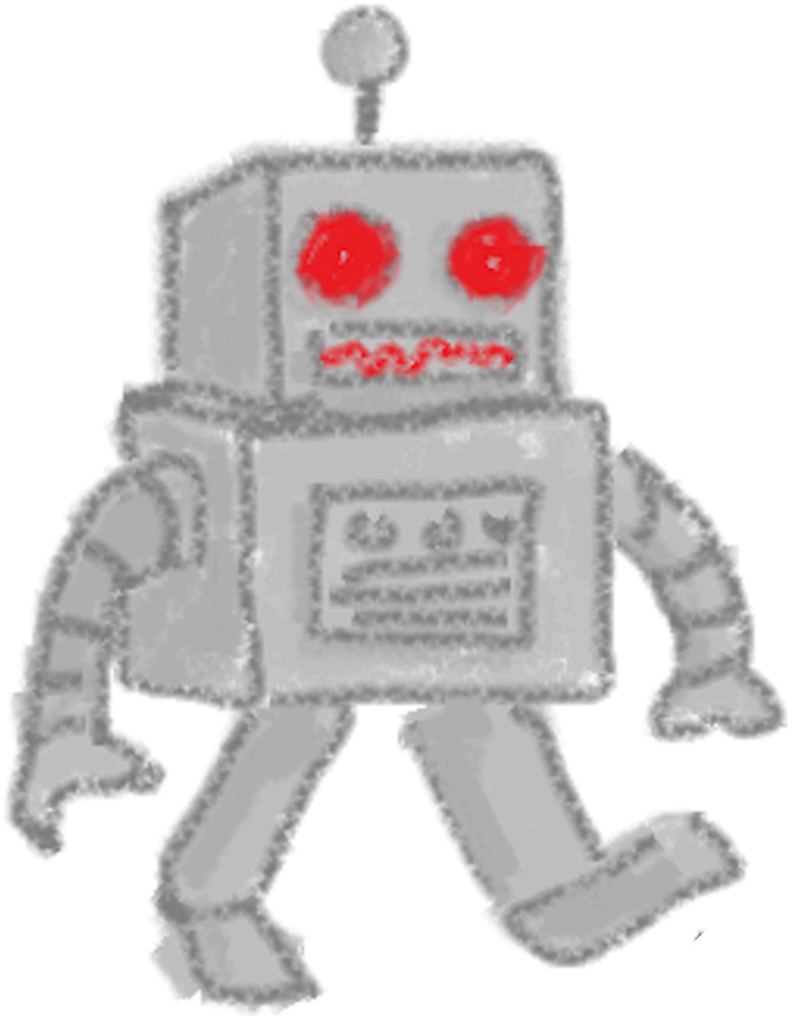

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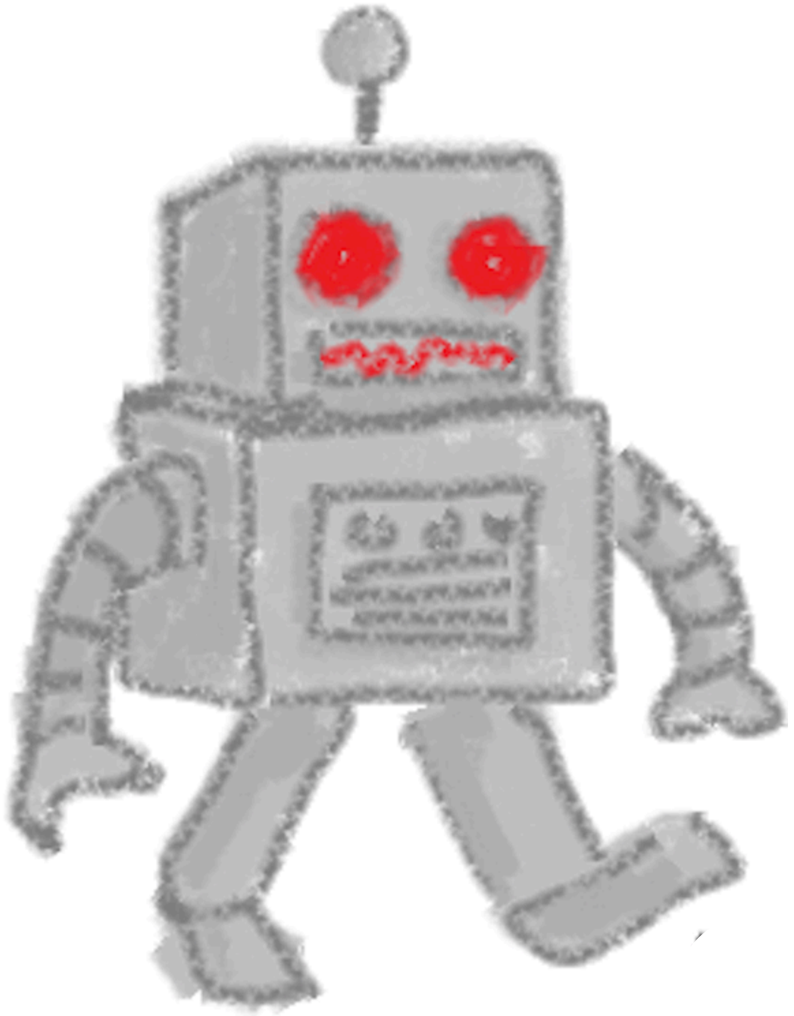

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$$s_0 \sim \mu_0, a_0 \sim \pi(s_0), r_0, s_1 \sim P(s_0, a_0), a_1 \sim \pi(s_1), r_1 \ldots$$

	Learn from Experience	Generalize	Interactive	Exploration	Credit assignment
Supervised Learning					
Reinforcement Learning					

Table content based on slides from Emma Brunskill

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Infinite horizon Discounted MDP

$$\mathcal{M} = \{S, A, P, r, \mu_0, \gamma\}$$

$$P : S \times A \mapsto \Delta(S), \quad r : S \times A \to [0,1], \quad \gamma \in [0,1)$$

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Value function $V^\pi(s) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \middle| s_0 = s, a_h \sim \pi(s_h), s_{h+1} \sim P(\cdot \mid s_h, a_h)\right]$

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Optimal Policy

For infinite horizon discounted MDP, there exists a deterministic stationary policy

$\pi^\star : S \mapsto A$, s.t., $V^{\pi^\star}(s) \geq V^\pi(s), \forall s, \pi$

[Puterman 94 chapter 6, also see theorem 1.4 in the RL monograph]

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Theorem 1: Bellman Optimality

$$V^\star(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^\star(s') \right]$$

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$$\leq \mathbb{E} \left[r(s, \widehat{\pi}(s)) + \gamma r(s', \widehat{\pi}(s')) + \ldots \right] = V^{\widehat{\pi}}(s)$$

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Proof of Bellman Optimality

Theorem 1: Bellman Optimality

$$V^\star(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^\star(s') \right]$$

Denote $\widehat{\pi}(s) := \arg\max_a Q^\star(s,a)$, we just proved $V^{\widehat{\pi}}(s) = V^\star(s), \forall s$

This implies that $\arg\max_a Q^\star(s,a)$ is an optimal policy

Proof of Bellman Optimality

Theorem 2:

For any $V : S \to \mathbb{R}$, if $V(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \right]$ for all s,

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$$|V(s) - V^\star(s)| = \left| \max_a (r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V(s')) - \max_a (r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V^\star(s')) \right|$$

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&\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left| V(s') - V^\star(s') \right| \\
&\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left(\max_{a'} \gamma \mathbb{E}_{s'' \sim P(s',a')} \left| V(s'') - V^\star(s'') \right| \right)
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$$\leq \max_{a_1, a_2, \ldots a_{k-1}} \gamma^k \mathbb{E}_{s_k} |V(s_k) - V^\star(s_k)|$$

Outline

1. Definition of infinite horizon discounted MDPs

2. Bellman Optimality

3. State-action distribution

Trajectory distribution and state-action distribution

Q: what is the probability of π generating trajectory $\tau = \{s_0, a_0, s_1, a_1, \ldots, s_h, a_h\}$?

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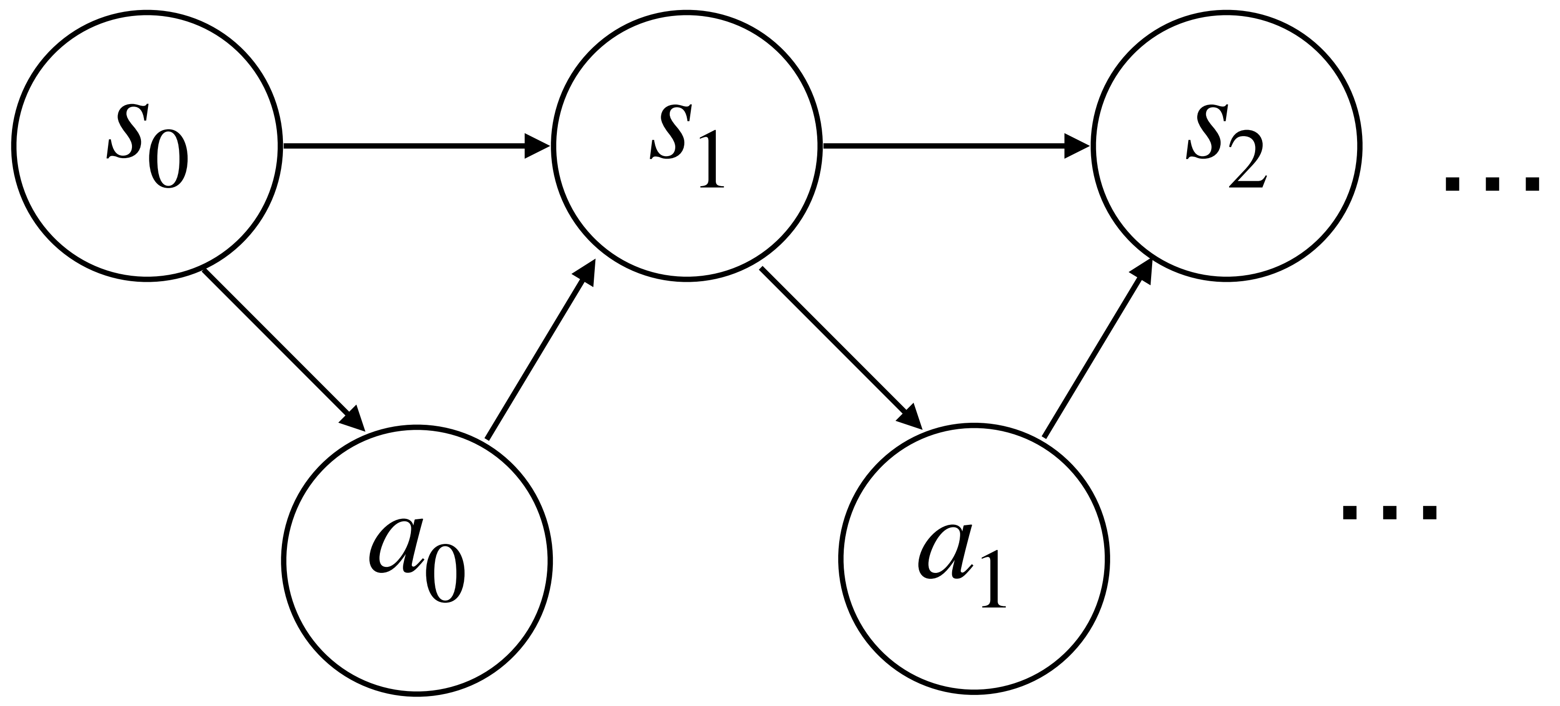

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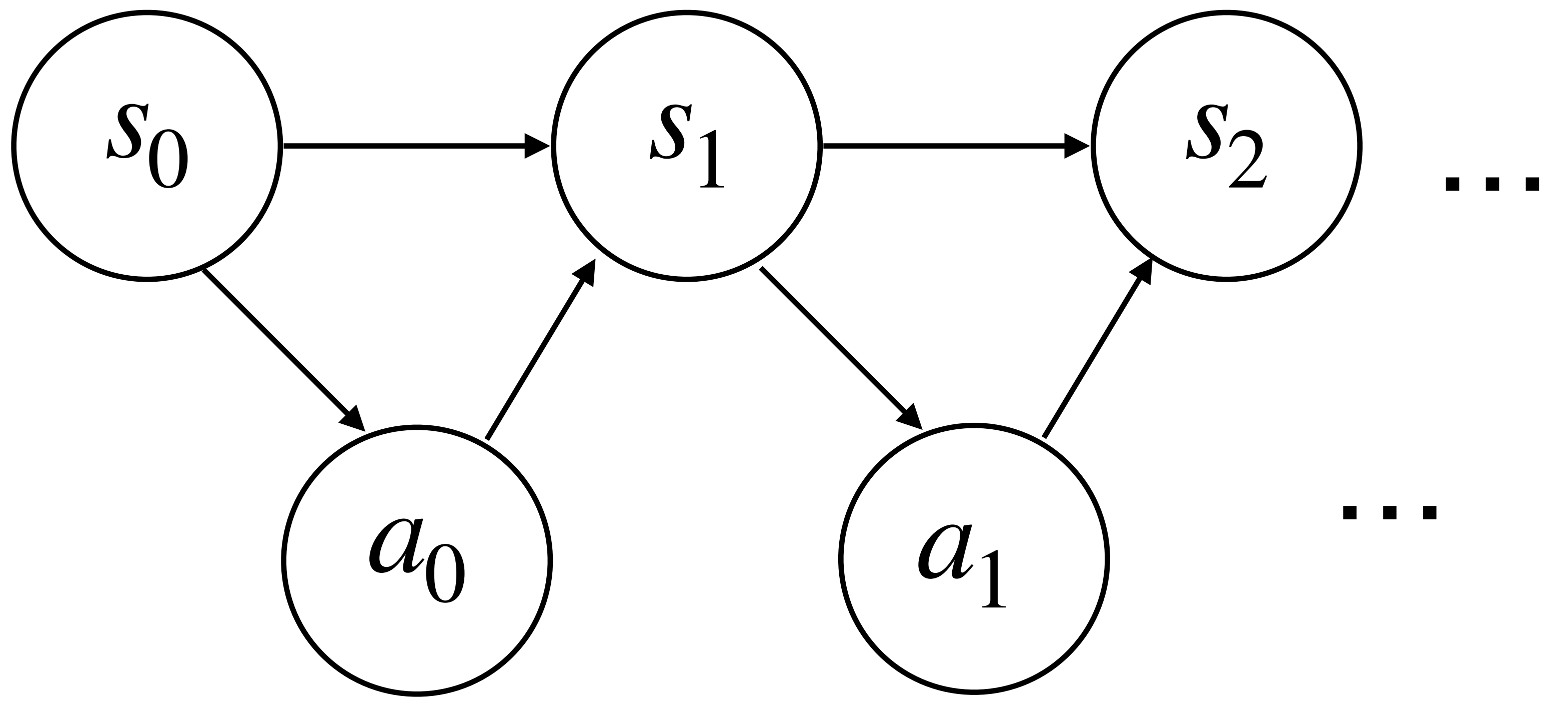

$$\mathbb{P}^\pi(s_0, a_0, \ldots, s_h, a_h)$$

$$= \pi(a_0 \mid s_0) P(s_1 \mid s_0, a_0) \pi(a_1 \mid s_1) P(s_2 \mid s_1, a_1) \ldots P(s_h \mid s_{h-1}, a_{h-1}) \pi(a_h \mid s_h)$$

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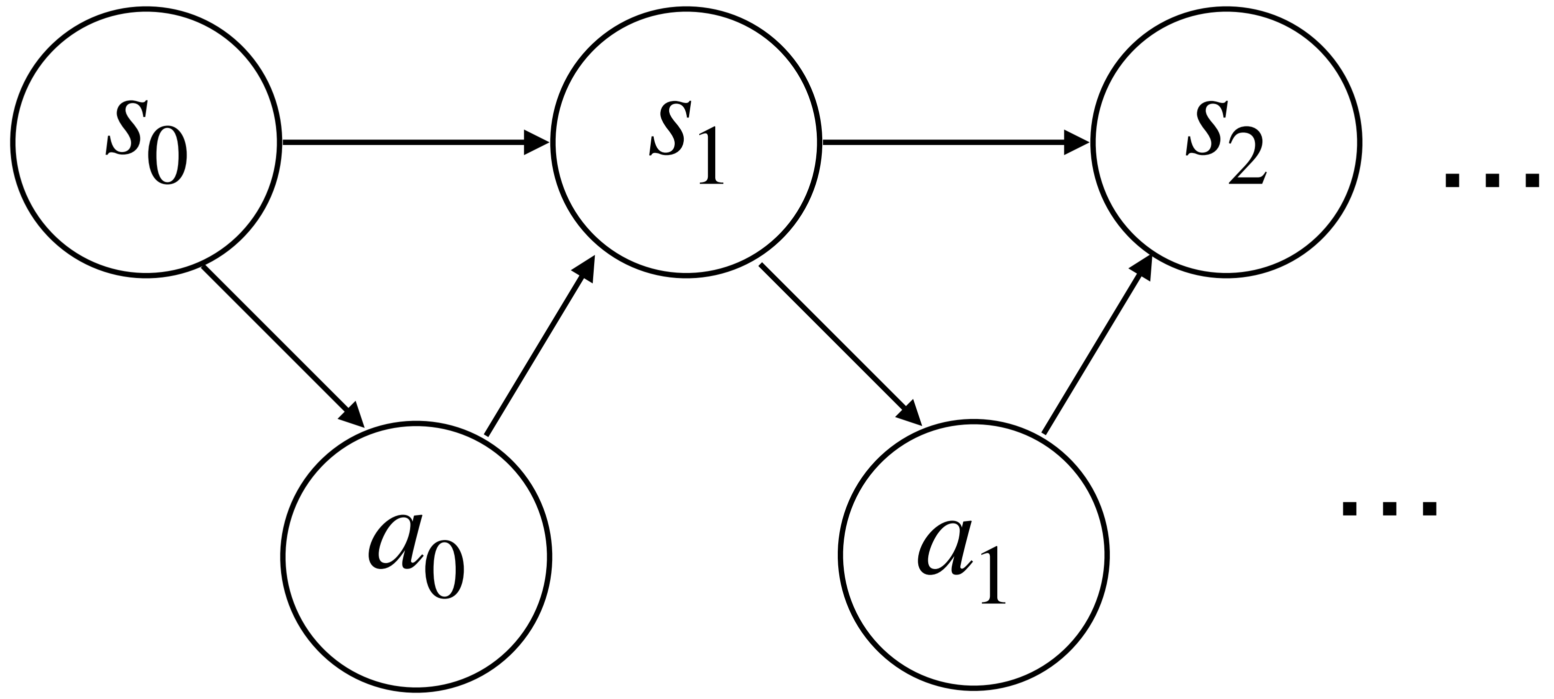

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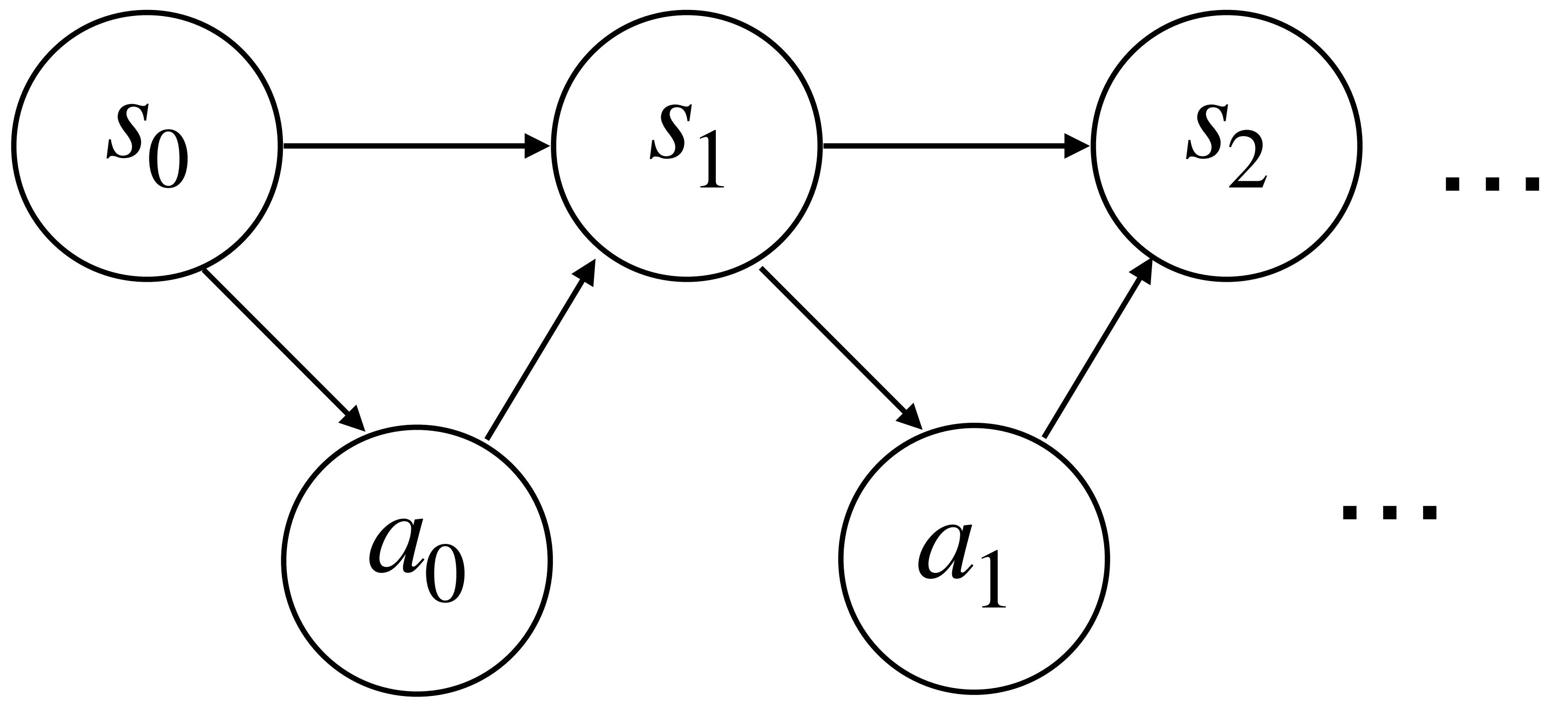

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Q: what's the probability of π visiting state (s,a) at time step h?

$$\mathbb{P}_h^\pi(s, a; s_0) = \sum_{a_0, s_1, a_1, \ldots, s_{h-1}, a_{h-1}} \mathbb{P}^\pi(s_0, a_0, \ldots, s_{h-1}, a_{h-1} s_h = s, a_h = a)$$

State action occupancy measure

$\mathbb{P}_h(s, a; s_0, \pi)$: probability of π visiting (s, a) at time step $h \in \mathbb{N}$, starting at s_0

$$d^{\pi}_{s_0}(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h(s, a; s_0, \pi)$$

$$V^{\pi}(s_0) = \frac{1}{1 - \gamma} \sum_{s,a} d^{\pi}_{s_0}(s, a) r(s, a)$$

Summary for today

Key definitions: MDPs, Value / Q functions, State-action distribution

Key property: Bellman optimality (the two theorems and their proofs)