Basics of MDPs and Planning in MDPs

Wen Sun

CS 6789: Foundations of Reinforcement Learning

Announcements

Office hours (mine and DAs) are coming

HW0 is due Feb 1st

Recap: RL versus Supervised Learning

Q: How to solve Supervised Learning via RL?

Recap: Infinite Horizon MDPs

$$\mathscr{M} = \{S, A, P, r, \gamma\}$$

$$P : S \times A \mapsto \Delta(S), \quad r : S \times A \to [0,1], \quad \gamma \in [0,1)$$

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Value function $V^{\pi}(s) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \,\middle|\, s_0 = s, a_h \sim \pi(s_h), s_{h+1} \sim P(\cdot \mid s_h, a_h)\right]$

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Q function $Q^{\pi}(s, a) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \,\middle|\, (s_0, a_0) = (s, a), a_h \sim \pi(s_h), s_{h+1} \sim P(\cdot \mid s_h, a_h)\right]$

Recap: Bellman Optimality

$$\mathscr{M} = \{S, A, P, r, \gamma\}$$

$$P : S \times A \mapsto \Delta(S), \quad r : S \times A \to [0,1], \quad \gamma \in [0,1)$$

Theorem 1: Bellman Optimality (Q-version)

$$Q^\star(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[\max_{a' \in A} Q^\star(s', a') \right]$$

Main Question for Today:

Given an MDP $\mathscr{M} = (S, A, P, r, \gamma)$, How to find $\pi^\star$ (stationary & deterministic)

Outline

1. Bellman optimality — property of $V^\star$

2. Optimal planning: Value Iteration

Bellman Optimality

Theorem 2:

For any $V : S \to \mathbb{R}$, if $V(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \right]$ for all s,

then $V(s) = V^\star(s), \forall s$

$$\min_V \sum_s \left(V(s) - \max_a \left[r(s,a) + \gamma \mathop{\mathbb{E}}_{s' \sim P(s,a)} V(s') \right] \right)^2$$

$$= 0$$

Bellman Optimality

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then $V(s) = V^\star(s), \forall s$

$$|\, V(s) - V^\star(s) \,| = \left| \max_a (r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V(s')) - \max_a (r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V^\star(s')) \right|$$

V(s)

Bell-opt for $V^\star$

Bellman Optimality

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$$\leq \max_a \left| (r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V(s')) - (r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V^\star(s')) \right|$$

$$\left| \max_x f(x) - \max_x g(x) \right|$$

$$\leq \max_x \left| f(x) - g(x) \right|$$

Bellman Optimality

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$$
\begin{aligned}
|V(s) - V^\star(s)| &= \left| \max_a (r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V(s')) - \max_a (r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V^\star(s')) \right| \\
&\leq \max_a \left| (r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V(s')) - (r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V^\star(s')) \right| \\
&\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left| V(s') - V^\star(s') \right|
\end{aligned}
$$

$|\mathbb{E}_x f(x)| \leq \mathbb{E}_x |f(x)|$

Bellman Optimality

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&\leq \max_a \left| (r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V(s')) - (r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V^\star(s')) \right| \\
&\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left| V(s') - V^\star(s') \right| \\
&\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left(\max_{a'} \gamma \mathbb{E}_{s'' \sim P(s',a')} \left| V(s'') - V^\star(s'') \right| \right)
\end{aligned}
$$

Bellman Optimality

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For any $V : S \to \mathbb{R}$, if $V(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \right]$ for all s,

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&\leq \max_a \gamma \mathbb{E}_{s' \sim P(s,a)} \left(\max_{a'} \gamma \mathbb{E}_{s'' \sim P(s',a')} \left| V(s'') - V^\star(s'') \right| \right) \\
&\leq \max_{a_1, a_2, \ldots a_{k-1}} \gamma^k \mathbb{E}_{s_k} |V(s_k) - V^\star(s_k)|
\end{aligned}
$$

$= 0$ $\lim_{k \to \infty}$

set $k \to \infty$ then $\gamma^\infty = 0$

Bellman Optimality for $Q^\star$

What about $Q^\star$?

$$Q^*(s,a) = r(s,a) + \gamma \mathop{\mathbb{E}}_{s' \sim p(s,a)} \max_{a'} Q^*(s',a')$$

Bellman Optimality for $Q^\star$

What about $Q^\star$?

We should have:

For any $Q : S \times A \to \mathbb{R}$, if $Q(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} \max_{a'} Q(s',a')$

for all s, then $Q(s,a) = Q^\star(s,a), \forall s, a$

once we find Q^*, $\pi^*(s) = \arg\max_a Q^*(s,a)$

Outline

1. Bellman optimality — property of $V^\star$

2. Optimal planning: Value Iteration

Define Bellman Operator $\mathcal{T}$:

Given a function $f : S \times A \mapsto \mathbb{R}$,

$$\mathcal{T}f : S \times A \mapsto \mathbb{R},$$

$$(\mathcal{T}f)(s,a) := r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} \max_{a' \in A} f(s',a'), \forall s, a \in S \times A$$

$\in \mathbb{R}$

Value Iteration Algorithm:

$\sum_h \gamma^h r_h \leq \sum_h \gamma^h = \frac{1}{1-\gamma}$

1. Initialization: $Q^0 : \|Q^0\|_\infty \in (0, \frac{1}{1-\gamma})$

2. Iterate until convergence: $Q^{t+1} = \mathcal{T} Q^t$

$\forall s,a \quad Q^{t+1}(s,a) := r(s,a) + \gamma \mathop{\mathbb{E}}_{s' \sim p(s,a)} \max_{a'} Q^t(s',a')$

Intuition:

Via Bellman optimality theorem:

$$Q^\star = \mathcal{T} Q^\star$$

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Consider the simple problem: finding fixed point solution $x^\star = \ell(x^\star)$

$\ell : \mathbb{R} \to \mathbb{R}$

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$$|x_t - x^\star| =$$

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$$x_0, x_{t+1} = \ell(x_t), t = 0, \ldots,$$

$$|x_t - x^\star| = |\ell(x_{t-1}) - \ell(x^\star)| \leq L |x_{t-1} - x^\star|$$

$$\leq L^2 |x_{t-2} - x^\star|$$

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$$|x_t - x^\star| = |\ell(x_{t-1}) - \ell(x^\star)| \leq L |x_{t-1} - x^\star|$$

If $L < 1$ (i.e., contraction), then it converges exponentially fast

Convergence of Value Iteration:

Lemma [contraction]: Given any Q, Q', we have:

$$\|\mathcal{T}Q - \mathcal{T}Q'\|_\infty \leq \gamma \|Q - Q'\|_\infty$$

Proof:

$\|f(x)\|_\infty$

$= \max_x |f(x)|$

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$$|\mathcal{T}Q(s,a) - \mathcal{T}Q'(s,a)| = \left| r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \max_{a'} Q(s',a') - \left(r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \max_{a'} Q'(s',a') \right) \right|$$

$\mathcal{T}Q(s,a)$ $\qquad$ $\mathcal{T}Q'(s,a)$

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$$\leq \gamma \sum_{s'} P(s' \mid s,a) \left| \left(\max_{a'} Q(s',a') - \max_{a'} Q'(s',a') \right) \right|$$

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&\leq \gamma \sum_{s'} P(s' \mid s,a) \left| \left(\max_{a'} Q(s',a') - \max_{a'} Q'(s',a') \right) \right| \\
&\leq \gamma \sum_{s'} P(s' \mid s,a) \max_{a'} \left| \left(Q(s',a') - Q'(s',a') \right) \right|
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\end{aligned}$$

$\forall s, a$

$|\mathcal{T}Q(s,a) - \mathcal{T}Q'(s,a)| \leq \gamma \|Q - Q'\|_\infty$

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&\leq \gamma \max_{s'} \max_{a'} \left| \left(Q(s',a') - Q'(s',a') \right) \right| \quad = \gamma \|Q - Q'\|_\infty
\end{aligned}$$

Convergence of Value Iteration:

Lemma [Convergence]: Given Q^0, we have:

$$\|Q^t - Q^\star\|_\infty \leq \gamma^t \|Q^0 - Q^\star\|_\infty$$

Proof:

$\mathcal{T} Q^\star = Q^\star$

Convergence of Value Iteration:

Lemma [Convergence]: Given Q^0, we have:

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Proof:

$$\|Q^{t+1} - Q^\star\|_\infty = \|\mathcal{T}Q^t - \mathcal{T}Q^\star\|_\infty \leq \gamma \|Q^t - Q^\star\|_\infty$$

$Q^{t+1} \quad Q^\star \quad \leq \gamma^2 \|Q^{t-1} - Q^\star\|_\infty$

Convergence of Value Iteration:

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Proof:

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$$\cdots \leq \gamma^{t+1} \|\widehat{Q}^0 - Q^\star\|_\infty$$

$\hat{\pi}^t(s) = \operatorname{argmax}_a \hat{Q}^t(sa)$ when $t \to \infty$

Final Quality of the Policy:

$$\pi^t : \pi^t(s) = \arg\max_a Q^t(s,a)$$

Theorem: $V^{\pi^t}(s) \geq V^\star(s) - \frac{2\gamma^t}{1-\gamma}\|Q^0 - Q^\star\|_\infty \forall s \in S$

Proof:

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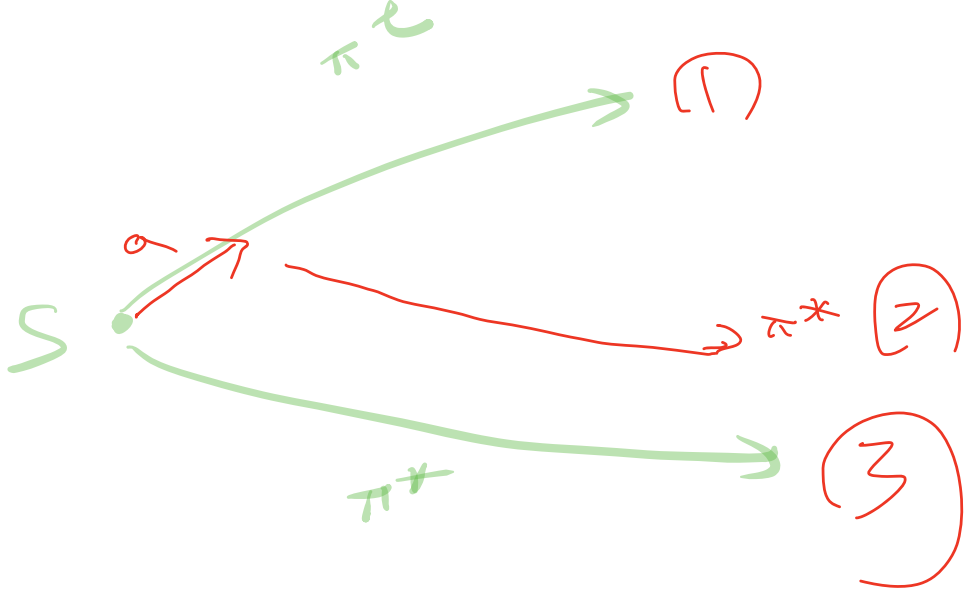
$$V^{\pi^t}(s) - V^\star(s) = Q^{\pi^t}(s, \pi^t(s)) - Q^\star(s, \pi^\star(s))$$



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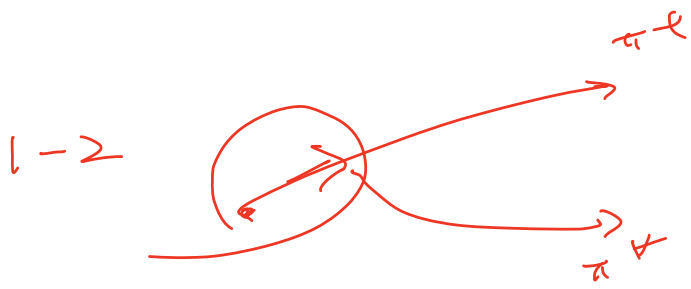
$$V^{\pi^t}(s) - V^\star(s) = Q^{\pi^t}(s, \pi^t(s)) - Q^\star(s, \pi^\star(s))$$

$$= Q^{\pi^t}(s, \pi^t(s)) - Q^\star(s, \pi^t(s)) + Q^\star(s, \pi^t(s)) - Q^\star(s, \pi^\star(s))$$

1 - 2

2 - 3



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Proof:

$$V^{\pi^t}(s) - V^\star(s) = Q^{\pi^t}(s,\pi^t(s)) - Q^\star(s,\pi^\star(s))$$

$$= Q^{\pi^t}(s,\pi^t(s)) - Q^\star(s,\pi^t(s)) + Q^\star(s,\pi^t(s)) - Q^\star(s,\pi^\star(s))$$

$$= \gamma \mathbb{E}_{s' \sim P(s,\pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s,\pi^t(s)) - Q^\star(s,\pi^\star(s))$$

$Q^{\pi^t}(s,\pi^t(s))$

$= R(s,\pi^t(s)) + \gamma \mathbb{E}_{s'} V^{\pi^t}(s')$

$Q^\star(s,\pi^t(s)) = R(s,\pi^t(s)) + \gamma \mathbb{E}_{s'} V^\star(s')$

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≤ 0

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Proof:

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$$\geq \gamma \mathbb{E}_{s' \sim P(s,\pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s,\pi^t(s)) - Q^t(s,\pi^t(s)) + Q^t(s,\pi^\star(s)) - Q^\star(s,\pi^\star(s))$$

$$\geq \gamma \mathbb{E}_{s' \sim P(s,\pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) - 2\gamma^t\|Q^0 - Q^\star\|_\infty$$

$\geq \gamma \mathbb{E}_{s' \sim P(s,\pi^t(s))}$ $\gamma \mathbb{E}_{s'' \sim P(s',\pi^t(s'))}\left(V^{\pi^t}(s'') - V^\star(s'')\right) - 2\gamma^t\|Q^0 - Q^\star\|_\infty$ $\leq 2\gamma^t\|Q^0 - Q^\star\|_\infty$

Final Quality of the Policy:

$$\pi^t : \pi^t(s) = \arg\max_a Q^t(s,a)$$

Theorem: $V^{\pi^t}(s) \geq V^\star(s) - \frac{2\gamma^t}{1-\gamma}\|Q^0 - Q^\star\|_\infty \forall s \in S$

Proof:

$$V^{\pi^t}(s) - V^\star(s) = Q^{\pi^t}(s,\pi^t(s)) - Q^\star(s,\pi^\star(s))$$

$$= Q^{\pi^t}(s,\pi^t(s)) - Q^\star(s,\pi^t(s)) + Q^\star(s,\pi^t(s)) - Q^\star(s,\pi^\star(s))$$

$$= \gamma\mathbb{E}_{s'\sim P(s,\pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s,\pi^t(s)) - Q^\star(s,\pi^\star(s))$$

$$\geq \gamma\mathbb{E}_{s'\sim P(s,\pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s,\pi^t(s)) - Q^t(s,\pi^t(s)) + Q^t(s,\pi^\star(s)) - Q^\star(s,\pi^\star(s))$$

$$\geq \gamma\mathbb{E}_{s'\sim P(s,\pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) - 2\gamma^t\|Q^0 - Q^\star\|_\infty \quad \text{...Recursion}$$

$$\geq -2\gamma^t\|Q^0 - Q^\star\|_\infty \left(\sum_{h=0}^{\infty}\gamma^h\right)$$

Summary for today

Planning algorithm (no learning so far):

VI: fixed point iteration $Q^{t+1} = \mathcal{T} Q^t$

1. Bellman operator is a contraction map

2. $\|Q^t - Q^\star\|_\infty$ being small implies V^{π^t} & $V^\star$ are close