Generalization in Large scale MDPs

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CS 6789: Foundations of Reinforcement Learning

Recap on Bellman Error and Bellman Operator

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If $BE(s,a) \neq 0$, then $f \neq Q^\star$

Notations

Probability of π visiting (s, a) at time step h: $d_h^\pi(s, a)$

Question for Today

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In other words, what structural conditions permit RL generalization, provably?

Outline for Today

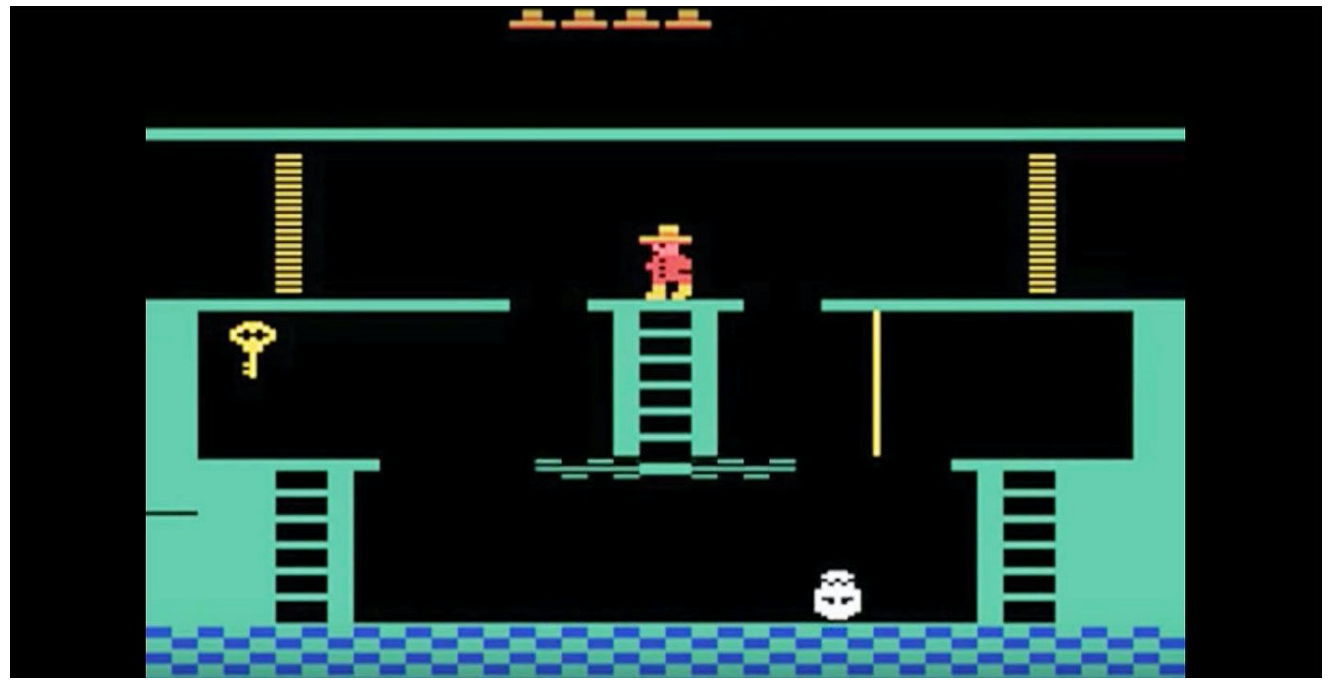
1. Bellman rank Definitions

2. Examples that are captured by the Bellman rank framework

Setting

Finite horizon episodic MDP $\left\{\{S_h\}_{h=0}^{H}, \{A_h\}_{h=0}^{H-1}, H, s_0, r, P\right\}$

State space S_h is extremely large:

$s =$ 

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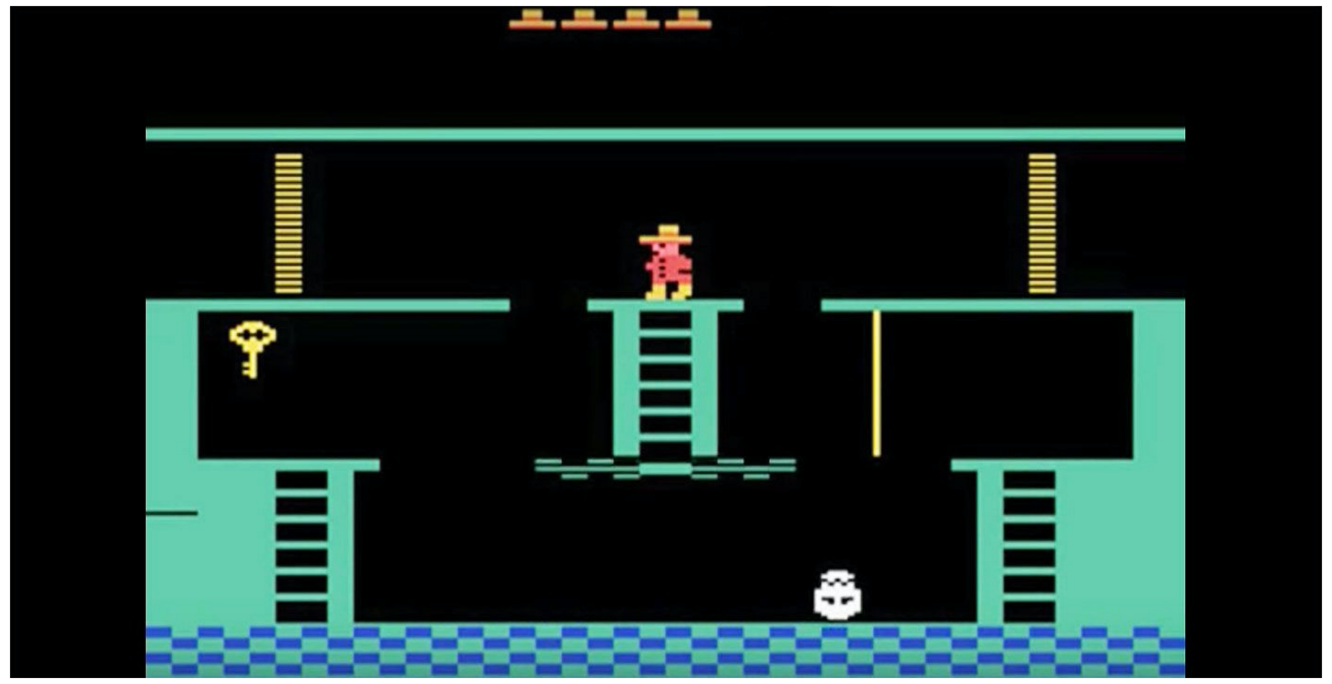
$s =$ 

Not acceptable: poly $(|S|)$

Need to generalize via (nonlinear) function approximation

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We will consider **Q function class**

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Define **value function class**: $\mathcal{V} = \{V_f : V_f(s) = \max_a f(s,a) \,|\, f \in \mathcal{F}\}$

Learning Goal:

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Given approximation error ϵ and failure prob δ,
can we learn ϵ *near optimal policy* (i.e., $V^{\widehat{\pi}} \geq V^{\star} - \epsilon$) in # of samples scaling
poly* with all relevant parameters *(here, we need poly in* $\ln(|\mathscr{F}|)$*)

How to check if a Q-approximator is good?

We define **average** Bellman error of a Q-estimate g below:

$$\mathscr{E}(g; f, h) = \mathbb{E}_{s_h, a_h \sim d_h^{\pi_f}} \left[g(s_h, a_h) - r(s_h, a_h) - \mathbb{E}_{s_{h+1} \sim P(\cdot | s_h, a_h)} \left[\max_{a \in \mathscr{A}} g(s_{h+1}, a) \right] \right]$$

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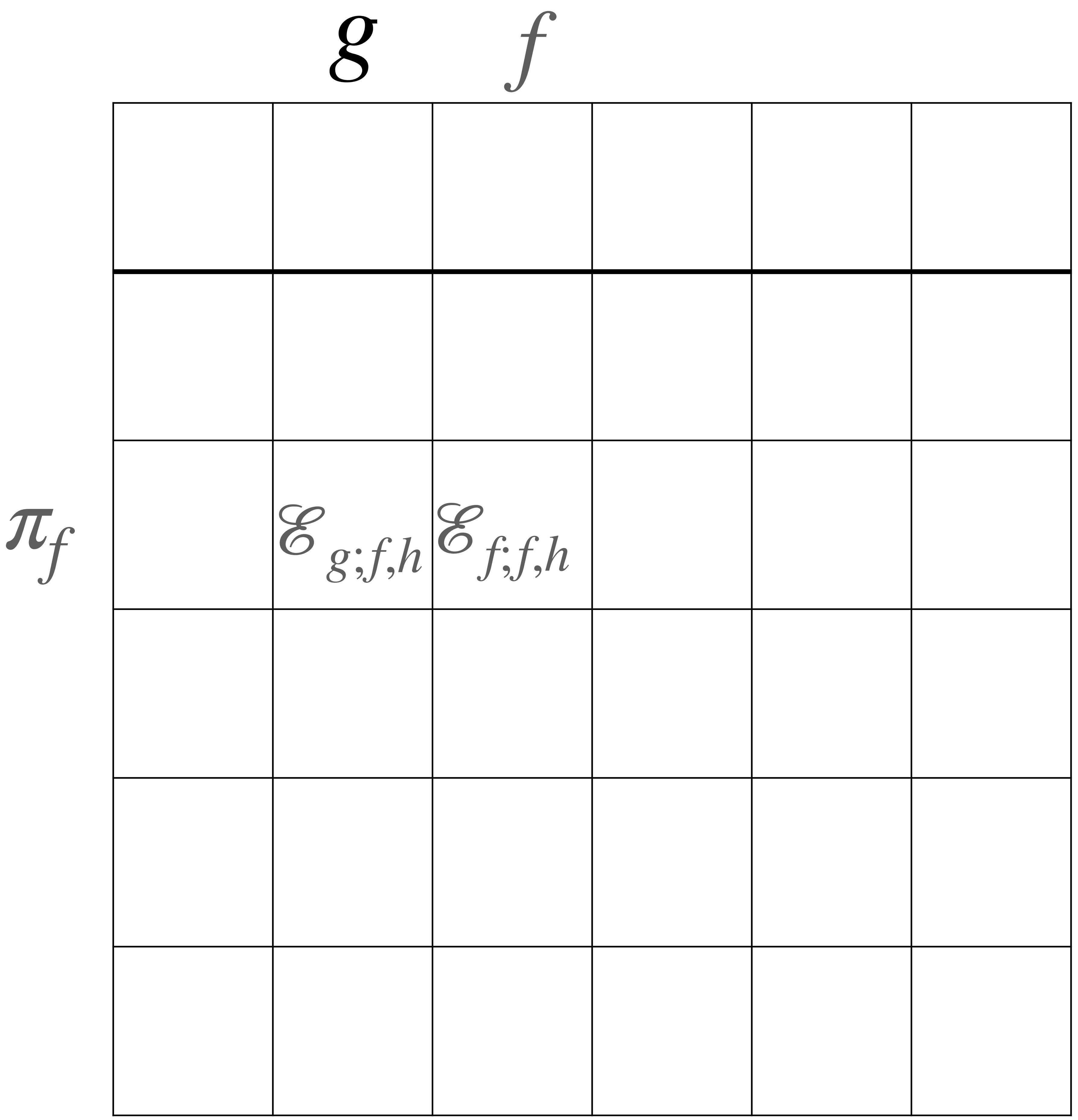
The Q / V-Bellman rank

$$\forall h : \mathcal{E}_h \in \mathbb{R}^{|\mathcal{F}|\times|\mathcal{F}|}$$



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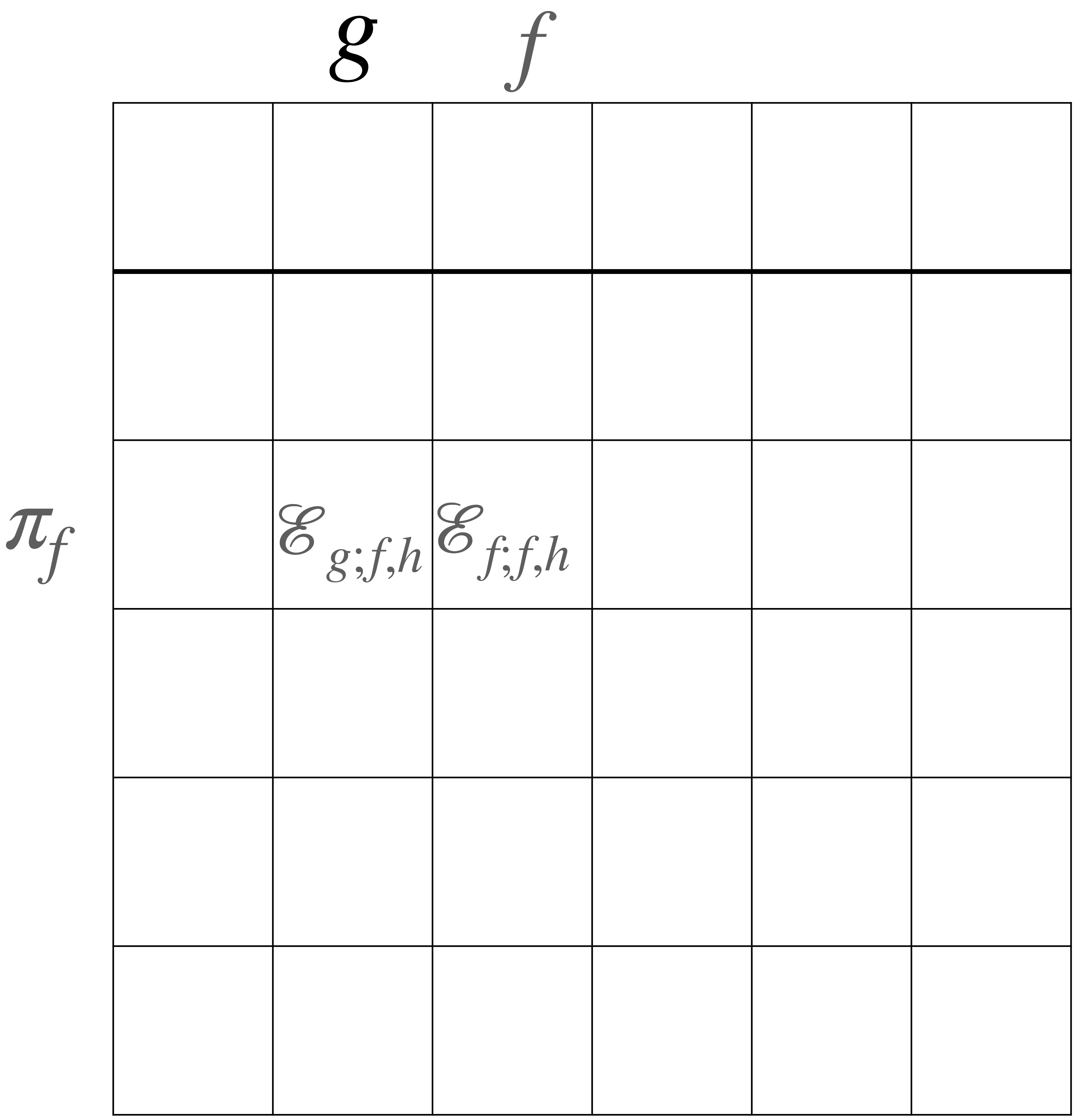

Rank of this Matrix is defined as Bellman Rank

The Q / V-Bellman rank

In other words, there are two mappings $W_h : \mathcal{F} \mapsto \mathbb{R}^d, \quad X_h : \mathcal{F} \mapsto \mathbb{R}^d$ (d = Bellman-rank)

$$\forall f, g \in \mathcal{F} : \mathcal{E}(g; f, h) = \left\langle W_h(g), X_h(f) \right\rangle$$

Note, we just assume the existence of W, X, but they are unknown

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Note linear Bell-completion captures tabular / linear mdp already

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As we will see, linear Q*&V* is learnable, and recall linear Q* is not…

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Claim: this model has Q-Bellman rank $|Z||A| + |Z|$

We can show that this model is captured by linear $Q^\star$ & $V^\star$

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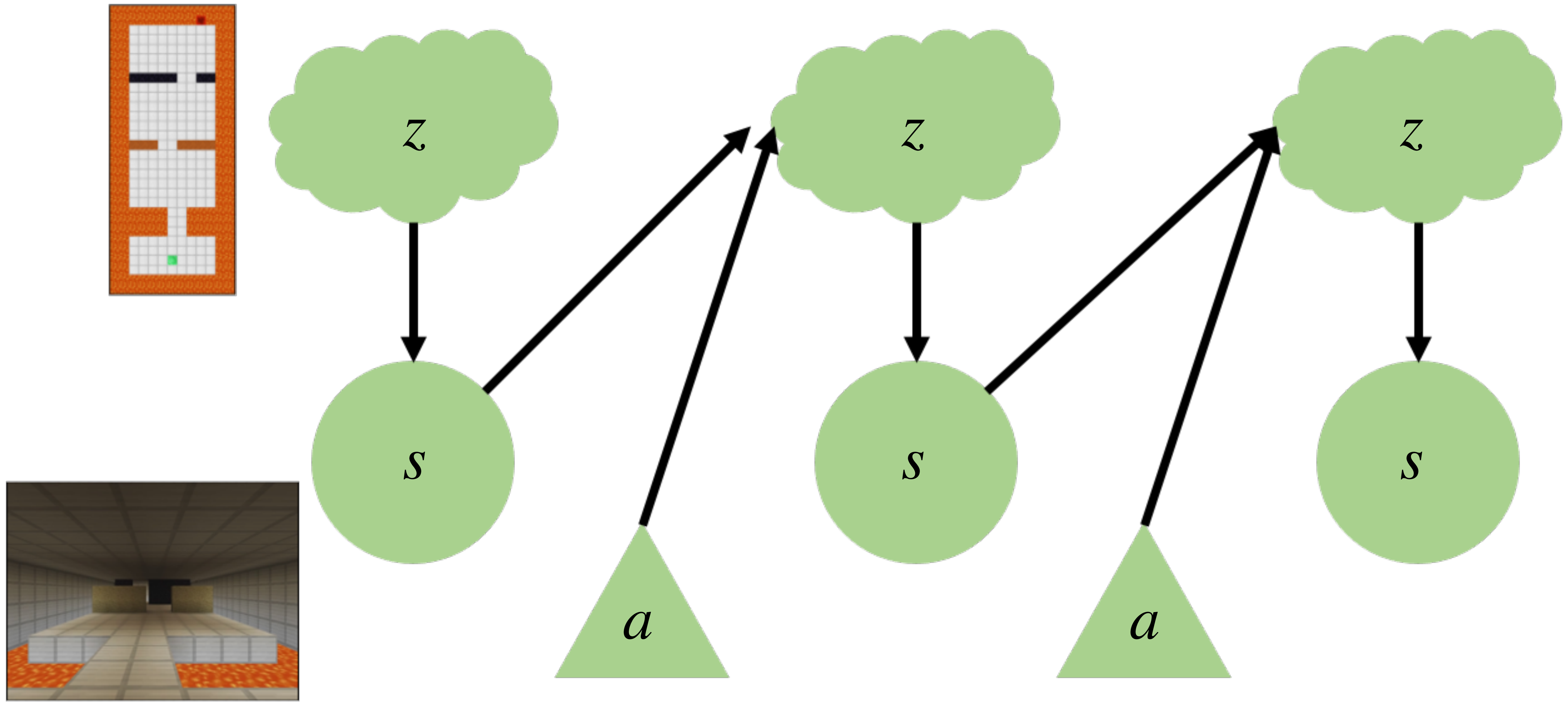
Latent variable MDP

Latent variable MDP is captured by low-rank MDP, so it has small V-Bellman rank…



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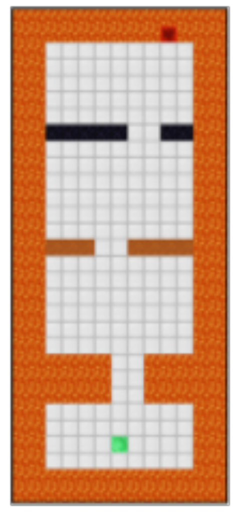

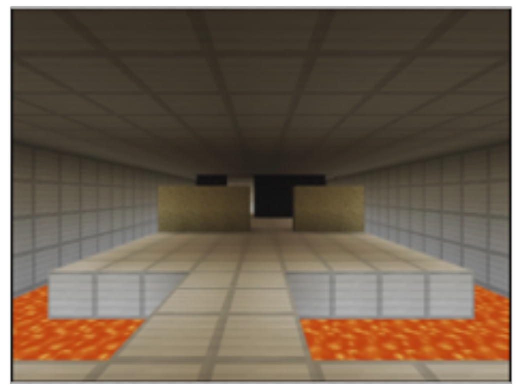

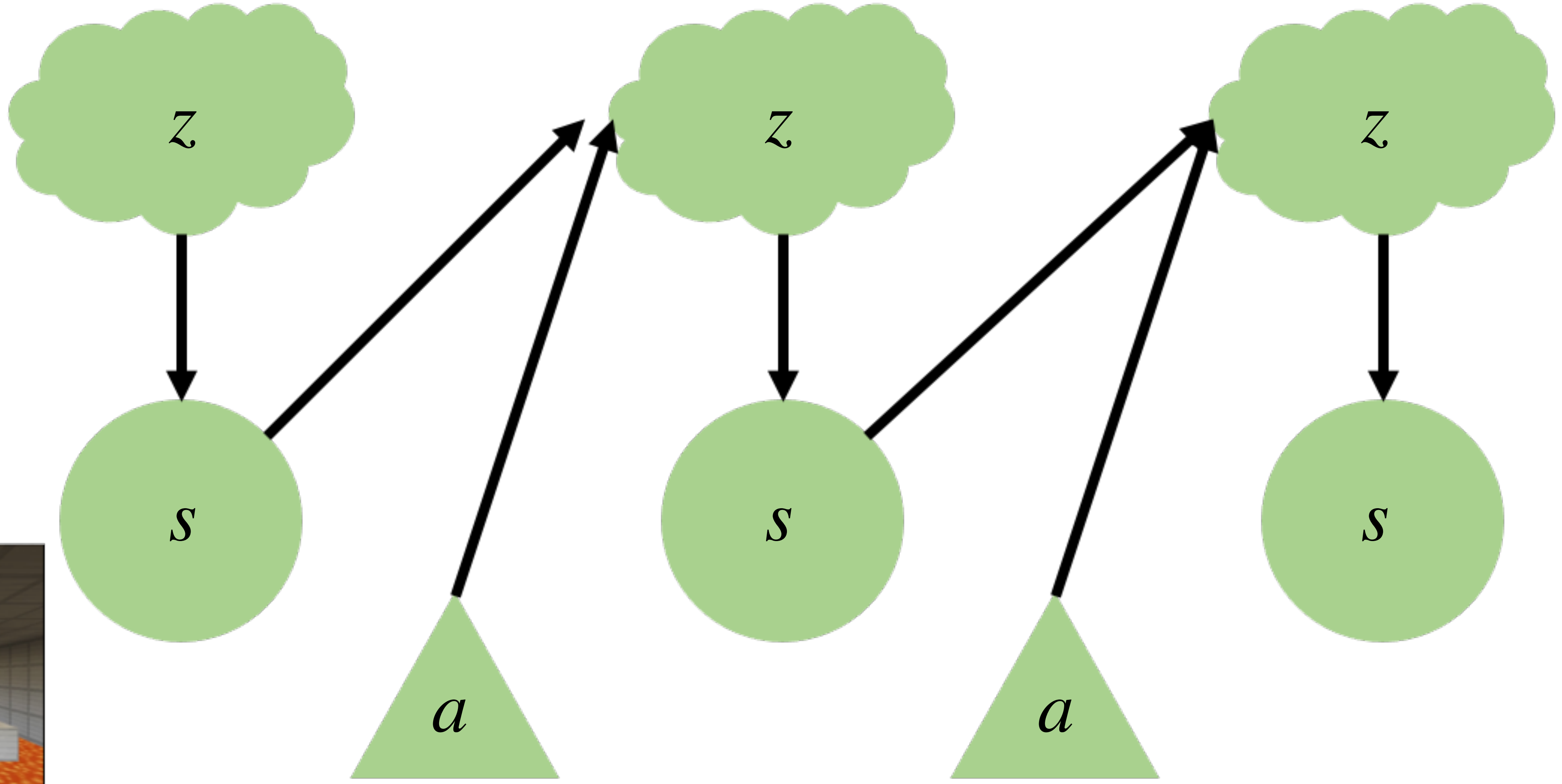

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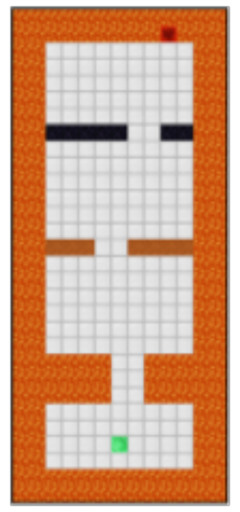

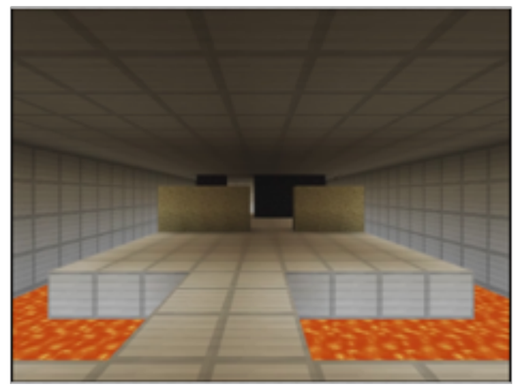

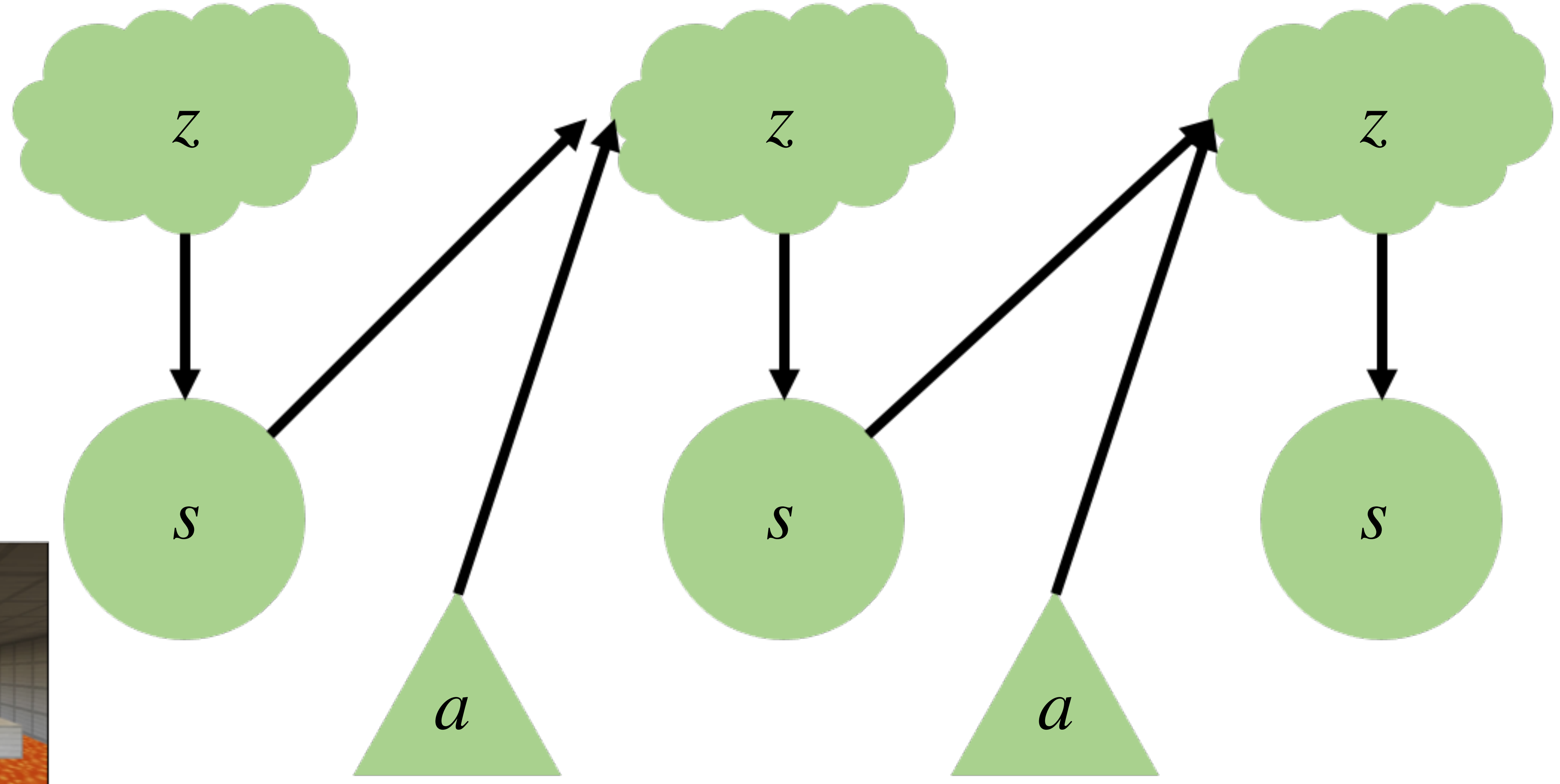

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V-Bellman rank = Number of latent states

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4. Many models (more in the book chapter) indeed have low-Q or V Bellman rank

Next week:

A general algorithm that can learn an ϵ near optimal policy w/ # of samples

$$\text{poly}(H, 1/\epsilon, \ln(|\mathcal{H}|), \text{b-rank})$$