Planning in MDPs

Wen Sun

CS 6789: Foundations of Reinforcement Learning

Announcements

HW0 is due tmr

Recap: Value iteration

$$Q^{t+1} = \mathcal{T} Q^t$$

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Q: when will π^t be the optimal policy?

Outline

1. Policy Iteration

2. Computation complexity of VI and PI

3. Linear Programming formulation

Policy Iteration Algorithm:

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Closed-form for PE
(see 1.1.3 in Monograph)

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Analysis of Policy Iteration

Recall: Policy Improvement $\pi^{t+1}(s) = \arg\max_a Q^{\pi^t}(s, a), \forall s$

Lemma: Monotonic improvement $Q^{\pi^{t+1}}(s, a) \geq Q^{\pi^t}(s, a), \forall s, a$

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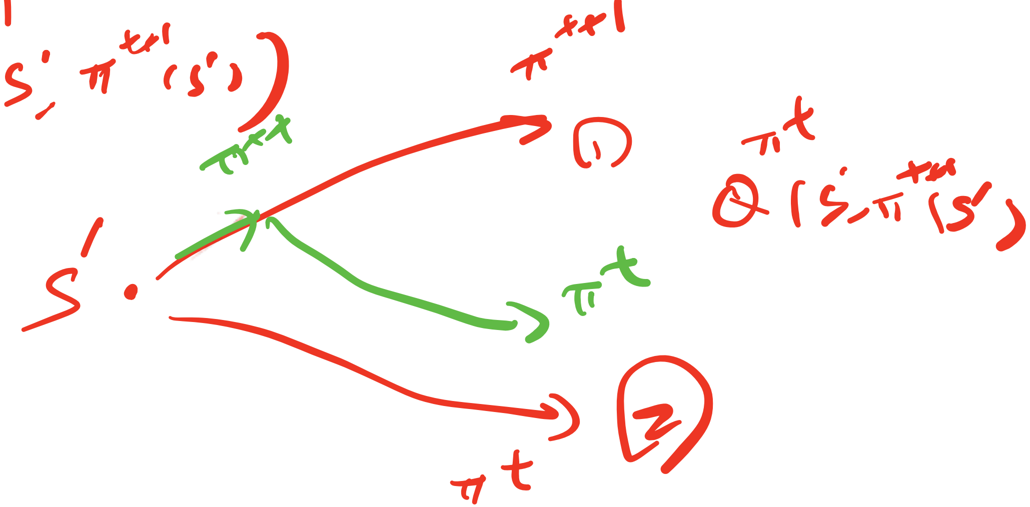

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≥ 0

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$$\geq \gamma \mathbb{E}_{s' \sim P(s,a)} \left[Q^{\pi^{t+1}}(s', \pi^{t+1}(s')) - Q^{\pi^t}(s', \pi^{t+1}(s')) \right]$$

$$\geq \gamma \mathbb{E}_{s' \sim P(s,a)} \left[\gamma \mathbb{E}_{s'' \sim P(s', \pi^{t+1}(s'))} \left[Q^{\pi^{t+1}}(s'', \pi^{t+1}(s'')) - Q^{\pi^t}(s'', \pi^{t+1}(s'')) \right] \right]$$

$$= \gamma^{\infty} \cdot (\cdots)$$

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$V^{\pi^{t+1}}(s) \geq V^{\pi^t}(s), \forall s$

$$V^{\pi^{t+1}}(s) = Q^{\pi^{t+1}}(s, \pi^{t+1}(s)) \geq Q^{\pi^t}(s, \pi^{t+1}(s)) \geq Q^{\pi^t}(s, \pi^t(s)) = V^{\pi^t}(s)$$

Analysis of Policy Iteration

Recall: Policy Improvement $\pi^{t+1}(s) = \arg\max_a Q^{\pi^t}(s, a), \forall s$

Theorem: Convergence $\|V^{\pi^{t+1}} - V^\star\|_\infty \leq \gamma \|V^{\pi^t} - V^\star\|_\infty$

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$$V^\star(s) - V^{\pi^{t+1}}(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V^\star(s') \right] - \left[r(s, \pi^{t+1}(s)) + \gamma \mathbb{E}_{s' \sim P(s,\pi^{t+1}(s))} V^{\pi^{t+1}}(s') \right]$$

Bell-opt

Bell-eqn for π^{t+1}

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$$Q^{\pi^t}(s, \pi^{t+1}(s))$$

$$= \max_a Q^{\pi^t}(s,a)$$

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$$\leq \max_a \left(r(s,a) + \gamma \mathbb{E}_{s'\sim P(s,a)} V^\star(s') - \left(r(s,a) + \gamma \mathbb{E}_{s'\sim P(s,a)} V^{\pi^t}(s') \right)\right)$$

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$$\leq \gamma \|V^\star - V^{\pi^t}\|_\infty$$

Analysis of Policy Iteration

Q: what happens when π^{t+1} and π^{t} are exactly the same?

Show that π^{t} is an optimal policy $\pi^{\star}$

Q: does this imply that the algorithm will terminate?

S A A^S

Outline

1. Policy Iteration

2. Computation complexity of VI and PI

3. Linear Programming formulation

Computation complexity of VI and PI

Given an MDP $\mathscr{M} = (S, A, P, r, \gamma)$ can we exactly compute $Q^\star$ (or find $\pi^\star$) in time polynomial wrt $S, A, 1/(1-\gamma)$?

$\frac{1}{1-\gamma} \leftarrow 1+\gamma+\gamma^2 + \dots \gamma^\infty$

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$$(S^3 + S^2A) \cdot \min \left\{ \frac{A^S}{S}, \frac{S^2 A \log \frac{S^2}{1-\gamma}}{1-\gamma} \right\}$$

per iter - complexity

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What about poly(S, A) algs?

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The primal linear programming

Recall the Bellman consistency:

$$V(s) = \max_a \left\{ r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[V(s') \right] \right\}, \forall s$$

$V(s) = V^*(s)$

$V(s) \geq r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V(s'), \ \forall a$

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$$\min \sum_s \mu(s) V(s) \qquad \mu(s) > 0, \forall s$$

$$\text{s.t.} \quad V(s) \geq r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \quad \forall s, a \in S \times A$$

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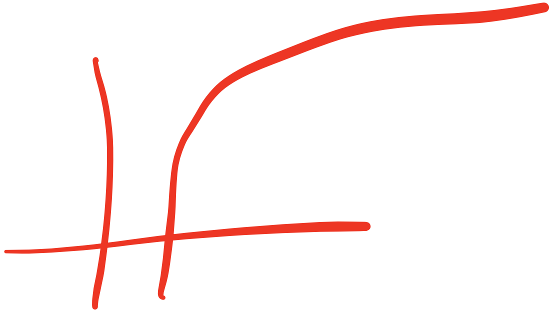
s.t. $V(s) \geq r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V(s') \quad \forall s, a \in S \times A$

(Proof in HW1)

LP Runtime

[Ye, '05]: there is an interior point algorithm (CIPA)
which is ("nearly") strongly polynomial, i.e., no poly dependence on $1/(1-\gamma)$

$$S^4 A^4 \ln\left(\frac{S}{1-\gamma}\right)$$



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  - It is also very helpful conceptually.
  - In some cases, it also provides a reasonable algorithmic approach

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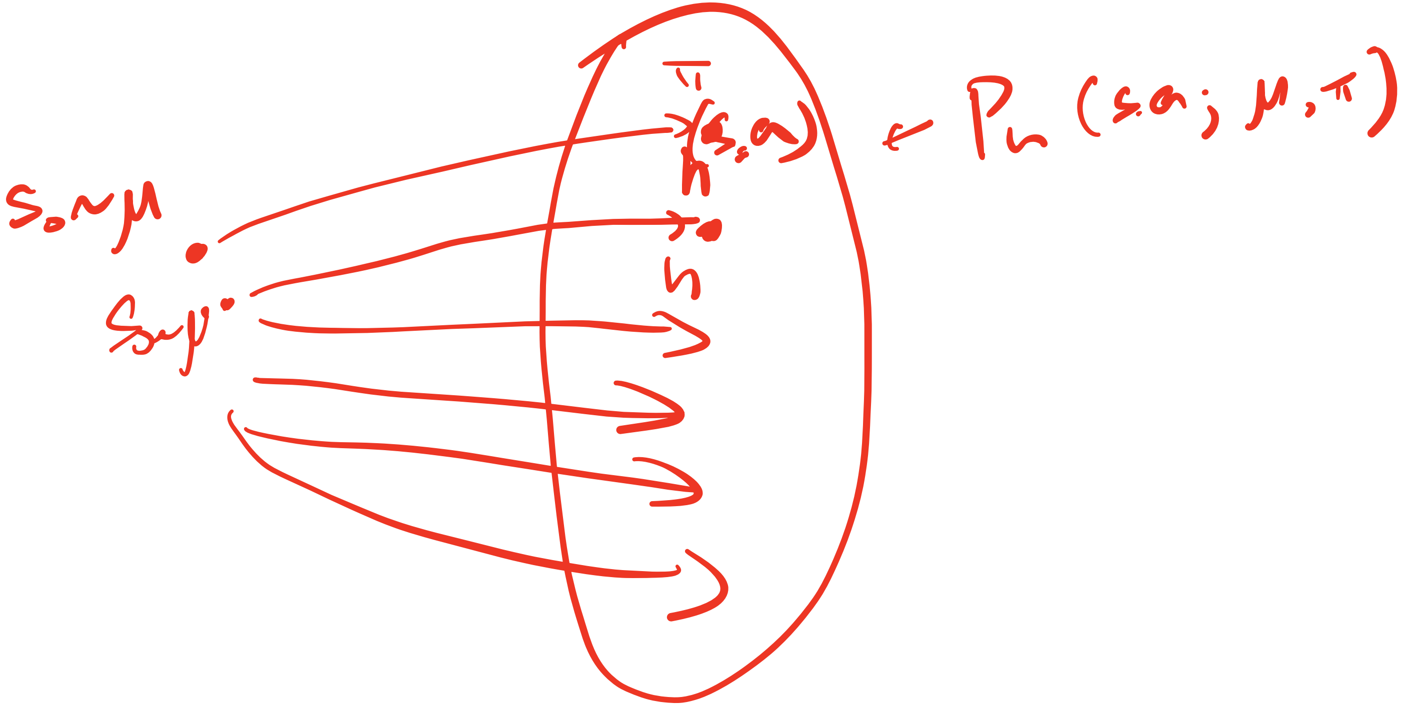
  - It is also very helpful conceptually.
  - In some cases, it also provides a reasonable algorithmic approach

- Let us start by understanding the dual variables

State action occupancy measure

$\mathbb{P}_h(s, a; \mu, \pi)$: probability of π visiting (s, a) at time step $h \in \mathbb{N}$, starting at $s_0 \sim \mu$



State action occupancy measure

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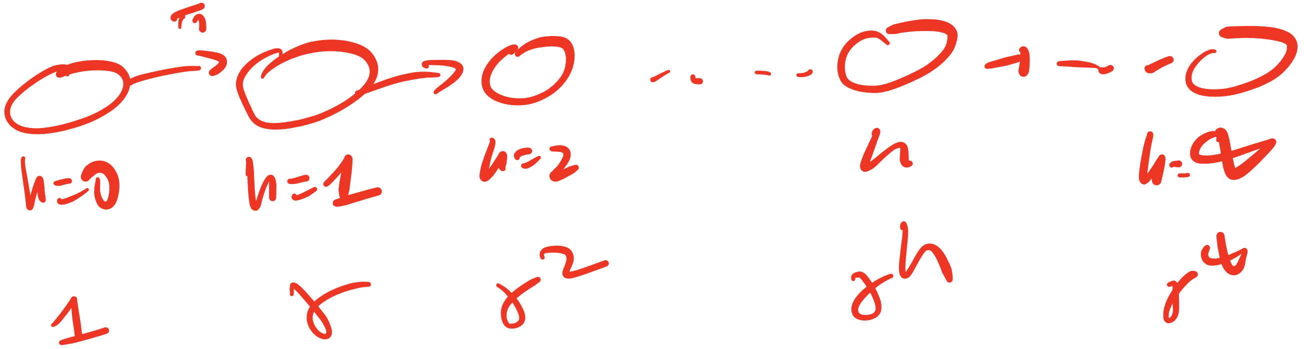
$$d^\pi_\mu(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h(s, a; \mu, \pi)$$

$$\sum_{s,a} d^\pi_\mu(s, a) = 1$$



State action occupancy measure

$\mathbb{P}_h(s, a; \mu, \pi)$: probability of π visiting (s, a) at time step $h \in \mathbb{N}$, starting at $s_0 \sim \mu$

$$d_\mu^\pi(s, a) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}_h(s, a; \mu, \pi)$$

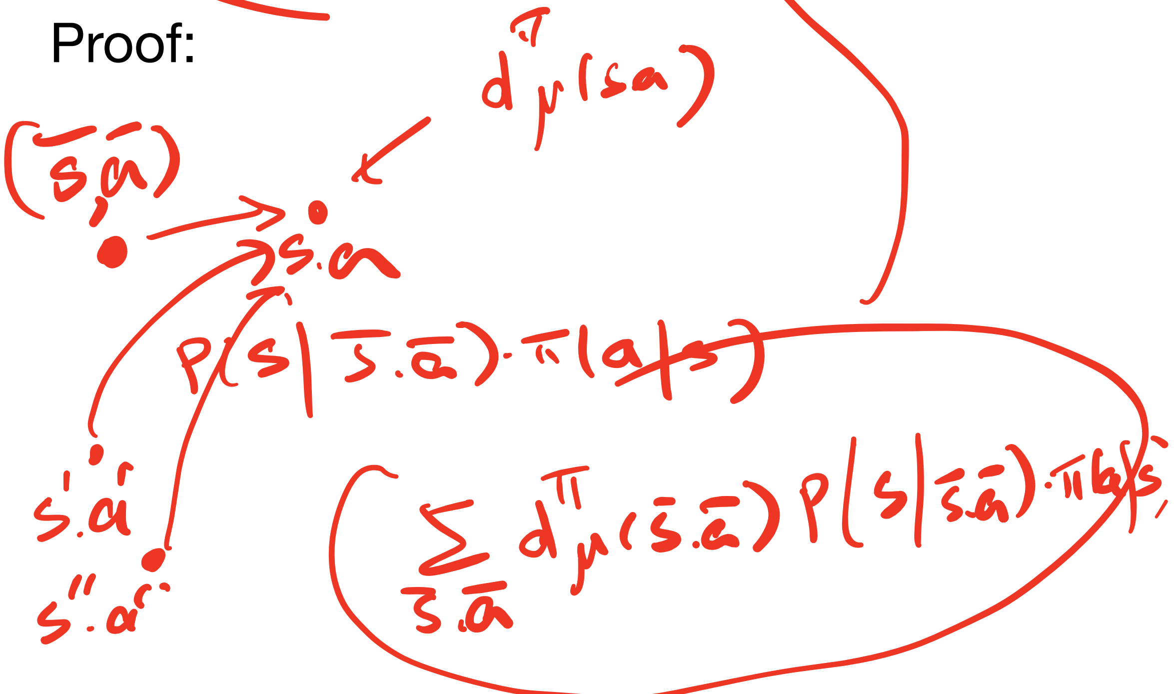
$$\mathbb{E}_{s_0 \sim \mu} V^\pi(s_0) = \frac{1}{1 - \gamma} \sum_{s,a} d_\mu^\pi(s, a) r(s, a)$$

$$= \frac{1}{1-\gamma} \left(\mathbb{E}_{s,a \sim d_\mu^\pi(s,a)} r(s,a) \right)$$

A Bellman equation like property for $d_\mu^\pi(s,a)$

$$\sum_a d_\mu^\pi(s,a) = (1-\gamma)\mu(s) + \gamma \sum_{\bar{s},\bar{a}} P(s \mid \bar{s},\bar{a}) d_\mu^\pi(\bar{s},\bar{a})$$

Proof:



The "State-Action" Polytope

- Let us define the state-action polytope K as follows:

$$K_\mu := \left\{ d \,|\, d \geq 0 \text{ and } \sum_a d(s,a) = (1-\gamma)\mu(s) + \gamma \sum_{s',a'} P(s|s',a')d(s',a') \right\}$$

$d \geq 0 \iff d(s,a) \geq 0, \ \forall s,a$

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Lemma: $d \in K_\mu$ if and only if there exists a (possibly randomized) policy π s.t. $d_\mu^\pi = d$

(Proof in HW1)

The Dual LP

$$\max_{d} \sum_{s,a} d(s,a) r(s,a)$$

$$\text{s.t. } \; d \in K_\mu$$

- One can verify that this is the dual of the primal LP.

Summary

Notations: Value / Q functions, state-action occupant measures, Bellman equation / optimality

Planning algorithms: VI, PI, LP (primal and dual)