Learning in Generative Model

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CS 6789: Foundations of Reinforcement Learning

Today:

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Two natural models for learning in an unknown MDP

- Episodic setting:
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- Sample complexity of RL:
  how many transitions do we need observe in order to find a near optimal policy?
  - Episodic setting: we must actively explore to gather information
  - Generative model setting: lets us disentangle the issue of fundamental statistical limits from exploration.

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- What is the minmax optimal sample complexity, with generative modeling access? (using *any* algorithm)
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    i.e. estimate P accurately (using $O(S^2A)$ samples) and then use $\widehat{P}$ for planning.
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    (i.e. learn with fewer than $\Omega(S^2A)$ samples)
- If sublinear learning is possible, then we do not need an accurate model of the world in order to act near-optimally?

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    where $\text{count}(s',s,a)$ is the #times (s,a) transitions to state s'.
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- The total number of calls to our generative model is SAN.

Attempt 1:

the naive model based approach

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(HW1 will walk you through the proof)

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Given policy π, does $P \approx \widehat{P}$ imply $V^\pi \approx V^\pi_{\widehat{P}}$?

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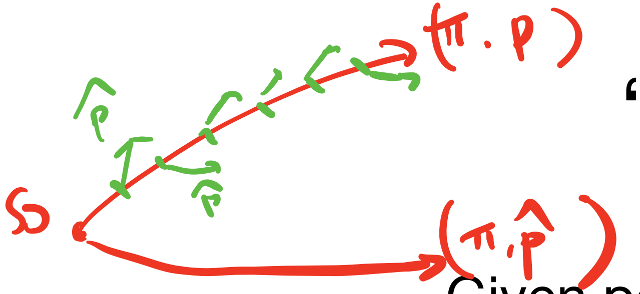

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- Given any two transitions P and $\widehat{P}$, and any policy π, we have:

$$\forall s_0 : V^{\pi}(s_0) - V^{\pi}_{\widehat{P}}(s_0) = \frac{\gamma}{1-\gamma} \mathbb{E}_{s,a \sim d^{\pi}_{s_0}} \left[\left(P(s,a) - \widehat{P}(s,a) \right)^{\top} V^{\pi}_{\widehat{P}} \right]$$

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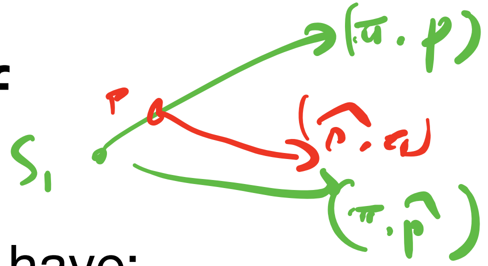
“Simulation” Lemma: proof



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$$V^\pi(s_0) - V^\pi_{\widehat{P}}(s_0) = Q^\pi(s_0, \pi(s_0)) - Q^\pi_{\widehat{P}}(s_0, \pi(s_0))$$

$$= \gamma \mathbb{E}_{s_1\sim P(s_0,\pi(s_0))} V^\pi(s_1) - \gamma \mathbb{E}_{s_1\sim \widehat{P}(s_0,\pi(s_0))} V^\pi_{\widehat{P}}(s_1)$$

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$$\Rightarrow \gamma \left(P(s_0, \pi(s_0)) - \widehat{P}(s_0, \pi(s_0))\right)^\top V^\pi_{\widehat{P}}$$

Combine Model Accuracy and Simulation

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Set it to ϵ, solve for N

Conclusion on the naive approach

Given δ. If we draw $\frac{S^2 A}{\epsilon^2(1-\gamma)^4} \cdot \ln \frac{SA}{\delta}$ many total samples, w/ prob at at least $1-\delta$, we have $V^{\pi^\star}(s) - V^{\widehat{\pi}^\star}(s) \leq \epsilon, \forall s$

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Q: can we do better than a linear scaling in $S^2 A$?

Attempt 2:

obtaining sublinear sample complexity

idea: use concentration only on $V^{\star}$

Model error projected on $V^\star$

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- Recall $\|V^\star\|_\infty \leq 1/(1-\gamma)$.

Model error projected on $V^\star$

- Recall $\|V^\star\|_\infty \leq 1/(1-\gamma)$.
- By Hoeffding's inequality and the union bound,

$$\max_{s,a} \left| E_{s' \sim P(\cdot|s,a)}[V^\star(s')] - E_{s' \sim \widehat{P}(\cdot|s,a)}[V^\star(s')] \right|$$

$$\leq O\left(\frac{1}{1-\gamma} \sqrt{\frac{\log(SA/\delta)}{N}} \right)$$

which holds with probability greater than $1-\delta$.

Bounding error $\|Q^\star - \widehat{Q}^\star\|_\infty$

Recall $Q^\star$ the optimal Q function in P, $\widehat{Q}^\star$ the optimal Q in $\widehat{P}$, we have (proof next slide)

$$\|Q^\star - \widehat{Q}^\star\|_\infty \leq \frac{1}{(1-\gamma)^2}\sqrt{\frac{2\log(2SA/\delta)}{N}}$$

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Recall $\widehat{\pi}^\star(s) = \arg\max_a \widehat{Q}^\star(s,a)$, we immediately have (recall VI analysis):

$$\|V^\star - V^{\widehat{\pi}^\star}\|_\infty \leq \frac{1}{(1-\gamma)^3}\sqrt{\frac{\ln(SA/\delta)}{N}}$$

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