Learning with Linear Bellman Completion & Generative Model

CS 6789: Foundations of Reinforcement Learning

Announcements

1. HW1 is going to be out Thursday.

2. Wen’s office hour: after lectures

Recap: Generative model + Tabular

1. Generative model assumption:

At any (s, a), we can sample $s' \sim P(\cdot \mid s, a)$

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Q: why this could be a strong assumption in practice?

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Algorithm:

1. For each (s, a), i.i.d sample N next states, $s_i' \sim P(\cdot \mid s, a)$

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3. Find optimal policy under $\hat{P}$, i.e., $\hat{\pi}^\star = \mathrm{PI}(\hat{P}, r)$

MPC

Recap: Generative model + Tabular

Result:

When $N \geq \frac{\ln(SA/\delta)}{\epsilon^2(1-\gamma)^6}$, then w/ prob $1-\delta$, we will learn a $\hat{\pi}^\star$, such that $\|Q^\star - Q^{\hat{\pi}^\star}\|_\infty \leq \epsilon$

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Recap: Generative model + Tabular

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Remarks:

1. Horizon factor is not tight at all (Ch2 in AJKS optimizes it to $1/(1-\gamma)^5$)

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$SA \cdot N \approx SA$

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1. Horizon factor is not tight at all (Ch2 in AJKS optimizes it to $1/(1-\gamma)^5$)

2. Remarkably, our learned model $\hat{P}$ in this case is not necessarily accurate at all

Today: Generative model + linear function approximation

Key question: what happens when state-action space is large or even continuous?

Outline:

1. The Linear Bellman Completion Condition

2. The Least Square Value Iteration Algorithm

3. Guarantee and the proof sketch

Finite Horizon MDPs and DP

$$\mathcal{M} = \{S, A, P_h, r, H\}$$

h=0, 1, 2, ... H-1

$$P_h : S \times A \mapsto \Delta(S), \quad r : S \times A \to [0,1]$$

Compute $\pi^\star$ via DP (backward in time):

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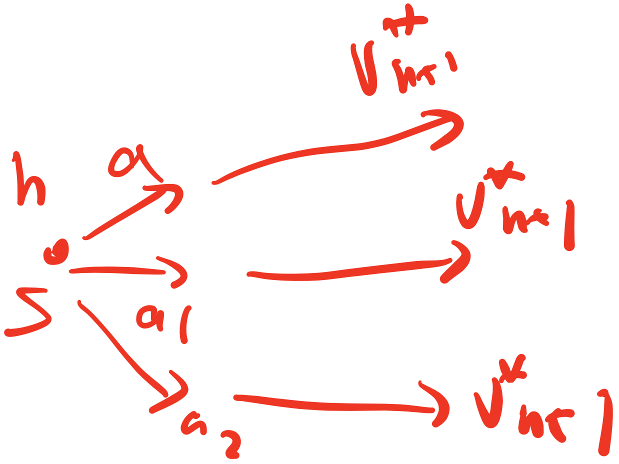
1. set $Q^\star_{H-1}(s,a) = r(s,a)$, $\pi^\star_{H-1}(s) = \arg\max_a Q^\star_{H-1}(s,a)$, $V^\star_{H-1}(s) = \max_a Q^\star_{H-1}(s,a)$



$V^\star_{h+1}$ & $\pi^\star_{h+1}$ $\Rightarrow$ $V^\star_h$ & $\pi^\star_h$

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2. At h, set $Q^\star_h(s,a) = r(s,a) + \mathbb{E}_{s' \sim P_h(\cdot|s,a)} V^\star_{h+1}(s')$,
$\pi^\star_h(s) = \arg\max_a Q^\star_h(s,a)$, $V^\star_h(s) = \max_a Q^\star_h(s,a)$

$h+1 \Rightarrow h$

Recall Error amplification

1. **Bellman optimality:** $\|Q - \mathcal{T}Q\|_\infty = 0$, then $Q = Q^\star$

$$Q = \mathcal{T}Q$$

$$(\mathcal{T}Q)(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \max_{a'} Q(s',a')$$

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$\hat{\pi} = \arg\max_a Q(s,a)$

Then we have error amplification:

$\|Q - Q^\star\|_\infty \leq \epsilon/(1-\gamma)$, => $V^\star - V^{\hat{\pi}} \leq \epsilon/(1-\gamma)^2$

$TQ^\star = Q^\star$

$\|Q - Q^\star\|_\infty$

$\leq \|Q - TQ\|_\infty + \|TQ - TQ^\star\|_\infty$

$\leq \epsilon + \gamma\|Q - Q^\star\|_\infty \leq \epsilon + \gamma\epsilon + \gamma^2\epsilon + \ldots = \frac{\epsilon}{1-\gamma}$

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Then we have error amplification:

$\|Q - Q^\star\|_\infty \leq \epsilon/(1-\gamma)$, => $V^\star - V^{\hat{\pi}} \leq \epsilon/(1-\gamma)^2$

Similar results hold in finite horizon, with the effective horizon $1/(1-\gamma)$ being replaced by H

Linear Bellman Completion

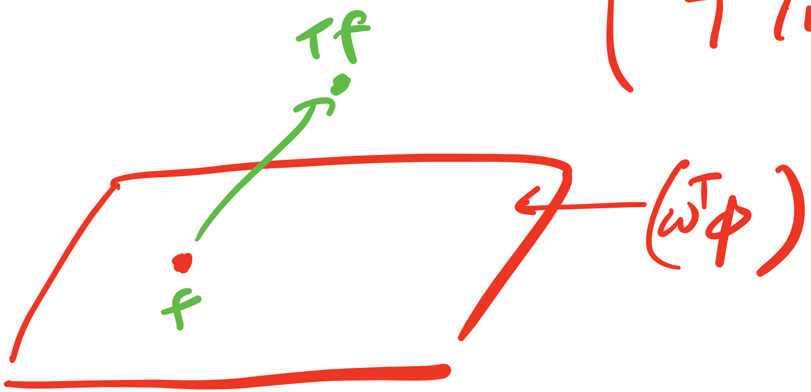
$\phi(s,a) \in \mathbb{R}^d$

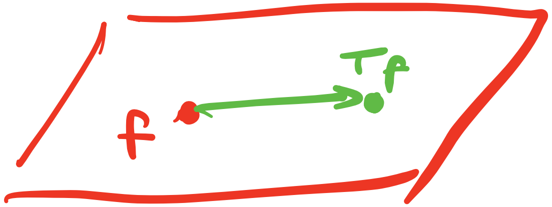
Given feature ϕ, take any linear function $w^\top \phi(s,a)$:

$$\forall h, \exists \theta \in \mathbb{R}^d, s.t., \theta^\top \phi(s,a) = r(s,a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} w^\top \phi(s',a'), \forall s, a$$

$T(w^\top \phi)$

$\left(T(w^\top \phi) \right)$ is linear in ϕ





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Notation: we will denote such $\theta := \mathcal{T}_h(w)$, where $\mathcal{T}_h : \mathbb{R}^d \mapsto \mathbb{R}^d$

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$w = 0$

$\vec{\theta}^\top \phi(s,a)$

It implies that $Q_h^\star$ is linear in ϕ:

$$Q_h^\star = (\theta^\star)^\top \phi, \forall h$$

Why?

$r(s,a) = \theta_r^\top \phi(s,a) \Rightarrow Q_H^\star(s,a)$

$Q_h^\star \leftarrow \mathcal{T}_h Q_{h+1}^\star$

$Q_{h+1}^\star(s,a) = \theta^\top \phi(s,a) = \theta_r^\top \phi(s,a)$

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Why?

reward $r(s, a)$ is linear in ϕ, i.e., $Q_{H-1}^\star(s, a)$ is linear,

now recursively show that $Q_h^\star$ is linear

Why this is a reasonable assumption?

It captures at least two special cases: tabular MDP and linear dynamical systems

1. Tabular MDP:

Set $\phi(s,a)$ to be a one-hot encoding vector in $\mathbb{R}^{SA}$, i.e., $\phi(s,a) = [0,\ldots,0,1,0,\ldots0]^\top$

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2. Linear System with Quadratic feature ϕ

$$s \in \mathbb{R}^2, a \in \mathbb{R}, P_h(\cdot \mid s, a) = \mathcal{N}\left(As + ba, \sigma^2 I\right)$$

$s' \leftarrow s, a$

$s' = As + ba + \varepsilon, \ \varepsilon \sim N(0, \sigma^2 I)$

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$$\phi(s, a) = [s_1, s_2, s_1^2, s_2^2, s_1 s_2, s_1 a, s_2 a, a, a^2, 1]^\top$$

$$s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in \mathbb{R}^2$$

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Claim: $r(s,a) + \mathbb{E}_{s' \sim P(s,a)} \max_{a'} w^T \phi(s', a')$ is a linear function in ϕ

$r(s,a) = w^T \phi(s,a)$

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Assume the given feature ϕ has linear Bellman completion, i.e.,

$$\forall h, \exists \theta \in \mathbb{R}^d, s.t., \theta^\top \phi(s,a) = r(s,a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} w^\top \phi(s',a'), \forall s, a$$

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This is counter-intuitive: in SL (e.g., linear regression), adding elements to features is ok!

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~~poly(H, d)~~

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i.e., polynomial bound poly(d, H) is not possible for linear $Q^\star$ (Ch5 AJKS)

What we will show today:

1. Generative Model

(i.e., we can reset system to any (s, a), query $r(s, a), s' \sim P(. \mid s, a)$)

$+$

2. Linear Bellman Completion

$=$

Sample efficient Learning
(poly time)

Outline:

1. The Linear Bellman Completion Condition

2. Learning: The Least Square Value Iteration Algorithm

3. Guarantee and the proof sketch

LSVI: Least-Square Value Iteration

Recall linear bellman-completion implies $Q_h^\star(s,a) = (\theta_h^\star)^\top \phi(s,a), \forall s,a,h$

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$\mathscr{D}_h = \{s, a, r, s'\}, r = r(s,a), s' \sim P_h(\cdot \mid s,a)$

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linear function to approximate $Q^\star$

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$h=0, \ldots H-1$

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Set $V_H(s) = 0, \forall s$

Given datasets $\mathcal{D}_0, \ldots, \mathcal{D}_{H-1}$, w/
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For h = H-1 to 0:

$$\theta_h = \arg\min_\theta \sum_{\mathcal{D}_h} \left(\theta^T \phi(s,a) - \left(r + V_{h+1}(s')\right)\right)^2$$

linear predictor

Regress Target

Given datasets $\mathcal{D}_0, \ldots, \mathcal{D}_{H-1}$, w/ $\mathcal{D}_h = \{s, a, r, s'\}, r = r(s,a), s' \sim P_h(\cdot \mid s, a)$

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$Q_h(s,a) \approx Q_h^\star(s,a)$

Why LSVI may work?

When we do linear regression at step h:

$$x := \phi(s,a), \quad y := r + V_{h+1}(s')$$

$s' \sim P_h(s,a)$

$E[y|x]$

$E[r + V_{h+1}(s') | s a]$

Set $V_H(s) = 0, \forall s$

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When we do linear regression at step h:

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We note that:

$$\mathbb{E}[y \mid x] = \underbrace{r(s,a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} \theta_{h+1}^{\top} \phi(s', a')}_{\mathcal{T}_h(\theta_{h+1})^{\top}\phi(s,a) \text{ due to Linear BC}}$$

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Return $\hat{\pi}_h(s) = \arg\max_a \theta_h^{\top} \phi(s,a), \forall h$

$V_{h+1}(s') = \max_{a'} Q_{h+1}(s', a')$

$= \max_{a'} \theta_{h+1}^{\top} \phi(s', a')$

Bayes opt at h is linear in ϕ

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i.e., our regression target is indeed linear in ϕ, and it is close to $Q_h^\star$ if

$$V_{h+1} \approx V_{h+1}^\star$$

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$$\mathbb{E}[y \mid x] = \underbrace{r(s,a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} \theta_{h+1}^\top \phi(s',a')}_{\mathcal{T}_h(\theta_{h+1})^\top \phi(s,a) \text{ due to Linear BC}}$$

i.e., our regression target is indeed linear in ϕ, and it is close to $Q_h^\star$ if $V_{h+1} \approx V_{h+1}^\star$

Set $V_H(s) = 0, \forall s$

For h = H-1 to 0:

$$\theta_h = \arg\min_\theta \sum_{\mathcal{D}_h} \left(\theta^T \phi(s,a) - \left(r + V_{h+1}(s')\right)\right)^2$$

Set $V_h(s) := \max_a \theta_h^\top \phi(s,a), \forall s$

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If $V_{h+1} \approx V_{h+1}^\star$, and linear regression succeeds (e.g., $\theta_h \approx \mathcal{T}_h(\theta_{h+1})$),

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Why LSVI may work?

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Then we should hope $\theta_h^\top \phi(s,a) \approx Q_h^\star(s,a)$

Outline:

1. The Linear Bellman Completion Condition

2. Learning: The Least Square Value Iteration Algorithm

3. Guarantee and the proof sketch

Sample complexity of LSVI

Theorem: There exists a way to construct datasets $\{\mathscr{D}_h\}_{h=0}^{H-1}$, such that with probability at least $1-\delta$, we have:

$$V^{\hat{\pi}} - V^\star \leq \epsilon$$

w/ total number of samples in these datasets scaling $\widetilde{O}\left(d^2 + H^6 d^2/\epsilon^2\right)$

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Plans: (1) OLS and D-optimal design; (2) construct $\mathscr{D}_h$ using D-optimal design; (3) transfer regression error to $\|\theta_h^\top \phi - Q_h^\star\|_\infty$

Detour: Ordinary Linear Squares

Consider a dataset $\{x_i, y_i\}_{i=1}^N$, where $y_i = (\theta^\star)^\top x_i + \epsilon_i, \quad \mathbb{E}[\epsilon_i | x_i] = 0$, ϵ_i are independent with $|\epsilon_i| \leq \sigma,$ assume $\Lambda = \sum_{i=1}^N x_i x_i^\top / N$ is full rank;

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Standard OLS guarantee: with probability at least $1 - \delta$, we have:

$$(\hat{\theta} - \theta^\star)^\top \Lambda (\hat{\theta} - \theta^\star) \leq O\left(\frac{\sigma^2 d \ln(1/\delta)}{N}\right)$$

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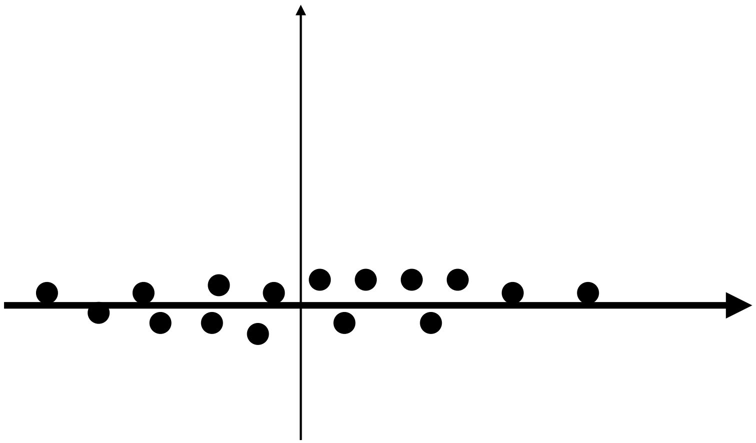

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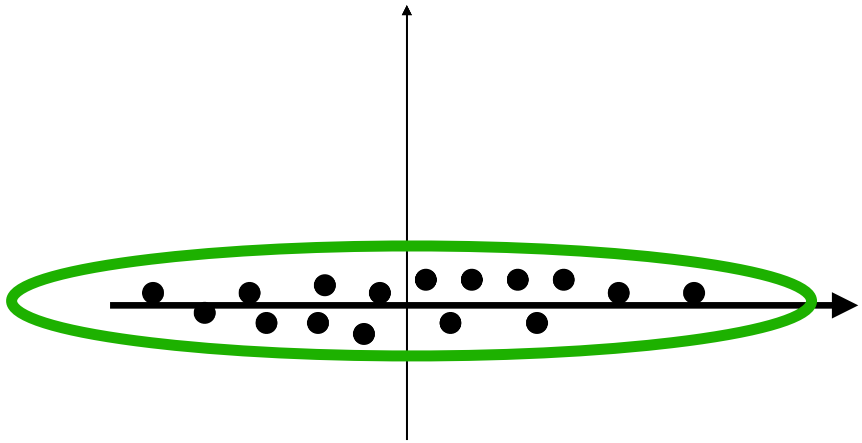

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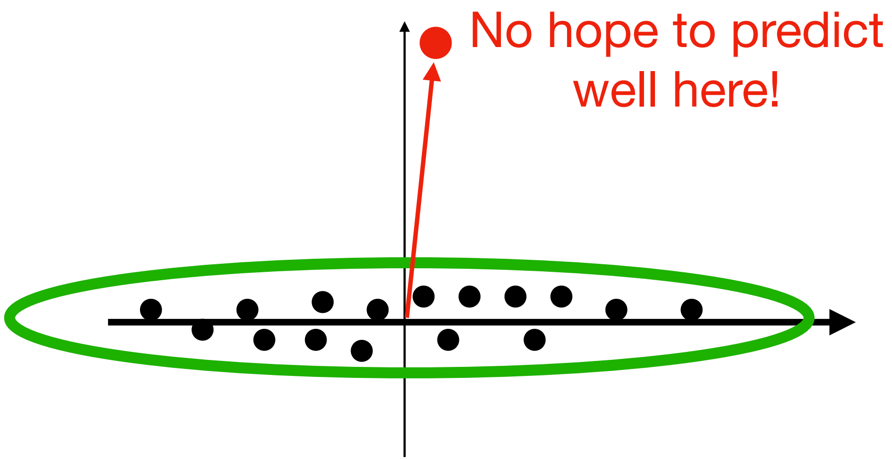

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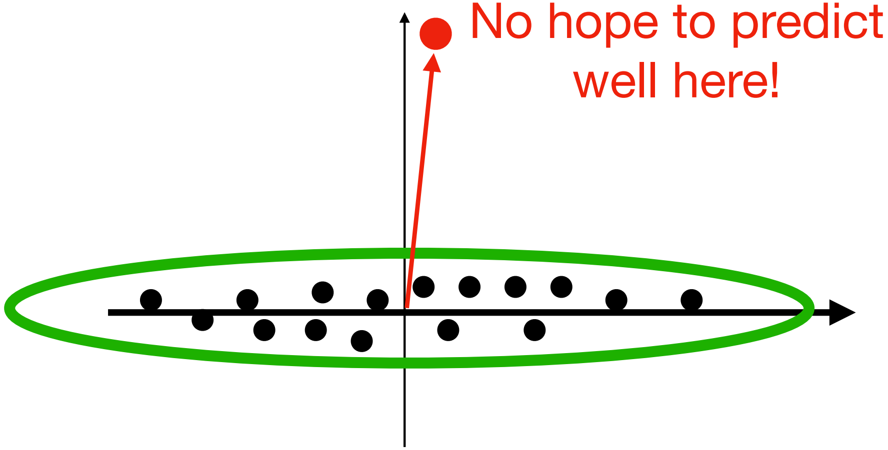

Let's actively design a diverse dataset !
(D-optimal Design)

Detour: D-optimal Design

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support$(\rho^\star) \leq d(d+1)/2$

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$$\max_{y\in\mathscr{X}} y^\top \left[\mathbb{E}_{x\sim\rho^\star} xx^\top\right]^{-1} y \leq d$$

Detour: OLS w/ D-optimal Design

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The OLS solution $\hat{\theta}$ on $\mathscr{D}$ has the following point-wise guarantee: w/ prob $1-\delta$

$$\max_{x\in\mathscr{X}} \left| \langle \hat{\theta} - \theta^\star, x \rangle \right| \leq \frac{\sigma d \ln(1/\delta)}{\sqrt{N}}$$

Summary so far on OLS & D-optimal Design

D-optimal Design $\rho^\star \in \Delta(\mathcal{X})$: $\rho^\star = \arg\max_{\rho \in \Delta(\mathcal{X})} \ln\det\left(\mathbb{E}_{x\sim\rho}\left[xx^\top\right]\right)$

Summary so far on OLS & D-optimal Design

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D-optimal design allows us to **actively** construct a dataset $\mathcal{D} = \{x, y\}$, such that OLS solution is **POINT-WISE** accurate:

$$\max_{x\in\mathcal{X}} \left| \langle \hat{\theta} - \theta^\star, x \rangle \right| \leq \frac{\sigma d \ln(1/\delta)}{\sqrt{N}}$$