Learning with Linear Bellman Completion & Generative Model

CS 6789: Foundations of Reinforcement Learning

Recap: Linear Bellman Completion

Given feature ϕ, take any linear function $w^\top \phi(s, a)$:

$$\forall h, \exists \theta \in \mathbb{R}^d, s.t., \theta^\top \phi(s, a) = r(s, a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} w^\top \phi(s', a'), \forall s, a$$

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But adding additional elements may just break the condition

Recap: Least-Square Value Iteration

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$$\theta_h = \arg\min_{\theta} \sum_{\mathscr{D}_h} \left(\theta^T \phi(s, a) - \left(r + V_{h+1}(s') \right) \right)^2$$

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BC always ensures linear regression is realizable:

i.e., our regression target

$$r(s, a) + \mathbb{E}_{s' \sim P_h(s,a)} \max_{a'} \theta_{h+1}^\top \phi(s', a')$$

is always linear:

Outline for Today

1. Linear regression, D optimal Design (active learning)

2. Proof sketch for LSVI

Theorem

Theorem: There exists a way to construct datasets $\{\mathscr{D}_h\}_{h=0}^{H-1}$, such that with probability at least $1-\delta$, we have:

$$V^{\hat{\pi}} - V^{\star} \leq \epsilon$$

w/ total number of samples in these datasets scaling $\widetilde{O}\left(d^2 + H^6 d^2/\epsilon^2\right)$

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2. Show that our estimators are near-bellman consistent: $\|\theta_h^\top \phi - \mathcal{T}_h(\theta_{h+1}^\top \phi)\|_\infty$ is small
3. Near-Bellman consistency implies near optimal performance (s.t. H error amplification)

Detour: Ordinary Linear Squares

Consider a dataset $\{x_i, y_i\}_{i=1}^{N}$, where $y_i = (\theta^\star)^\top x_i + \epsilon_i, \quad \mathbb{E}[\epsilon_i \mid x_i] = 0$, ϵ_i are independent with $|\epsilon_i| \leq \sigma,$ assume $\Lambda = \sum_{i=1}^{N} x_i x_i^\top / N$ is full rank;

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Standard OLS guarantee: with probability at least $1 - \delta$:

$$(\hat{\theta} - \theta^\star)^\top \Lambda (\hat{\theta} - \theta^\star) \leq O\left(\frac{\sigma^2 d \ln(1/\delta)}{N} \right)$$

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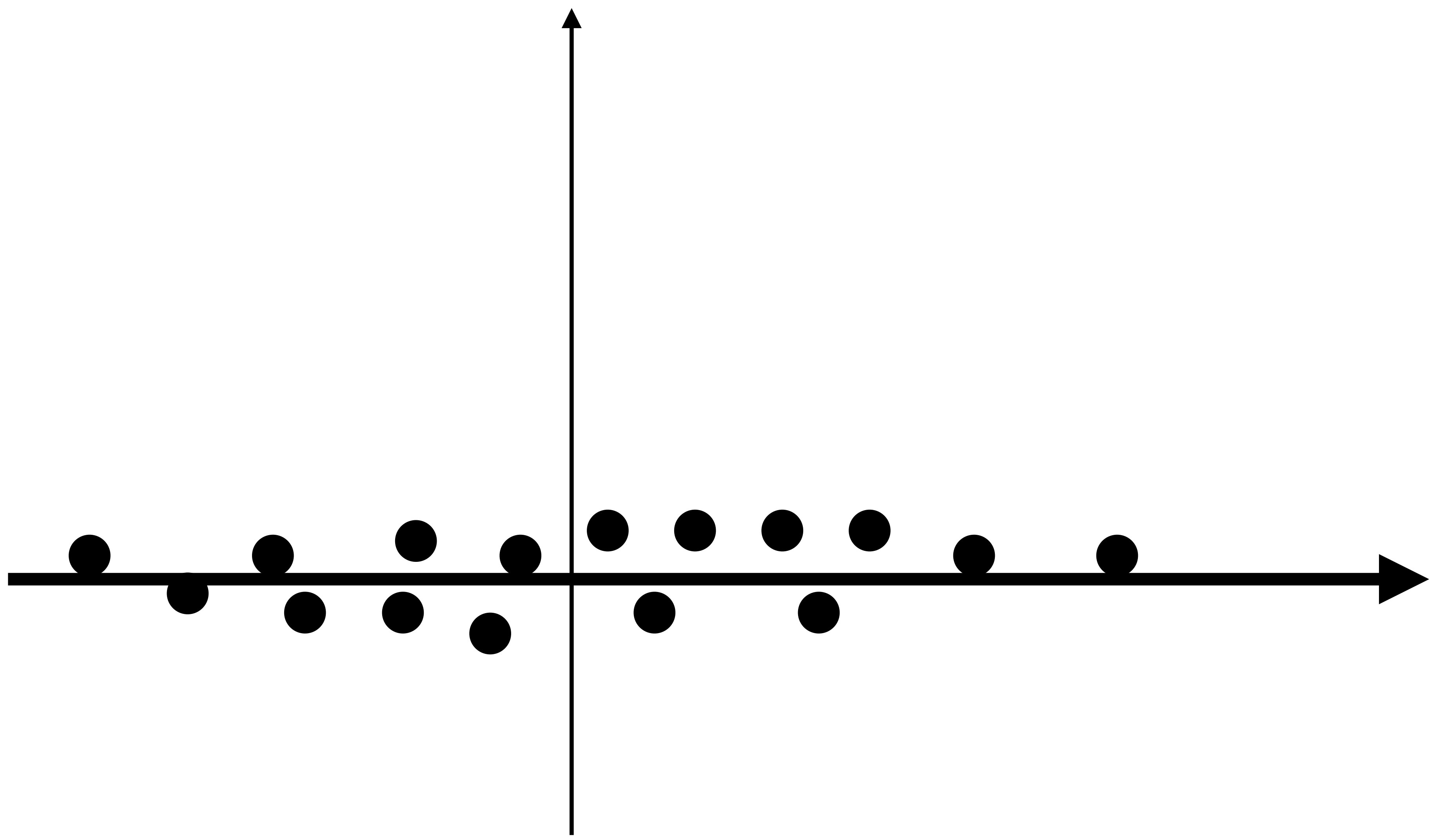

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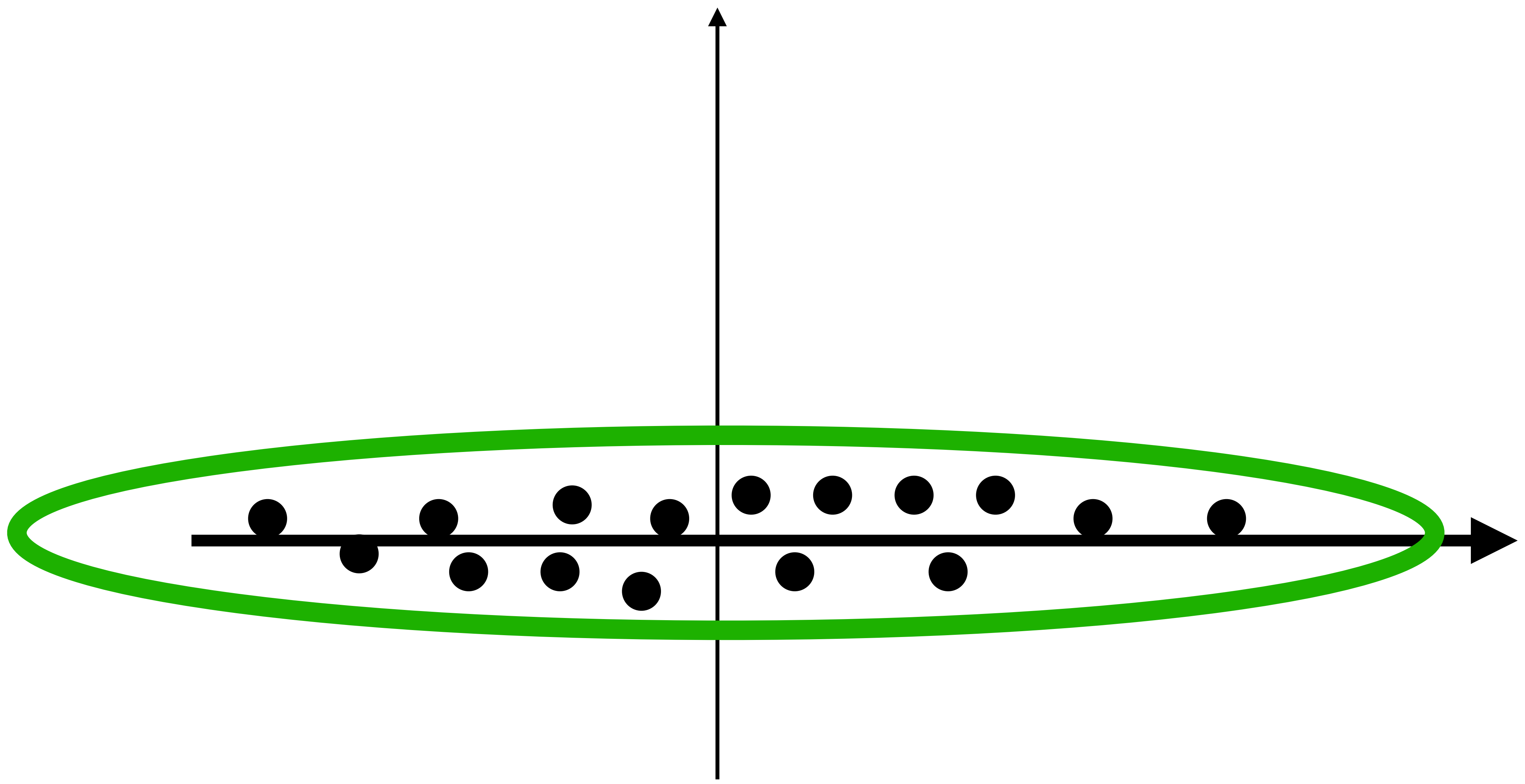

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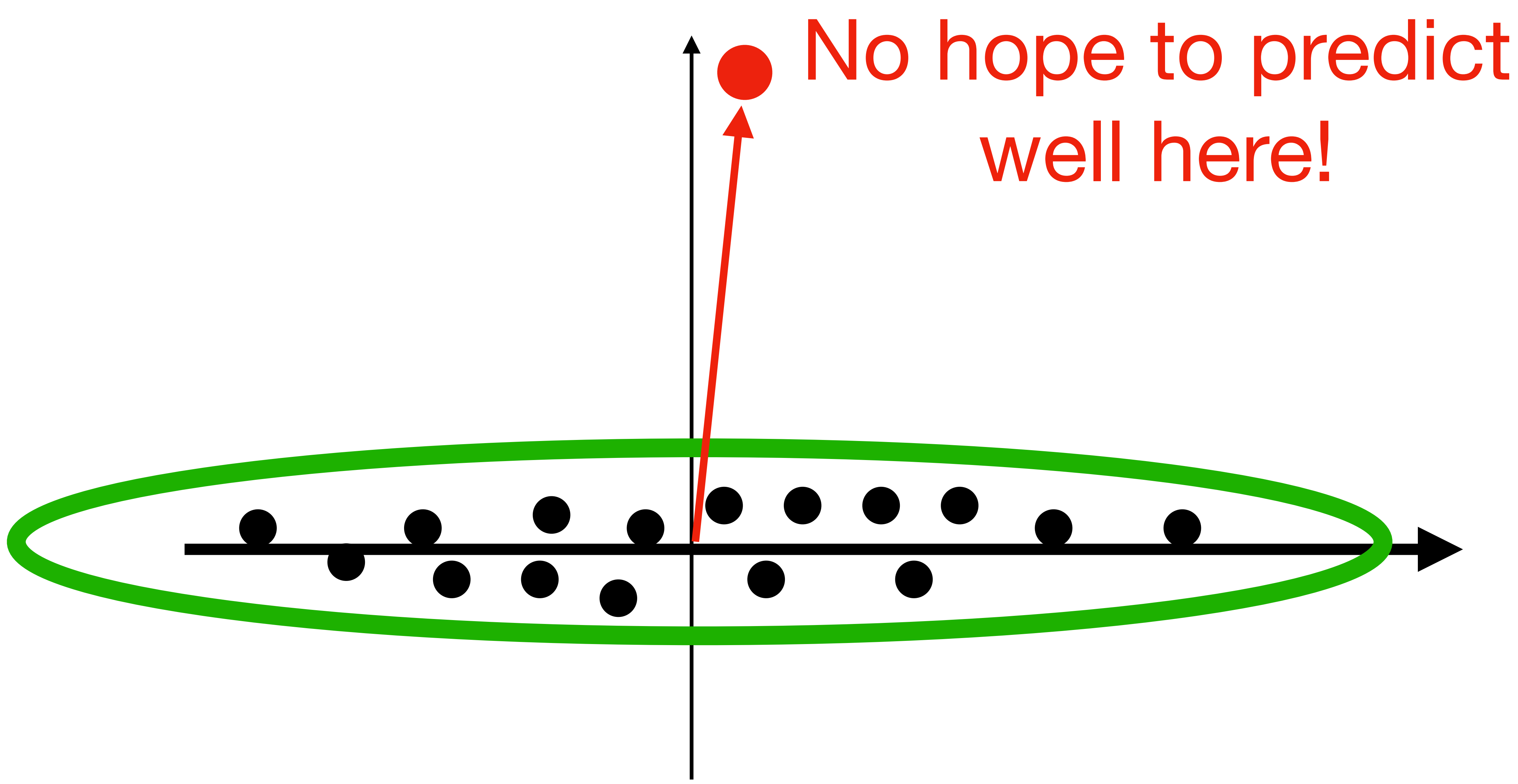

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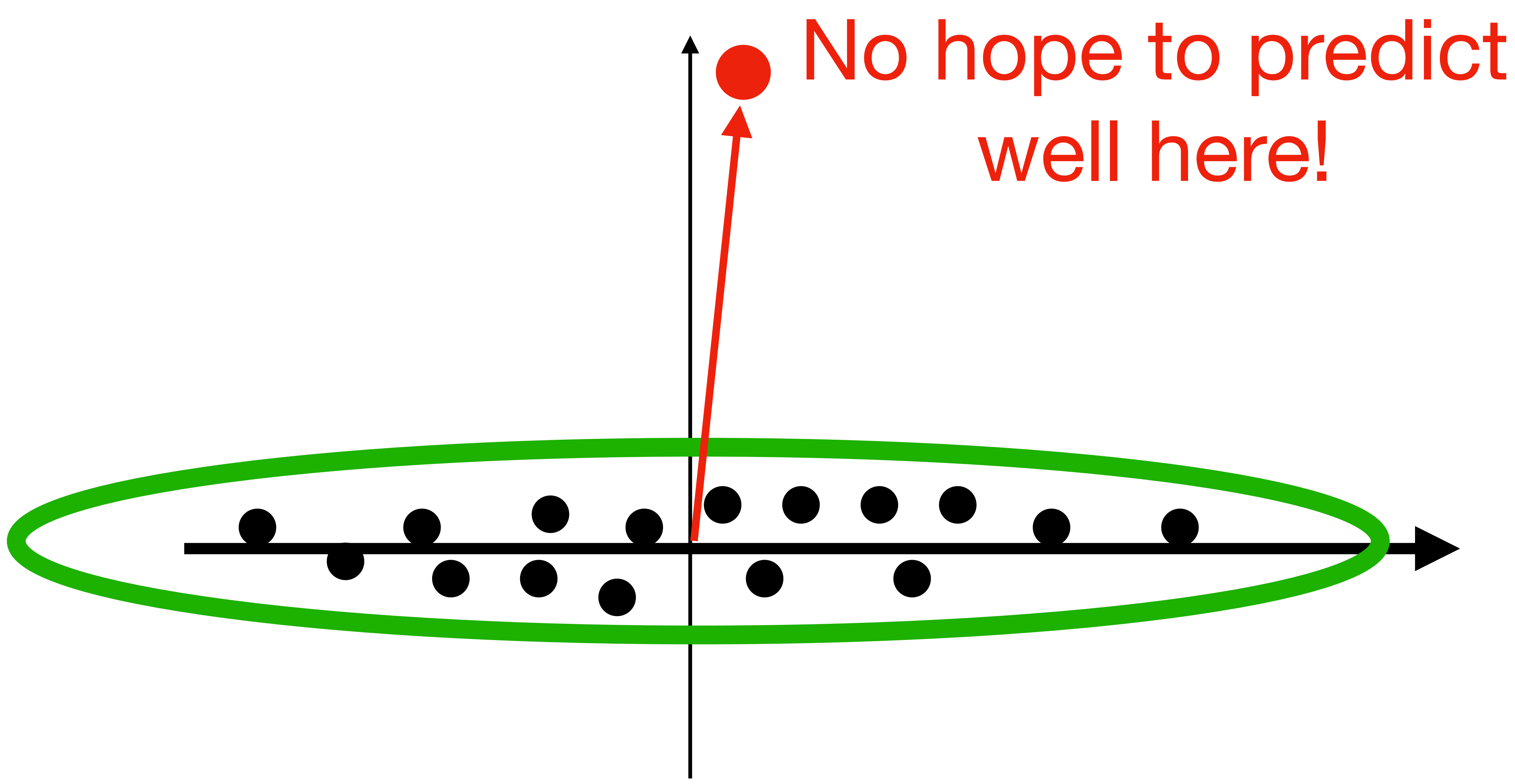

Let's actively design a diverse dataset !
(D-optimal Design)

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$$\max_{y\in\mathscr{X}} y^\top \left[\mathbb{E}_{x\sim\rho^\star} xx^\top\right]^{-1} y \leq d$$

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$$\left| (\hat\theta - \theta^\star)^\top x \right| \leq \left\| \Lambda^{1/2}(\hat\theta - \theta^\star) \right\|_2 \left\| \Lambda^{-1/2} x \right\|_2$$

Summary so far on OLS & D-optimal Design

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D-optimal design allows us to **actively** construct a dataset $\mathcal{D} = \{x, y\}$, such that OLS solution is **POINT-WISE** accurate:

$$\max_{x\in\mathcal{X}} \left|\langle \hat{\theta} - \theta^\star, x\rangle\right| \leq \frac{\sigma d \ln(1/\delta)}{\sqrt{N}}$$

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2. Proof sketch for LSVI

Using D-optimal design to construct $\mathscr{D}_h$ in LSVI

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$$\max_{s,a}\left|\theta_h^\top\phi(s,a) - \mathscr{T}_h(\theta_{h+1})^\top\phi(s,a)\right| \leq \widetilde{O}\left(Hd/\sqrt{N}\right).$$

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$$\Rightarrow V^\star - V^{\hat{\pi}} \leq \widetilde{O}(H^3 d/\sqrt{N})$$

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4. Near-Bellman consistency implies small approximation error of Q_h (holds in general)

Next week

Exploration: Multi-armed Bandits and online learning in Tabular MDP