Natural Policy Gradient

Wen Sun

CS 6789: Foundations of Reinforcement Learning

Today:

Natural policy optimization

History:

A Natural Policy Gradient

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NeurIPS 2002

Covariant Policy Search

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IJCAI 2003

Trust Region Policy Optimization

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ICML 2015

Notations and Settings:

Finite horizon setting: $\mathscr{M} = \{S, A, H, r, P, \rho\}$

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Trajectory distribution:

$$\Pr^\pi(\tau) = \rho(s_0)\pi(a_0 \mid s_0)P(s_1 \mid s_0, a_0)\pi(a_1 \mid s_1)\ldots P(s_{H-1} \mid s_{H-2}, a_{H-2})\pi(a_{H-1} \mid s_{H-1})$$

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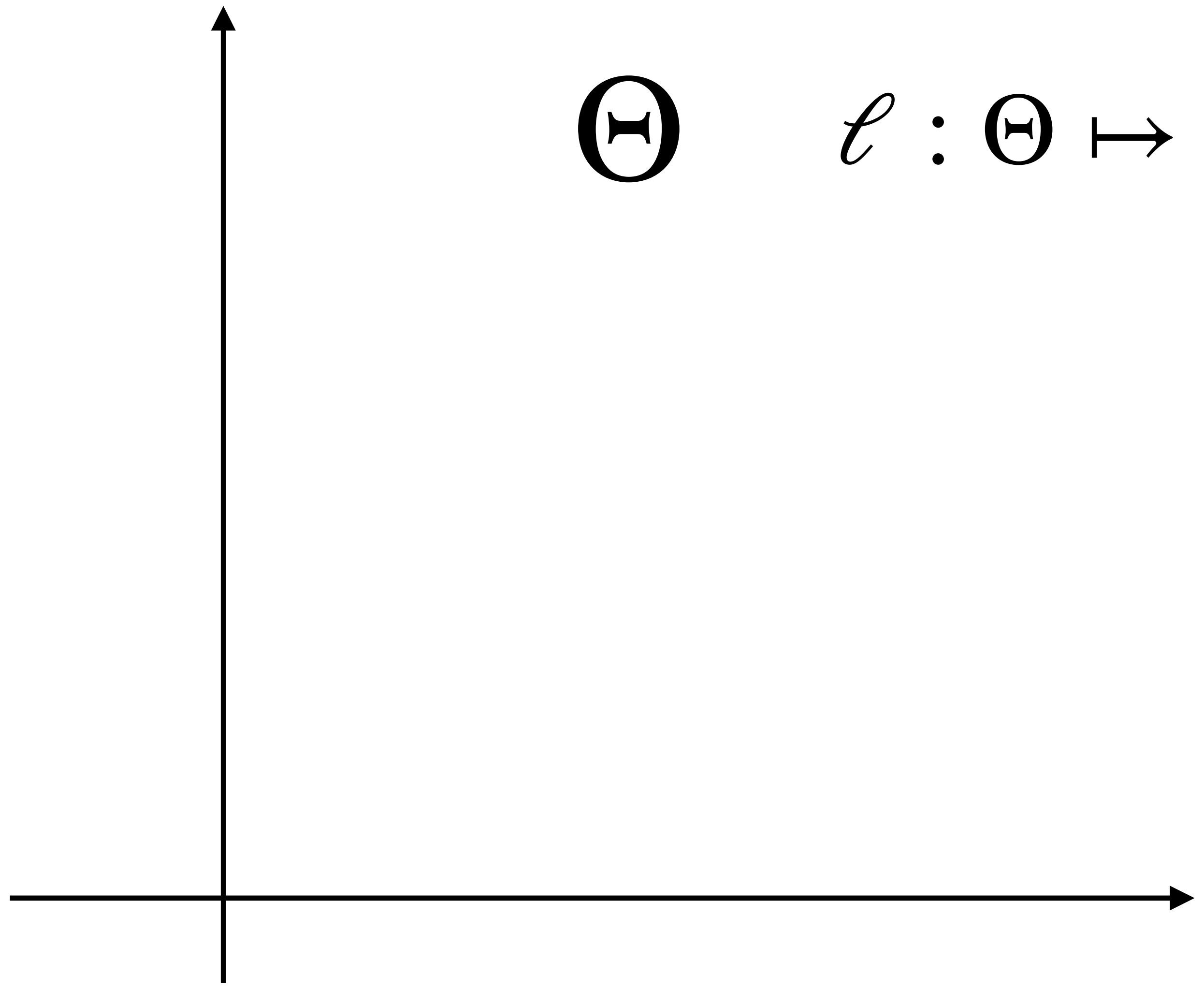
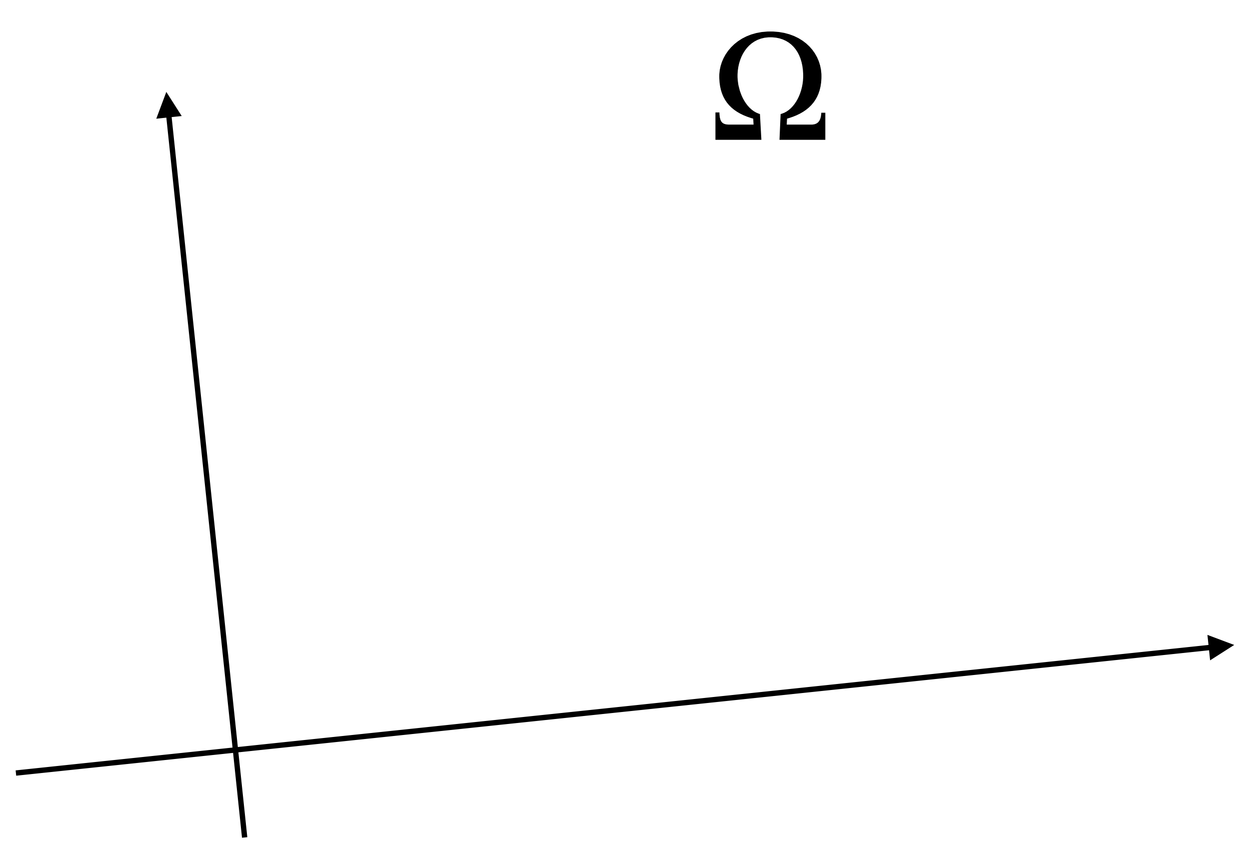
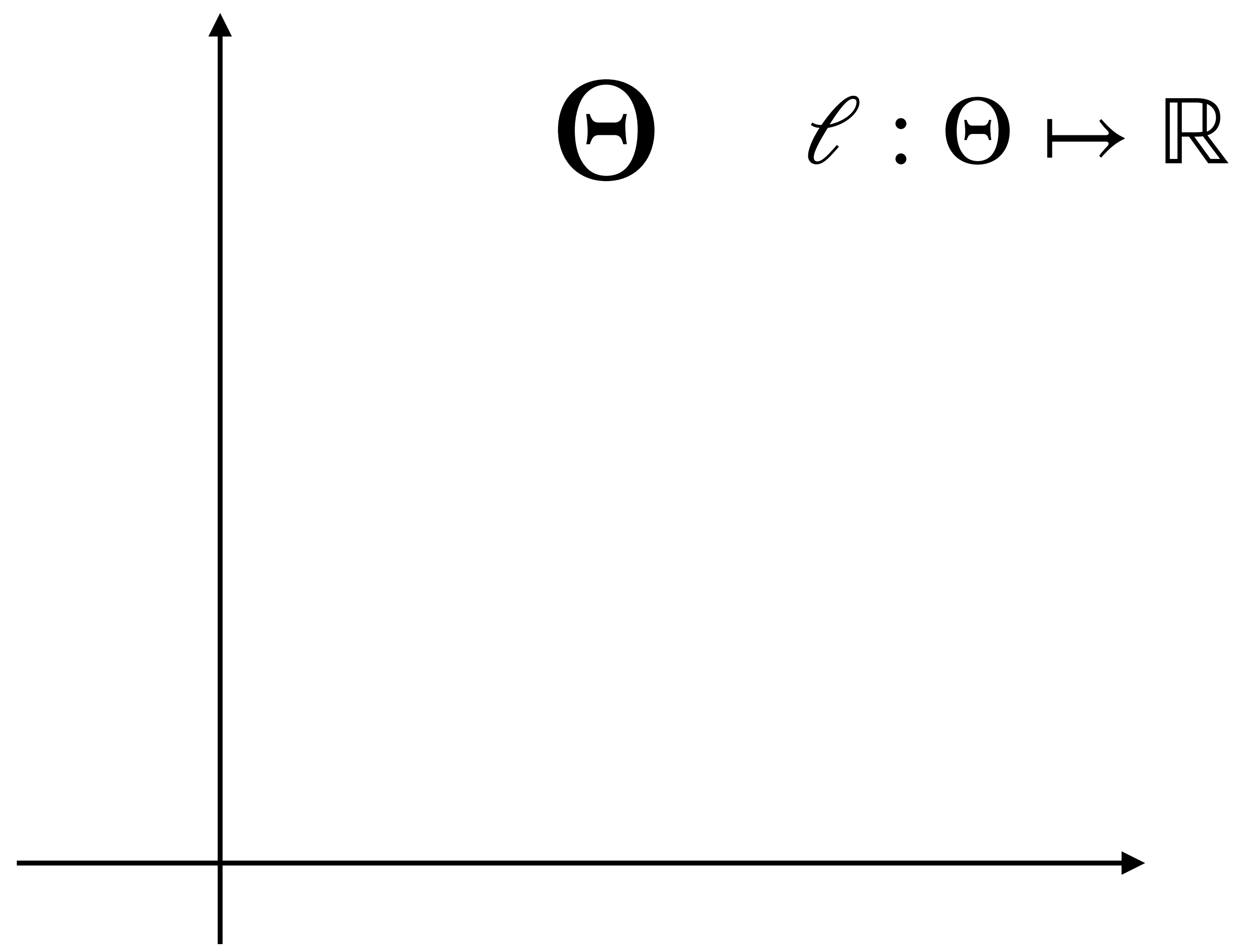
Different re-parameterization (scaling & translation) can lead to a quite different GD path



$\ell : \Theta \mapsto \mathbb{R}$



Ω



Θ
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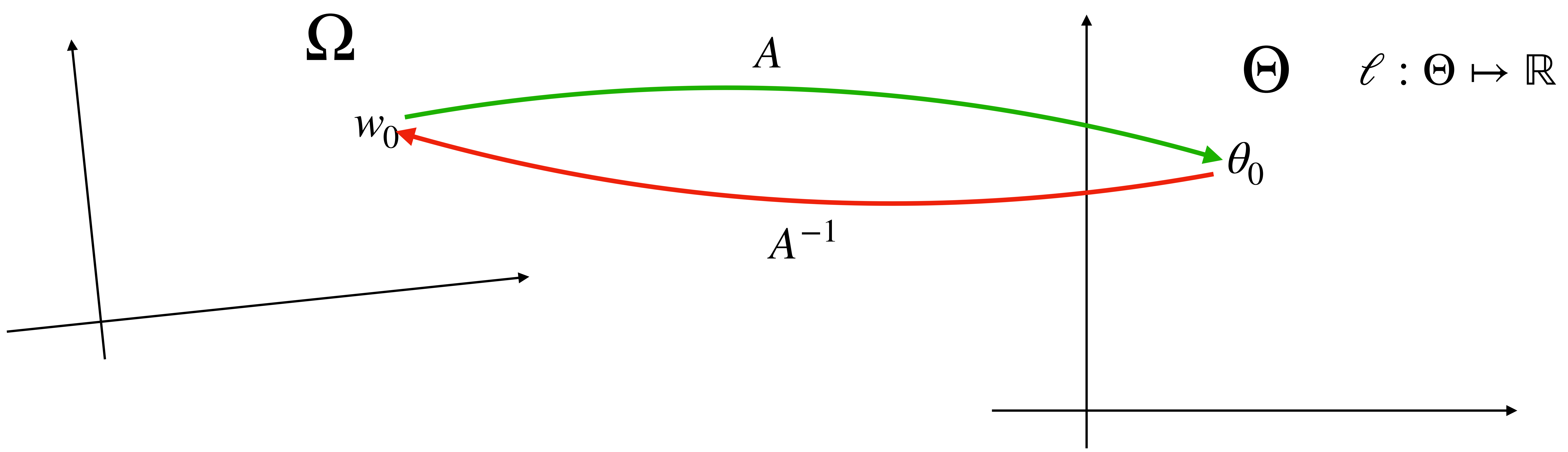

Ω
A
w_0
θ_0
A^{-1}
Θ
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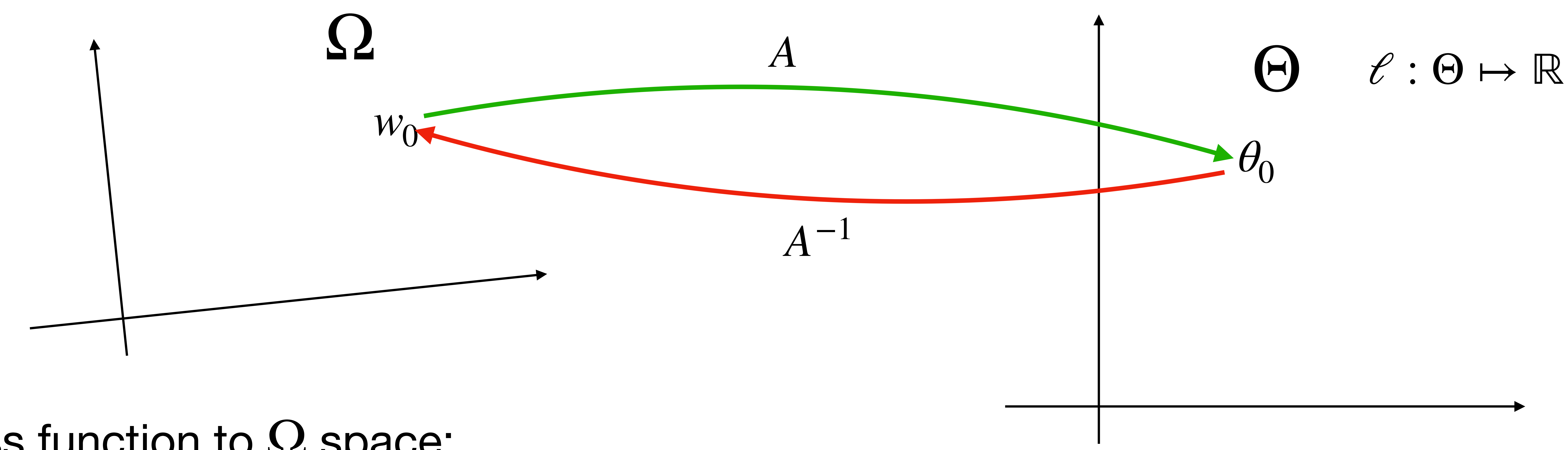
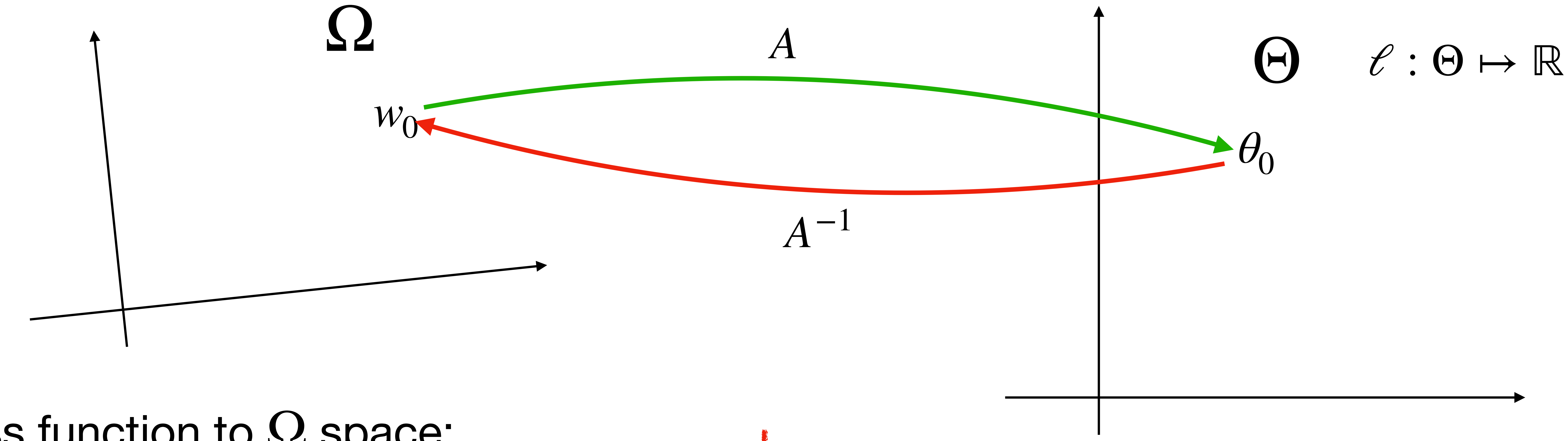


Map the loss function to Ω space:

$$g(w) := \ell(Aw)$$



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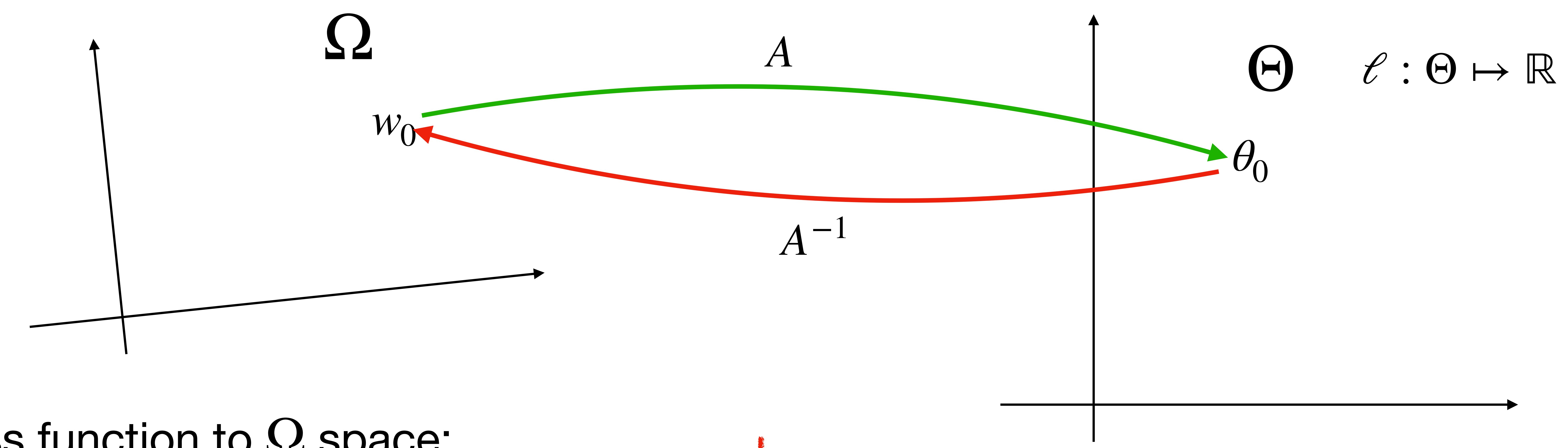

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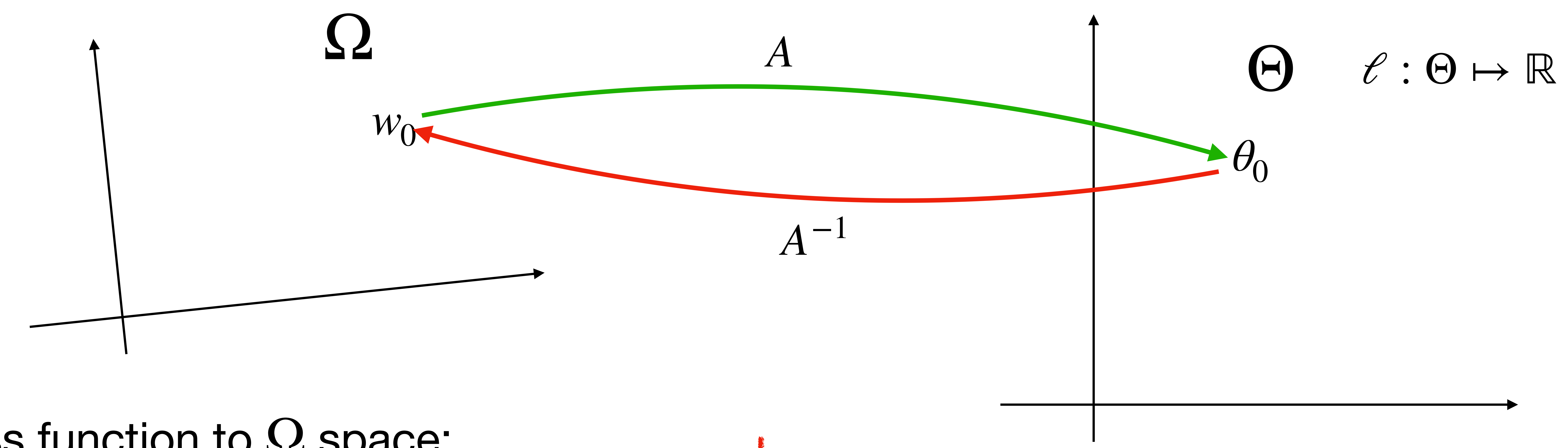

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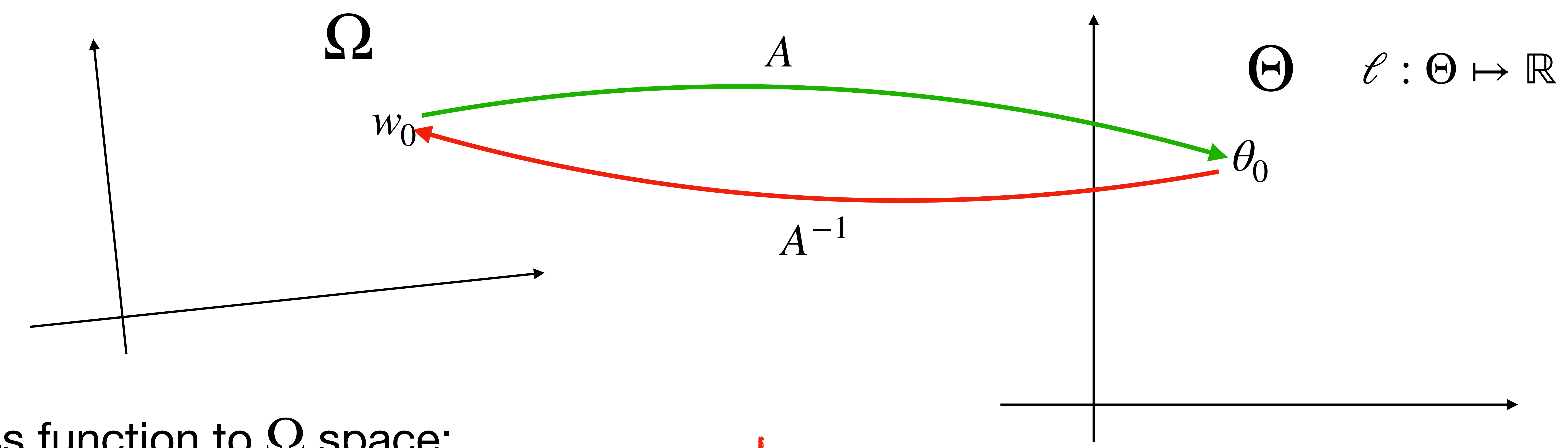

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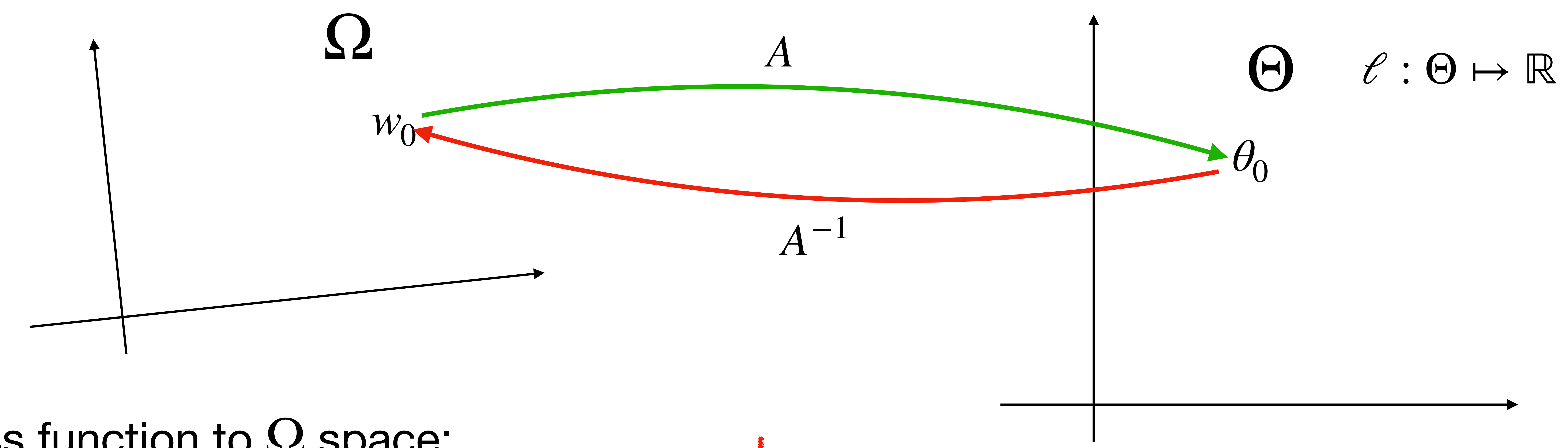

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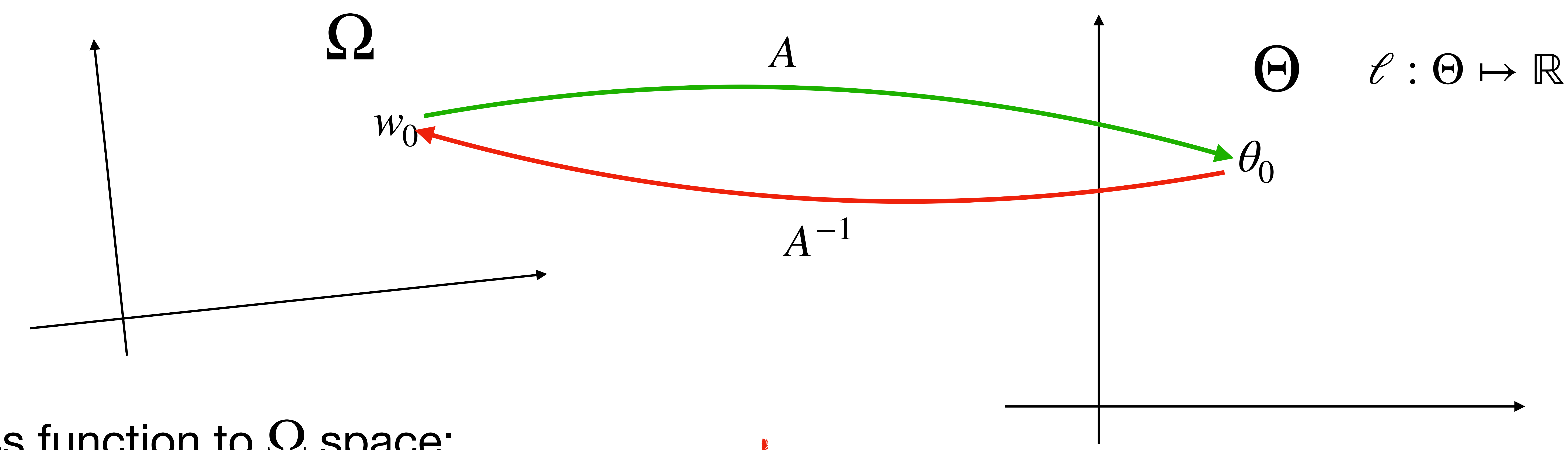

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Linear transformation A makes the GD path different!
i.e., not invariant wrt linear transformation (scaling, rotations, etc)

What would happen if we use a different distance metric…

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$$\max_{\pi_\theta} V^{\pi_\theta}(\rho)$$

$$\text{s.t., } KL\left(\mathrm{Pr}^{\pi_{\theta_0}} || \mathrm{Pr}^{\pi_\theta}\right) \leq \delta$$

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Q: How to do second-order Taylor expansion on the KL constraint?

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Fisher Information Matrix!

Second-order Taylor Expansion of KL at θ_0

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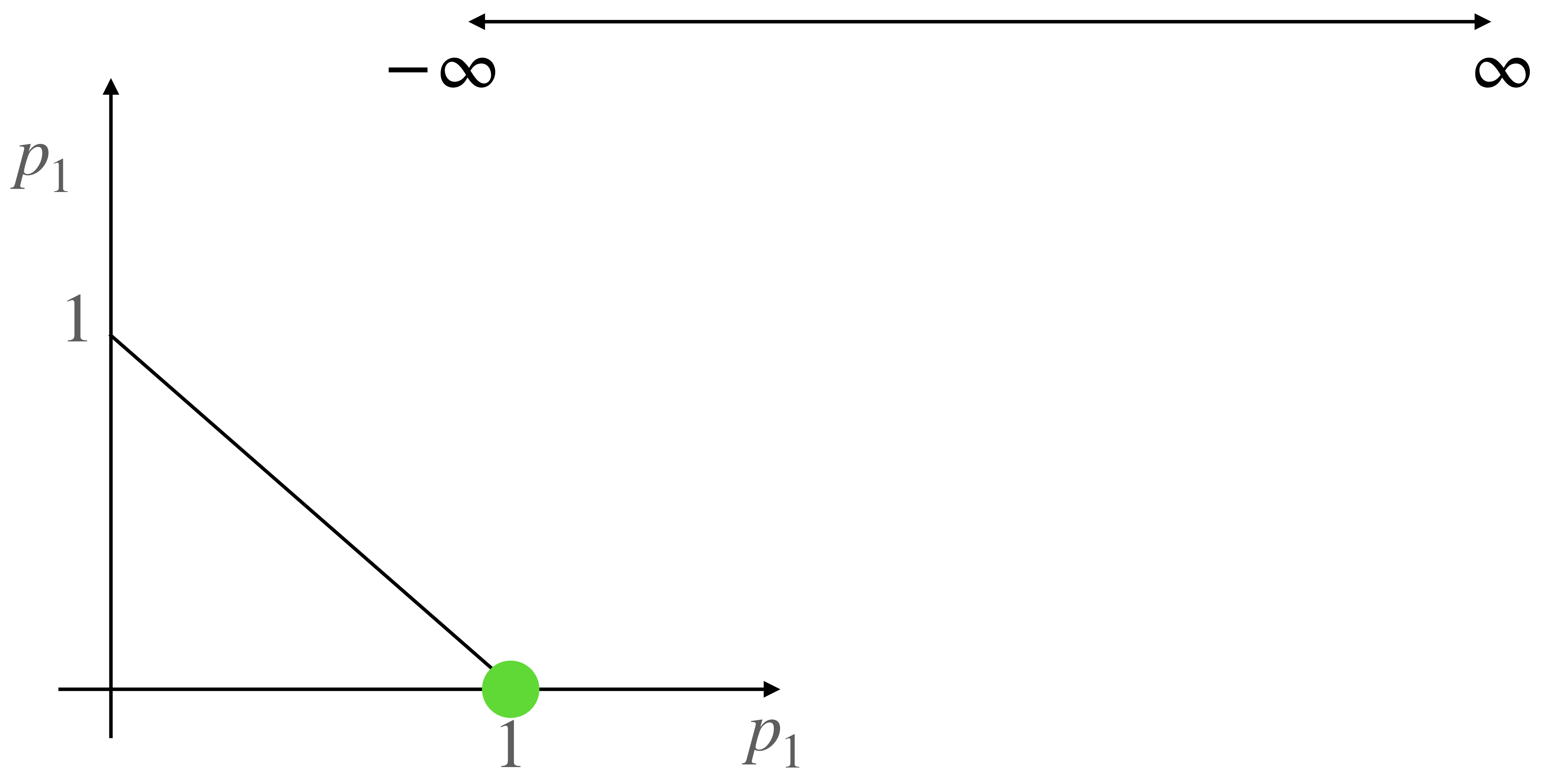

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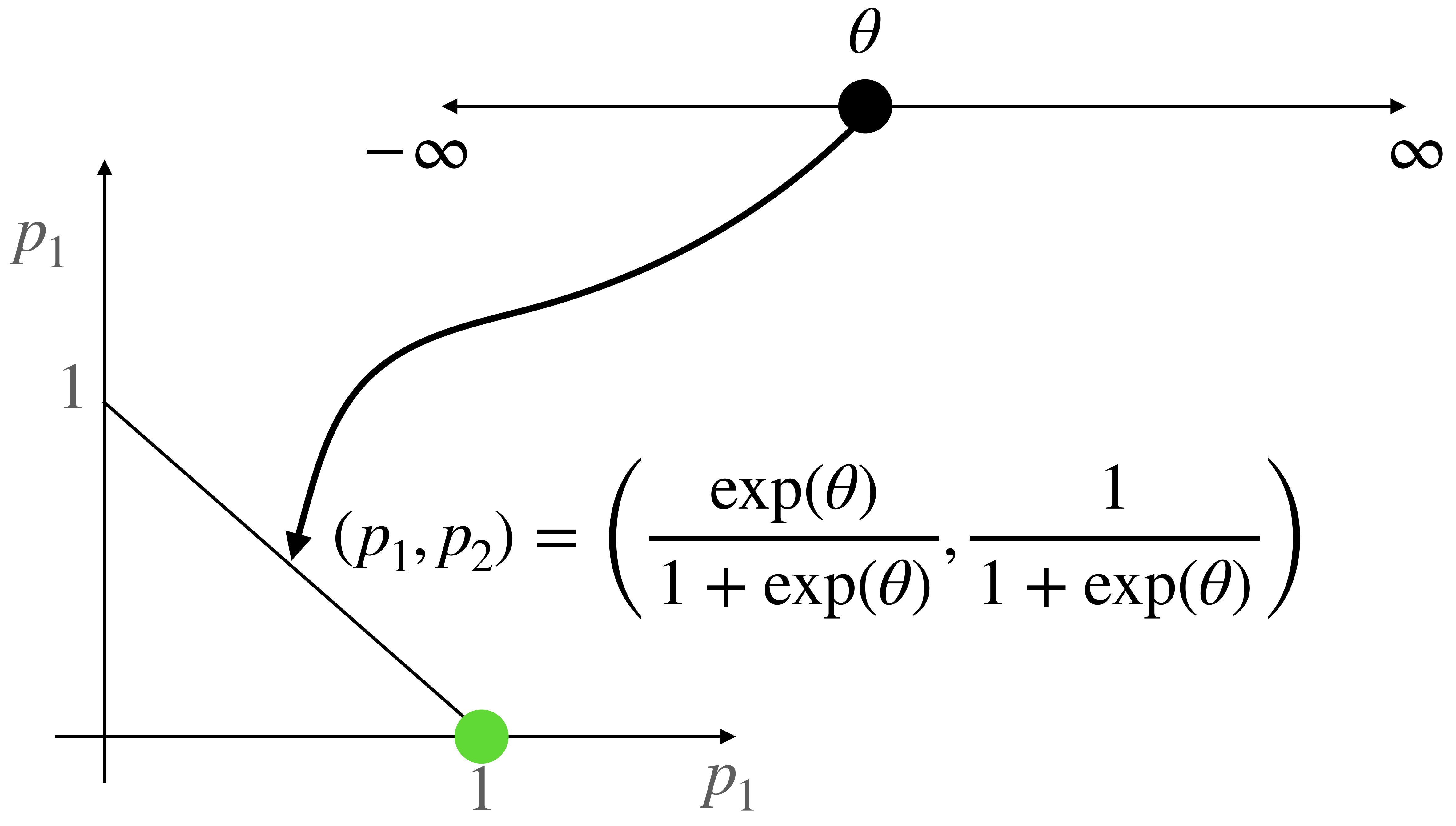

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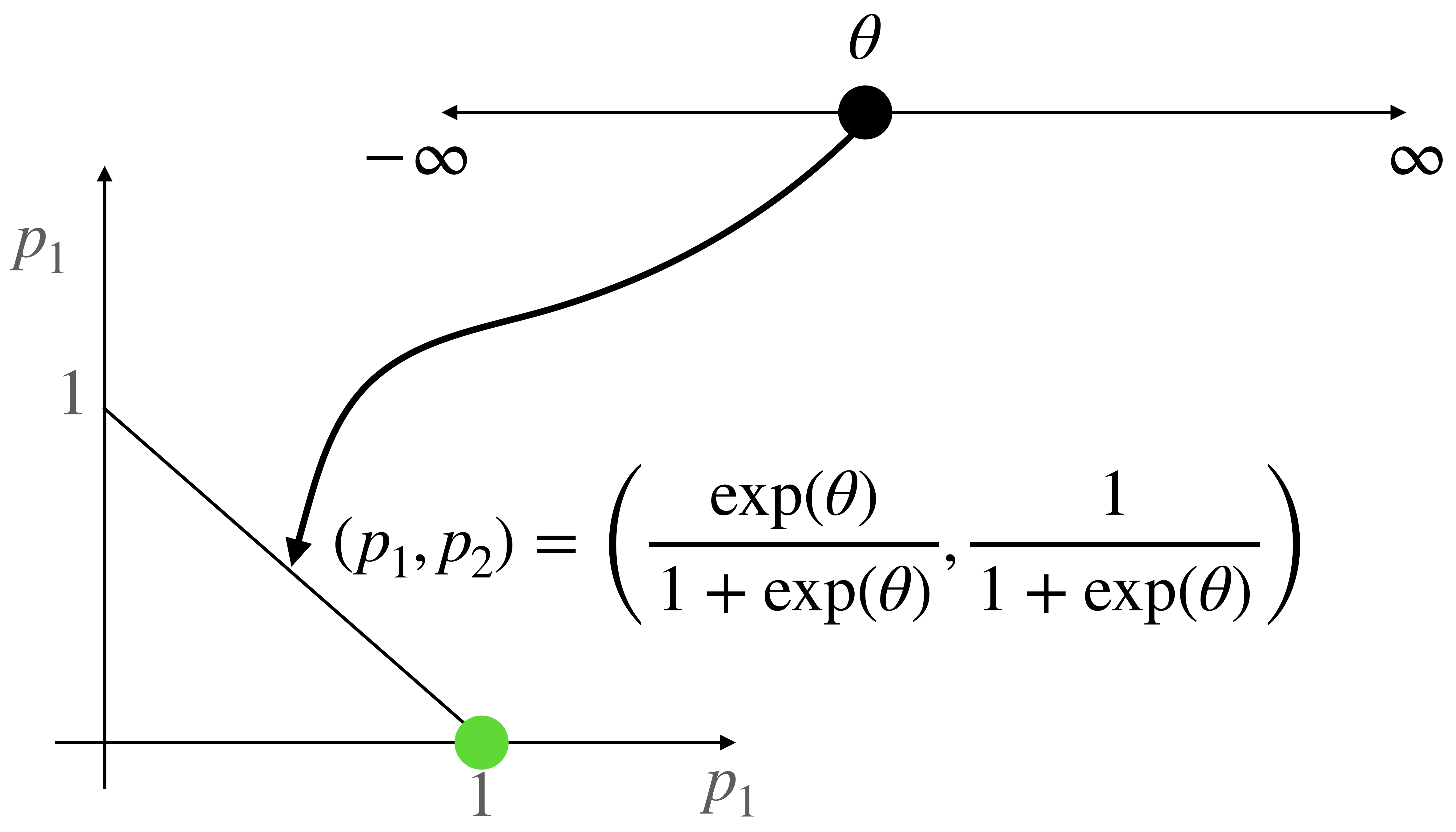

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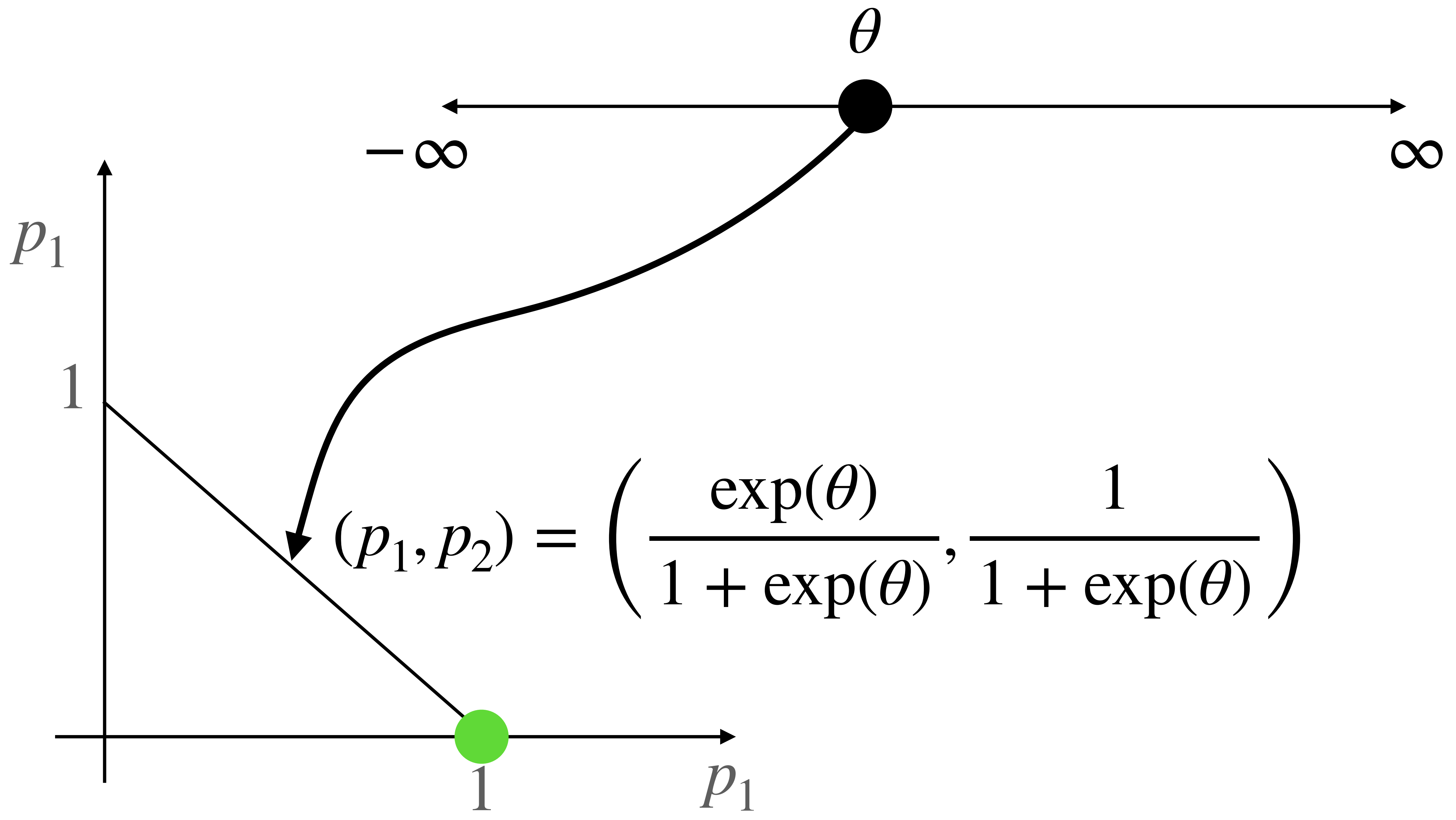

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$$F_{\theta_0}(\theta - \theta_0)^2 \leq \delta \Rightarrow (\theta - \theta_0)^2 \leq \frac{\delta}{F_{\theta_0}} \to \infty, \text{as } \theta_0 \to \infty$$

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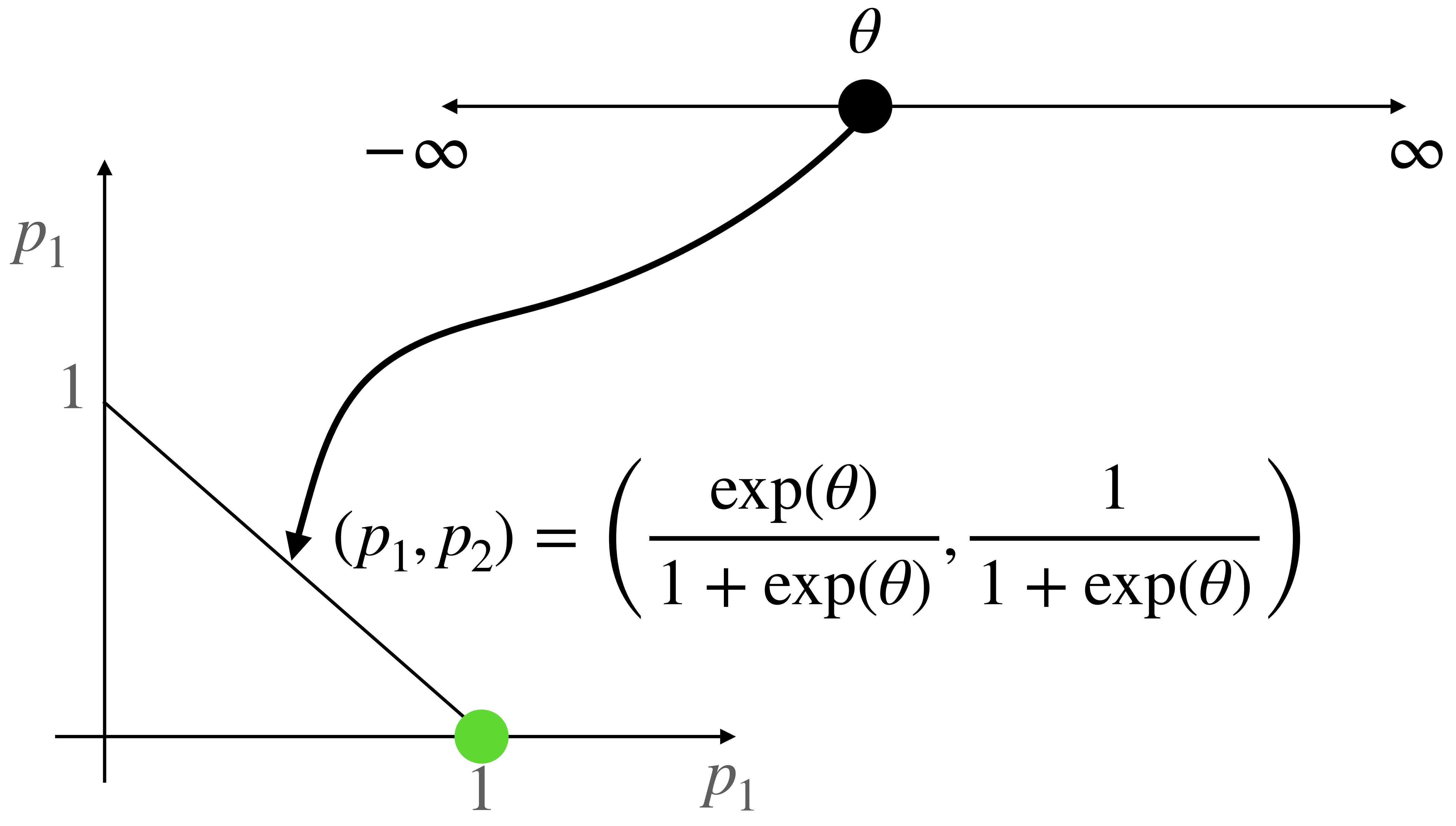

$$(p_1, p_2) = \left(\frac{\exp(\theta)}{1+\exp(\theta)}, \frac{1}{1+\exp(\theta)}\right)$$

$$F_\theta \to 0^+, \text{as } \theta \to \infty$$

$$F_{\theta_0}(\theta - \theta_0)^2 \leq \delta \Rightarrow (\theta - \theta_0)^2 \leq \frac{\delta}{F_{\theta_0}} \to \infty, \text{as } \theta_0 \to \infty$$

Plain GD in θ will move to $\theta = \infty$ at a constant speed, while Natural GD can traverse faster and faster when θ gets bigger

(Infinitely fast when $\theta \to \infty$)

Now we can solve the following quadratic programming:

$$\max_{\theta} \nabla V^{\pi_{\theta_0}}(\rho)^\top \left(\theta - \theta_0\right)$$

s.t. $\left(\theta - \theta_0\right)^\top F_{\theta_0}(\theta - \theta_0) \leq \delta$

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We have a closed form solution:

$$\theta = \theta_0 + \sqrt{\frac{\delta}{(\nabla V^{\pi_{\theta_0}})^\top F_{\theta_0}^{-1} \nabla V^{\pi_{\theta_0}}}} \cdot F_{\theta_0}^{-1} \nabla V^{\pi_{\theta_0}}$$

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Self-normalized step-size
(Learning rate is adaptive)

Summary

Natural Policy Gradient invariant to linear transformation
(Trust region constraint in terms KL on trajectory distributions)

Second order Taylor expansion of $\ell(\theta) := KL\left(\Pr^{\pi_{\theta_0}} || \Pr^{\pi_\theta}\right)$ at θ_0 is $(\theta - \theta_0)^\top F_{\theta_0}(\theta - \theta_0)$